NORM EQUALITY CONDITION IN TRIANGULAR INEQUALITY

RITSUO NAKAMOTO AND SIN-EI TAKAHASI

Received August 2, 2001

ABSTRACT. We characterize when the norm of the sum of elements in a Banach space is equal to the sum of their norms and give its applications to unital Banach algebras. Related to a recent result due to Barraa and Boumazgour, we give another equivalent condition of it in terms of linear functional on a unital C^*-algebra. Consequently, we have the following result.

For elements a and b in a unital C^*-algebra, the following statements are mutually equivalent:

(i) $\|a+b\| = \|a\| + \|b\|$.

(ii) There exists a norm one linear functional f such that $f(a) = \|a\|$ and $f(b) = \|b\|$.

(iii) There exists a state f such that $f(a^*b) = \|a\|\|b\|$.

1. INTRODUCTION

In normed linear spaces, the most fundamental inequality is the triangular inequality, that is, for elements x, y in normed space,

$$\|x+y\| \leq \|x\| + \|y\|. \tag{1}$$

It is a problem when the equality in (1) holds. In the inner product spaces, it is well known that the equality holds if and only if $x = \lambda y$ or $y = \lambda x$ for some constant $\lambda \geq 0$.

For bounded linear operators A and B on a Banach space, many authors studied the equation

$$\|A+B\| = \|A\| + \|B\|, \tag{2}$$

see [1], [9] and the references cited therein.

In [1], for a bounded linear operator T on a uniformly convex Banach space, it is shown that the Daugavet equation $\|1+T\| = 1 + \|T\|$ holds if and only if its norm $\|T\|$ is an approximate proper value of T.

Recently, for bounded linear operators A and B on a Hilbert space , Barraa and Boumazgour [2] proved (2) holds if and only if

$$\|A\|\|B\| \in \overline{W(A^*B)}, \tag{3}$$

where $\overline{W(T)}$ denotes the closure of the numerical range $W(T)$ of an operator T.

We note that the condition (3) is equivalent to the existence of a normalized positive linear functional (i.e. state) f on the algebra of all bounded linear operators such that $f(A^*B) = \|A\|\|B\|$ (see [3]).

In this note, we give a necessary and sufficient condition to hold that the norm of the sum of elements in a Banach space is equal to the sum of their norms and its applications. Related to a recent result due to Barraa and Boumazgour, we give another equivalent condition of it in terms of linear functional on a unital C^*-algebra. Consequently, for elements a and b in a

2000 *Mathematics Subject Classification.* 46B99, 47A10, 47A12, 47A30.

Key words and phrases. triangular inequality, Banach space, norm, C^*-algebra, numerical range, state.

unital C^*-algebra, the following statements are mutually equivalent: (i) $\|a+b\| = \|a\| + \|b\|$. (ii) There exists a norm one linear functional f such that $f(a) = \|a\|$ and $f(b) = \|b\|$. (iii) There exists a state f such that $f(a^*b) = \|a\|\|b\|$.

2. Results.

From the operator theoretic viewpoint, the above result of Barraa and Boumazgour is very interesting. We give its C^*-algebraic version, which is essentially same as their result:

Theorem 1. *Let $\mathcal{A}$ be a unital C^*-algebra. Then for $a, b \in \mathcal{A}$, $\|a+b\| = \|a\| + \|b\|$ holds if and only if there exists a state f on $\mathcal{A}$ such that $f(a^*b) = \|a\|\|b\|$, or equivalently, $\|a\|\|b\| \in \overline{W(a^*b)}$ where $\overline{W(c)} = \{f(c); f : \text{state on } \mathcal{A}\}$ for $c \in \mathcal{A}$.*

For the completeness, we shall give a proof.

If $\|a+b\| = \|a\| + \|b\|$ is satisfied, then there exists a state f on $\mathcal{A}$ such that

$$f((a+b)^*(a+b)) = \|a+b\|^2 = (\|a\| + \|b\|)^2,$$

as $(a+b)^*(a+b) \geq 0$. Since $f(a^*a) \leq \|a\|^2, f(b^*b) \leq \|b\|^2$ and $f(a^*b + b^*a) \leq 2\|a\|\|b\|$, it follows that $f(a^*a) = \|a\|^2, f(b^*b) = \|b\|^2$ and $f(a^*b + b^*a) = 2\|a\|\|b\|$. Furthermore, as $|f(a^*b)| \leq \|a\|\|b\|$, we have $f(a^*b) = f(b^*a) = \|a\|\|b\|$.

Conversely, if there exists a state f with $f(a^*b) = \|a\|\|b\|$, then it is easy to see that $f(a^*a) = \|a\|^2$ and $f(b^*b) = \|b\|^2$ by the Cauchy-Schwarz inequality. Hence we have

$$\begin{aligned} f((a+b)^*(a+b)) &= f(a^*a) + f(a^*b) + f(b^*a) + f(b^*b) \\ &= \|a\|^2 + 2\|a\|\|b\| + \|b\|^2 \\ &= (\|a\| + \|b\|)^2 \\ &\geq \|a+b\|^2. \end{aligned}$$

Remark 1. For a bounded linear operator T on a Hilbert space, a complex number λ ia called a normal approximate propervalue of T if there exists a sequence of unit vectors x_n such that $\lim_{n\to\infty} \|(T-\lambda)x_n\| = 0$ and $\lim_{n\to\infty} \|(T-\lambda)^*x_n\| = 0$ ([8]). It is known that $\|T\| \in \overline{W(T)}$ if and only if $\|T\|$ is a normal approximate proper value of T (see [7, 6]). Therefore, for bounded linear operators A and B, $\|A+B\| = \|A\| + \|B\|$ if and only if $\|A\|\|B\|$ is a normal approximate propervalue of A^*B, or equivalently, there exists a character χ (on the unital C^*-algebra generated by A^*B) with $\chi(A^*B) = \|A\|\|B\|$([8]).

In Banach space setting, we have

Theorem 2. *Let X be a Banach space and X^* its dual space. Let $x_1, \cdots, x_n$ be nonzero elements in X. Then $\|x_1 + \cdots + x_n\| = \|x_1\| + \cdots + \|x_n\|$ if and only if there exists an extreme point f in the closed unit ball X_1^* of X^* such that $f(x_i) = \|x_i\| \quad (i = 1, \cdots, n)$.*

Proof. Suppose that $\|x_1 + \cdots + x_n\| = \|x_1\| + \cdots + \|x_n\|$ and let $\lambda_i = \|x_i\|, u_i = x_i/\lambda_i \quad (i = 1, \cdots, n)$. Then, by the Hahn-Banach extension theorem, we can choose a norm one functional g in X^* with $g(x_1 + \cdots + x_n) = \lambda_1 + \cdots + \lambda_n$, so that

$$\frac{\lambda_1}{\lambda_1 + \cdots + \lambda_n} g(u_1) + \cdots + \frac{\lambda_n}{\lambda_1 + \cdots + \lambda_n} g(u_n) = 1.$$

Since each $g(u_i)$ is a modulus one complex number, it follows that $g(u_i) = 1$ and hence $g(x_i) = \|x_i\| \quad (i = 1, \cdots, n)$. Now set

$$F = \{h \in X^* : \|h\| \leq 1, h(x_i) = \|x_i\| \quad (i = 1, \cdots, n)\}.$$

Then F is a non-empty weak*-compact subset of X^* from the above argument and hence we can choose an extreme point f of F by the Krein-Milman theorem. We show that f is also an extreme point in the closed unit ball of X^*. To do this, let f_1 and f_2 be two elements of the closed unit ball of X^* such that $f = \dfrac{f_1 + f_2}{2}$. Then $f_1(x_i) \leq \|x_i\|, f_2(x_i) \leq \|x_i\|$ and $\dfrac{f_1(x_i) + f_2(x_i)}{2} = \|x_i\|$ for $i = 1, \cdots, n$. Therefore we have $f_1(x_i) = f_2(x_i) = \|x_i\| \quad (i = 1, \cdots, n)$ and hence $f_1, f_2 \in F$, so that $f = f_1 = f_2$.

The converse is straightforward.

Remark 2. In Theorem 2, it is easy to see that $\|x_1 + \cdots + x_n\| = \|x_1\| + \cdots + \|x_n\|$ if and only if there exists an element $f \in X_1^*$ such that $f(x_i) = \|x_i\| \quad (i = 1, \cdots, n)$.

Recall that the numerical range $V(a)$ of an element a in a unital Banach algebra $\mathcal{A}$ is defined as follows [4]:

$$V(a) = \{f(a); f(1) = 1, f \in \mathcal{A}_1^*\}.$$

As a result of Theorem 2,

Corollary 1. *For an element a in a unital Banach algebra, $\|1 + a\| = 1 + \|a\|$ holds if and only if $\|a\|$ belongs to the numerical range $V(a)$.*

Proof. By using Theorem 2, for a in a unital Banach algebra, $\|1 + a\| = 1 + \|a\|$ holds if and only if there exists a norm one linear functional f such that $f(1) = 1$ and $f(a) = \|a\|$, that is, $\|a\| \in V(a)$.

Remark 3. In a C^*-algebra, if $\|1 + a\| = 1 + \|a\|$ is satisfied, then we can choose a norm one self-adjoint linear functional f such that $f(1) = 1$ and $f(a) = \|a\|$. As $f(1) = \|f\| = 1$, f itself is a positive linear functional, that is, f is a state.

Here we notice that if a bounded linear operator A on a Hilbert space satisfies $\|1 + A\| = 1 + \|A\|$, then A is a normaloid operator (i.e. $\|A\|$ is equal to the spectral radius of A). However the converse does not necessarily hold as easily seen by the example $A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \oplus i$.

A bridge between Theorem 1 and Theorem 2 is the following.

Theorem 3. *Let $\mathcal{A}$ be a unital C^*-algebra. The the following statements are equivalent:*

(i) For $a, b \in \mathcal{A}$, there exists a norm one linear functional f such that $f(a) = \|a\|$ and $f(b) = \|b\|$.

(ii) For $a, b \in \mathcal{A}$, there exists a state f such that $f(a^*b) = \|a\|\|b\|$.

Proof. (i)$\Rightarrow$ (ii). Let f be a norm one linear functional on A such that $f(a) = \|a\|$ and $f(b) = \|b\|$, and $f = u \cdot g$ be the enveloping polar decomposition of f where u is a partial isometry and g is a state (see [5]). If $f(a) = \|a\|$ and $f(b) = \|b\|$, then we have $g(a^*a) = \|a\|^2$ and $g(b^*b) = \|b\|^2$, because

$$\|a\|^2 = g(ua)^2 \leq g(a^*u^*ua) \leq g(a^*a) \leq \|a\|^2.$$

Also, we have

$$\begin{aligned}(\|a\| + \|b\|)^2 = g(ua + ub)^2 &\leq g((a + b)^*u^*u(a + b)) \\ &\leq g((a + b)^*(a + b)) \leq \|a + b\|^2.\end{aligned}$$

Hence it follows that $g(a^*b + b^*a) = 2\|a\|\|b\|$, and so, $g(a^*b) = \|a\|\|b\|$.

(ii)$\Rightarrow$ (i). Let $g(x) = \frac{1}{\|a\|} f(a^*x)$ for $x \in \mathcal{A}$. Then it is easy to see that g is a norm one linear functional with $g(a) = \|a\|$ and $g(b) = \|b\|$.

References

[1] Y.A.Abranovich, C.D.Aliprantis and O.Burkinshaw, *The Daugavet equation in uniformly convex Banach spaces*, J. Func. Anal, 97(1991), 215-230.

[2] M.Barraa and M.Boumazgour , *Inner derivations and norm equality*, to appear in Proc. Amer Math. Soc., (Article electronically published on May 25, 2001).

[3] S.K.Bernerian and G.H.Orland, *On the closure of the numerical range of an operator*, Proc. Amer. Math. Soc., 18(1967), 499-503.

[4] F.F.Bonsall and J.Duncan , *Complete normed algebras*, Springer-Verlag, 1973.

[5] J.Dixmier , *C^*-algebras*, North-Holland Publishing Company, 1977.

[6] M.Fujii and R.Nakamoto, *On normal approximate spectrum. II*, Proc. Japan Acad., 48(1972), 297-301.

[7] S.Hildebrandt, *Über den numerische Wertebereich eines Operators*, Math. Ann., 163(1966), 230-247.

[8] I.Kasahara and H.Takai, *Approximate propervalues and characters of C^*-algebra*, Proc. Japan Acad., 48(1972), 91-93.

[9] C.S.Lin, *Generalized Daugavet equations and invertible operators on uniformly convex Banach spaces*, J. Math. Anal. and Appl., 197(1996), 518-528.

Faculty of Engineering, Ibaraki University, Nakanarusawa, Hitachi, Ibaraki 316-0033, Japan.
E-mail address: `nakamoto@@base.ibaraki.ac.jp`

Department of Basic Technology, Applied Mathematics and Physics, Yamagata University, Yonezawa 992-8510, Japan.