# NEW DERIVATION OF CONSERVATION LAWS FOR MAXIMIZING PROBLEM UNDER CONSTRAINTS

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Abstract. In contrast with Noether theorem, we built up a new operative procedure for the derivation of conservation laws. The purpose of this paper is to give further generalization of discovering conservation laws for the maximizing problem under constraints given by a system of functions with arbitrary degree of homogeneity. An illustration is made in the generalized growth model of von Neumann type.

**Introduction.** Noether theorem (Noether [11]) concerning with symmetries of the action integral or its generalization (Bessel-Hagan [1]) with those up to divergence plays an effective role for discovering conservation laws from the Lagrangian or the Hamiltonian structures of considering problem. In contrast with Noether theorem, we built up a new operative procedure for the derivation of conservation laws (Mimura and Nôno [6]) without using either Lagrangian or Hamiltonian structures. It was discussed first for a system of second-order differential equations and then the system was supposed to be given in the form of the Euler-Lagrange equations with some Lagrangian. The results were applied to various economic growth models (Mimura and Nôno [7]; Mimura, Fujiwara and Nôno [8], [9]; Fujiwara, Mimura and Nôno [2]-[5]) to discover new economic conservation laws including non-Noether ones.

Particularly in [7], there was discussed the optimal control problem to maximize an integration over a finite ($0 < T < \infty$) or an infinite ($T = \infty$) period of time:

$$\int_0^T e^{-\rho t} U(\dot{q},\, q) dt, \tag{1}$$

under constraint $F(\dot{q},\, q) = 0$ given by a first degree homogeneous function $F(\dot{q},\, q)$ with respect to $q$ and $\dot{q}$, where $q = (q^i(t))$, $\dot{q} = (dq^i/dt)$ $(i = 1, \cdots, n)$ and $\rho$ is a constant. Neamţu [10] reformed some results for costructing conserved quantities in [7] by replacing $F(\dot{q},\, q) = 0$ with $F^a(\dot{q},\, q) = 0$ $(a = 1, ..., m)$, where $F^a(\dot{q},\, q)$ are first degree homogeneous functions.

The purpose of this paper is to give further generalization of discovering conservation laws for the maximizing problem of (1) under constraint given by a system of functions with arbitrary degree of homogeneity:

$$F^a(\dot{q},\, q) = 0 \quad (a = 1, ..., m). \tag{2}$$

Here recall that von Neumann [14] gave an analysis of a model which has a unique ray of balanced growth. Samuelson [12] generalized the analysis by studying transient approaches

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to the von Neumann balanced-growth ray, which was given now as the maximizing problem of the integration

$$\int_0^T \dot{K}^1 dt$$

under constraint $F(\dot{K}, K) = 0$ given by a first degree homogeneous function $F(\dot{K}, K)$, where $K = (K^i(t))$ and $\dot{K} = (\dot{K}^i) = (dK^i/dt)$ $(i = 1, \cdots, n)$ are understood respectively as the $n$ capital goods and the $n$ capital formations. This is the model called von Neumann growth model, which grew up in [7] to be the generalized growth model of von Neumann type by replacing the integrating function $\dot{K}^1$ with the total time derivative of first degree homogeneous function $G(K)$ with respect to $K^i$, while the constraint $F(\dot{K}, K) = 0$ is unchanged.

In the application of this paper, the model of von Neumann type in [7] is generalized moreover with respect to the constraint by replacing the first degree homogeneous function with a system of arbitrary degree homogeneus functions. In his model ($n = 2$: $i = 1, 2$), Samuelson [13] derived two conserved quantities, which conclude that the ratio of the national income and the national wealth is constant (income-wealth conservation law). In our generalized model, together with $\rho = 0$ and $n = 2$, the system of functions with arbitrary degree of homogeneity are reduced later to a homogeneous second order polynomial. Two conserved quantites are derived and then, by using of which, a class of optimal paths for the finite horizon are determined in the reduced situation. Finally, in view of the optimal paths, it is detailed the Samuelson's income-wealth conservation law in our model.

For convenience, differentiability is assumed to be of sufficiently high order and the summation convention is employed throughout.

**1. Maximizing problem for optimal economic growths.** Our discussion begins with an optimal control problem to maximize the integration (1) over a finite or an infinite period of time under the constraint (2) given by a system of functions $F^a(\dot{q}, q)$ $(a = 1, \cdots, m)$ with arbitrary degree of homogeneity with respect to $q$ and $\dot{q}$. In the variational principle with the multiplier technique, we set the following Lagrangian as usual

$$L(\dot{\lambda}, \dot{q}, \lambda, q, t) = e^{-\rho t} U(\dot{q}, q) + \lambda_a F^a(\dot{q}, q). \tag{3}$$

Here put $\lambda_a = q^{n+a}$ and arrange the variables $q^i$ and $\lambda_a$ as $(q^\alpha) = (q^1, \cdots, q^n, \lambda_1, \cdots, \lambda_m)$. Then, according to $n+1 \le \alpha \le n+m$ and $1 \le \alpha \le n$, the Euler-Lagrange equations of (3):

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^\alpha}\right) - \frac{\partial L}{\partial q^\alpha} = 0 \tag{4}$$

separate into $F^a = 0$ $(a = 1, \cdots, m)$ and

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = 0 : \quad \frac{d}{dt}\left(e^{-\rho t}\frac{\partial U}{\partial \dot{q}^i} + \lambda_a \frac{\partial F^a}{\partial \dot{q}^i}\right) = e^{-\rho t}\frac{\partial U}{\partial q^i} + \lambda_a \frac{\partial F^a}{\partial q^i}. \tag{5}$$

A conserved quantity (first integral) in question is a quantity $\Omega(\dot{\lambda}, \dot{q}, \lambda, q, t)$ satisfying $d\Omega/dt = 0$ (conservation law) on the optimal paths, i.e., on solutions to (4), or equivalently to $F^a = 0$ and (5). The theorem 6 in [6] is reformulated as follows ([10], Theorem 4; cf. [7], Theorem 1).

**Theorem 1.** *For the Lagrangian* (3), *let* $(\xi_1^i,\,\eta_a^1) = (\xi_1^i(\dot\lambda,\,\dot q,\,\lambda,\,q,\,t),\,\eta_a^1(\dot\lambda,\,\dot q,\,\lambda,\,q,\,t))$ *and* $(\xi_2^i,\,\eta_a^2) = (\xi_2^i(\dot\lambda,\,\dot q,\,\lambda,\,q,\,t),\,\eta_a^2(\dot\lambda,\,\dot q,\,\lambda,\,q,\,t))$ *satisfy the equations*

$$(6)\quad \frac{\partial F^a}{\partial \dot q^i}\frac{d\xi^i}{dt} + \frac{\partial F^a}{\partial q^i}\xi^i = 0,$$

$$(7)\quad \frac{d}{dt}\left(\frac{\partial^2 L}{\partial \dot q^i \partial \dot q^j}\frac{d\xi^j}{dt} + \frac{\partial^2 L}{\partial \dot q^i \partial q^j}\xi^j + \frac{\partial F^a}{\partial \dot q^i}\eta_a\right) - \left(\frac{\partial^2 L}{\partial \dot q^j \partial q^i}\frac{d\xi^j}{dt} + \frac{\partial^2 L}{\partial q^i \partial q^j}\xi^j + \frac{\partial F^a}{\partial q^i}\eta_a\right) = 0,$$

*on the optimal paths for the maximizing problem of* (1) *under the constraints* (2). *Then the following conserved quatity* $\Omega$ *for the problem is constructed from* $(\xi_1^i,\,\eta_a^1)$ *and* $(\xi_2^i,\,\eta_a^2)$:

$$(8)\qquad \Omega = \frac{\partial^2 L}{\partial \dot q^i \partial \dot q^j}\left(\xi_1^i\frac{d\xi_2^j}{dt} - \xi_2^j\frac{d\xi_1^i}{dt}\right) + \frac{\partial^2 L}{\partial \dot q^i \partial q^j}(\xi_1^i\xi_2^j - \xi_2^i\xi_1^j) + \frac{\partial F^a}{\partial \dot q^i}(\xi_1^i\eta_a^2 - \xi_2^i\eta_a^1).$$

In terms of $(\xi^i,\,\eta_a) = (\dot q^i,\,\dot\lambda_a + \rho\lambda_a)$ satisfying (6) and (7), the theorem 7 in [6] is also reformulated as follows ([10], Theorem 5; cf. [7], Theorem 2).

**Theorem 2**. *For the Lagrangian* (3), *let* $(\xi^i,\,\eta_a) = (\xi^i(\dot\lambda,\,\dot q,\,\lambda,\,q,\,t),\,\eta_a(\dot\lambda,\,\dot q,\,\lambda,\,q,\,t))$ *satisfy the equations* (6) *and* (7) *on the optimal paths for the maximizing problem of* (1) *under the constraints* (2). *Then the following conserved quantity* $\Omega$ *for the problem is constructed from* $(\xi^i,\,\eta_a)$:

$$(9)\qquad \Omega = \dot q^i\left(\frac{\partial^2 L}{\partial \dot q^i \partial \dot q^j}\frac{d\xi^j}{dt} + \frac{\partial^2 L}{\partial \dot q^i \partial q^j}\xi^j + \frac{\partial F^a}{\partial \dot q^i}\eta_a\right) - \left(\frac{\partial L}{\partial q^i} + \rho\frac{\partial L}{\partial \dot q^i}\right)\xi^i.$$

Here we impose an arbitrary degree $s$ of homogeneity on the system of functions $F^a(\dot q,\,q)$ $(a = 1,\cdots,m)$ with respect to $\dot q^j$ and $q^j$, i.e.,

$$(10)\qquad \dot q^j\frac{\partial F^a}{\partial \dot q^j} + q^j\frac{\partial F^a}{\partial q^j} = sF^a,$$

which guarantee that $\xi^i = q^i$ satisfy the equation (6) whenever $F^a = 0$. So, together with $\xi^i = q^i$, the Lagrangian (3) is substituted for the equation (7). Then, by the differentiations of (10) with respect to $\dot q^i$ and $q^i$:

$$(11)\qquad \begin{aligned} \dot q^j\frac{\partial^2 F^a}{\partial \dot q^i \partial \dot q^j} + q^j\frac{\partial^2 F^a}{\partial \dot q^i \partial q^j} &= (s-1)\frac{\partial F^a}{\partial \dot q^i},\\ \dot q^j\frac{\partial^2 F^a}{\partial \dot q^j \partial q^i} + q^j\frac{\partial^2 F^a}{\partial q^i \partial q^j} &= (s-1)\frac{\partial F^a}{\partial q^i}, \end{aligned}$$

the equations (7) are reduced to

$$\begin{aligned} &\frac{d}{dt}\left(e^{-\rho t}\left(\dot q^j\frac{\partial^2 U}{\partial \dot q^i \partial \dot q^j} + q^j\frac{\partial^2 U}{\partial \dot q^i \partial q^j}\right) + ((s-1)\lambda_a + \eta_a)\frac{\partial F^a}{\partial \dot q^i}\right)\\ &\quad - \left(e^{-\rho t}\left(\dot q^j\frac{\partial^2 U}{\partial \dot q^j \partial q^i} + q^j\frac{\partial^2 U}{\partial q^i \partial q^j}\right) + ((s-1)\lambda_a + \eta_a)\frac{\partial F^a}{\partial q^i}\right) = 0. \end{aligned}$$

Here assume also that $U(\dot q,\,q)$ is $r$-th degree homogeneous with respect to $\dot q^j$ and $q^j$, i.e.,

$$(12)\qquad \dot q^j\frac{\partial U}{\partial \dot q^j} + q^j\frac{\partial U}{\partial q^j} = rU,$$

and put $\eta_a = (r-s)\lambda_a$. Then, by the relations from (5):

$$\frac{d}{dt}\left(\lambda_a \frac{\partial F^a}{\partial \dot q^i}\right) - \lambda_a \frac{\partial F^a}{\partial q^i} = -\frac{d}{dt}\left(e^{-\rho t}\frac{\partial U}{\partial \dot q^i}\right) + e^{-\rho t}\frac{\partial U}{\partial q^i},$$

the equations (7) lead finally to

$$\frac{d}{dt}\left(e^{-\rho t}\left(\dot q^j \frac{\partial^2 U}{\partial \dot q^i \partial \dot q^j} + q^j \frac{\partial^2 U}{\partial \dot q^i \partial q^j} - (r-1)\frac{\partial U}{\partial \dot q^i}\right)\right)$$
$$-e^{-\rho t}\left(\dot q^j \frac{\partial^2 U}{\partial \dot q^j \partial q^i} + q^j \frac{\partial^2 U}{\partial q^i \partial q^j} - (r-1)\frac{\partial U}{\partial q^i}\right) = 0,$$

which are satisfied identically by virtue of the following differentiations of (12) with respect to $\dot q^i$ and $q^i$:

$$\text{(13)} \qquad \begin{aligned} \dot q^j \frac{\partial^2 U}{\partial \dot q^i \partial \dot q^j} + q^j \frac{\partial^2 U}{\partial \dot q^i \partial q^j} &= (r-1)\frac{\partial U}{\partial \dot q^i}, \\ \dot q^j \frac{\partial^2 U}{\partial \dot q^j \partial q^i} + q^j \frac{\partial^2 U}{\partial q^i \partial q^j} &= (r-1)\frac{\partial U}{\partial q^i}. \end{aligned}$$

Therefore $(\xi^i, \eta_a) = (q^i, (r-s)\lambda_a)$ satisfies (6) and (7) on the optimal paths. The solution is substituted for (9) to obtain the conserved quantity

$$\begin{aligned} \Omega &= \dot q^i \left(\dot q^j \frac{\partial^2 L}{\partial \dot q^i \partial \dot q^j} + q^j \frac{\partial^2 L}{\partial \dot q^i \partial q^j} + (r-s)\lambda_a \frac{\partial F^a}{\partial \dot q^i}\right) - q^i \left(\frac{\partial L}{\partial q^i} + \rho \frac{\partial L}{\partial \dot q^i}\right) \\ &= \dot q^i \left(\dot q^j \frac{\partial^2 L}{\partial \dot q^i \partial \dot q^j} + q^j \frac{\partial^2 L}{\partial \dot q^i \partial q^j}\right) - e^{-\rho t} q^i \frac{\partial U}{\partial q^i} - \rho q^i \frac{\partial L}{\partial \dot q^i} + (r-s)\lambda_a \dot q^i \frac{\partial F^a}{\partial \dot q^i} - \lambda_a q^i \frac{\partial F^a}{\partial q^i}. \end{aligned}$$

Since by (11), (12) and (13):

$$\begin{aligned} \dot q^i \left(\dot q^j \frac{\partial^2 L}{\partial \dot q^i \partial \dot q^j} + q^j \frac{\partial^2 L}{\partial \dot q^i \partial q^j}\right) - e^{-\rho t} q^i \frac{\partial U}{\partial q^i} &= (r-1)e^{-\rho t}\dot q^i \frac{\partial U}{\partial \dot q^i} - e^{-\rho t} q^i \frac{\partial U}{\partial q^i} + (s-1)\lambda_a \dot q^i \frac{\partial F^a}{\partial \dot q^i} \\ &= r e^{-\rho t}\left(\dot q^i \frac{\partial U}{\partial \dot q^i} - U\right) + (s-1)\lambda_a \dot q^i \frac{\partial F^a}{\partial \dot q^i}, \end{aligned}$$

$\Omega$ is written as

$$\Omega = r e^{-\rho t}\left(\dot q^i \frac{\partial U}{\partial \dot q^i} - U\right) - \rho q^i \frac{\partial L}{\partial \dot q^i} + (r-1)\lambda_a \dot q^i \frac{\partial F^a}{\partial \dot q^i} - \lambda_a q^i \frac{\partial F^a}{\partial q^i};$$

and since by (10):

$$(r-1)\lambda_a \dot q^i \frac{\partial F^a}{\partial \dot q^i} - \lambda_a q^i \frac{\partial F^a}{\partial q^i} = -r\lambda_a q^i \frac{\partial F^a}{\partial q^i} + (r-1)s\lambda_a F^a,$$

$\Omega$ is written finally as

$$\Omega = r\left(-\lambda_a q^i \frac{\partial F^a}{\partial q^i} + e^{-\rho t}\left(\dot q^i \frac{\partial U}{\partial \dot q^i} - U\right)\right) - \rho q^i \left(\lambda_a \frac{\partial F^a}{\partial \dot q^i} + e^{-\rho t}\frac{\partial U}{\partial \dot q^i}\right) + (r-1)s\lambda_a F^a.$$

Thus, including the result ([10], Theorem 6), the theorem 3 in [7] is generalized as follows.

**Theorem 3.** *Let the functions $U(\dot{q},\, q)$ in (1) and $F^a(\dot{q},\, q)$ in (2) be $r$-th and $s$-th degree homogeneous with respect to $\dot{q}^i$ and $q^i$, respectively. Then, there exists the following conserved quantity $\Omega$ for the maximizing problem of (1) under the constraints (2):*

$$\Omega = r\left(-\lambda_a q^i \frac{\partial F^a}{\partial q^i} + e^{-\rho t}\left(\dot{q}^i \frac{\partial U}{\partial \dot{q}^i} - U\right)\right) - \rho q^i \left(\lambda_a \frac{\partial F^a}{\partial \dot{q}^i} + e^{-\rho t}\frac{\partial U}{\partial \dot{q}^i}\right). \tag{14}$$

We consider now a function $\varphi(\dot{q},\, q)$ which can separate the conserved quantity (14) into two conserved ones

$$\Omega_1 = \lambda_a q^i \frac{\partial F^a}{\partial q^i} - e^{-\rho t}\left(\dot{q}^i \frac{\partial U}{\partial \dot{q}^i} - U\right) + \rho\varphi, \tag{15}$$

$$\Omega_2 = q^i \left(\lambda_a \frac{\partial F^a}{\partial \dot{q}^i} + e^{-\rho t}\frac{\partial U}{\partial \dot{q}^i}\right) - r\varphi, \tag{16}$$

i.e., a function $\varphi(\dot{q},\, q)$ satisfying $d\Omega_1/dt = 0$ and $d\Omega_2/dt = 0$, where $-\Omega = r\Omega_1 + \rho\Omega_2$. In $d\Omega_1/dt$, it follows the identity

$$\frac{d}{dt}\left(e^{-\rho t}\dot{q}^i \frac{\partial U}{\partial \dot{q}^i} - e^{-\rho t}U\right) = \dot{q}^i \frac{d}{dt}\left(e^{-\rho t}\frac{\partial U}{\partial \dot{q}^i}\right) - e^{-\rho t}\dot{q}^i \frac{\partial U}{\partial q^i} + \rho e^{-\rho t}U,$$

and, by (10), the identity

$$\begin{aligned}\frac{d}{dt}\left(\lambda_a q^i \frac{\partial F^a}{\partial q^i}\right) &= \frac{d}{dt}\left(-\lambda_a \dot{q}^i \frac{\partial F^a}{\partial \dot{q}^i} + s\lambda_a F^a\right) \\ &= -\dot{q}^i \frac{d}{dt}\left(\lambda_a \frac{\partial F^a}{\partial \dot{q}^i}\right) + (s-1)\lambda_a \ddot{q}^i \frac{\partial F^a}{\partial \dot{q}^i} + s\lambda_a \dot{q}^i \frac{\partial F^a}{\partial q^i} + s\dot{\lambda}_a F^a.\end{aligned}$$

So, in view of (3), $d\Omega_1/dt$ leads to

$$\frac{d\Omega_1}{dt} = \rho\left(\frac{d\varphi}{dt} - e^{-\rho t}U\right) - \dot{q}^i \left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i}\right) + s\dot{\lambda}_a F^a.$$

also, in view of (10) and (12), $d\Omega_2/dt$ leads to

$$\frac{d\Omega_2}{dt} = -r\left(\frac{d\varphi}{dt} - e^{-\rho t}U\right) + q^i \left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i}\right) + s\lambda_a F^a.$$

Therefore $\Omega_1$ and $\Omega_2$ become conserved quantities if $\varphi$ satisfies $d\varphi/dt = e^{-\rho t}U$, i.e.,

$$\varphi = \int e^{-\rho t}U\,dt;$$

which is substituted for (15) and (16) to deduce:

**Theorem 4**. *Let the functions $U(\dot{q},\, q)$ in (1) and $F(\dot{q},\, q)$ in (2) be $r$-th and $s$-th degree homogeneous with respect to $\dot{q}^i$ and $q^i$, respectively. Then, there exist the following two conserved quantities $\Omega_1$ and $\Omega_2$ for the maximizing problem of (1) under the constraints (2):*

$$\Omega_1 = \lambda_a q^i \frac{\partial F^a}{\partial q^i} - e^{-\rho t}\left(\dot{q}^i \frac{\partial U}{\partial \dot{q}^i} - U\right) + \rho\int e^{-\rho t}U\,dt, \tag{17}$$

(18) $$\Omega_2 = q^i\left(\lambda_a\frac{\partial F^a}{\partial \dot{q}^i} + e^{-\rho t}\frac{\partial U}{\partial \dot{q}^i}\right) - r\int e^{-\rho t}U\,dt.$$

**2. Reformation of the maximizing problem.** Here assume that $U$ in (1) is a function of $q = (q^i)$ which is a collection of state variables $x = (x^\mu)$ and control variables $u = (u^\sigma)$, i.e., $(q^i) = (x^\mu, u^\sigma)$ $(\mu = 1, \cdots, k;\ \sigma = 1, \cdots, \ell;\ k + \ell = n)$; and $F^a$ $(a = 1, \cdots, m)$ is the following system of first degree homogeneous functions:

$$\begin{cases} F^a = \dot{q}^a - f^a(q) & \text{if } 1 \le a \le k, \\ F^a = 0 & \text{if } k+1 \le a \le k+\ell. \end{cases}$$

Then the maximizing problem turns into that of an integration over a finite $(0 < T < \infty)$ or an infinite $(T = \infty)$ period of time:

(1)′ $$\int_0^T e^{-\rho t}U(x,\,u)dt,$$

under constraints

(2)′ $$\dot{x}^\mu = f^\mu(x,\,u),$$

where $f^\mu(x,\,u)$ $(\mu = 1, \cdots, k)$ is a system of first degree homogeneous functions. Moreover, the Lagrangian $L$ of (3) takes the form

(3)′ $$L = e^{-\rho t}U + \lambda_\mu(\dot{x}^\mu - f^\mu),$$

whose Euler-Lagrange equations consist of (2)′ and

(5a)′ $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^\mu}\right) - \frac{\partial L}{\partial x^\mu} = 0:\quad e^{-\rho t}\frac{\partial U}{\partial x^\mu} = \dot{\lambda}_\mu + \frac{\partial f^\nu}{\partial x^\mu}\lambda_\nu,$$

(5b)′ $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{u}^\sigma}\right) - \frac{\partial L}{\partial u^\sigma} = 0:\quad e^{-\rho t}\frac{\partial U}{\partial u^\sigma} = \frac{\partial f^\mu}{\partial u^\sigma}\lambda_\mu.$$

In the setting, the conserved quantity $\Omega$ of (14) leads to

(14)′ $$\begin{aligned}\Omega &= r\left(\lambda_\mu\left(x^\nu\frac{\partial f^\mu}{\partial x^\nu} + u^\sigma\frac{\partial f^\mu}{\partial u^\sigma}\right) - e^{-\rho t}U\right) - \rho\lambda_\mu x^\mu \\ &= r\left(\lambda_\mu f^\mu - e^{-\rho t}U\right) - \rho\lambda_\mu x^\mu.\end{aligned}$$

The equations (5a)′ and (5b)′ are used to see

$$\begin{aligned} re^{-\rho t}U &= re^{-\rho t}\left(x^\mu\frac{\partial U}{\partial x^\mu} + u^\sigma\frac{\partial U}{\partial u^\sigma}\right) = \dot{\lambda}_\mu x^\mu + \lambda_\mu\left(x^\nu\frac{\partial f^\mu}{\partial x^\nu} + u^\sigma\frac{\partial f^\mu}{\partial u^\sigma}\right) \\ &= \dot{\lambda}_\mu x^\mu + \lambda_\mu f^\mu;\end{aligned}$$

which is, together with (2)′, carried into (14)′ to have the same appearance of the conserved quantity obtained in ([2], Theorem 1.2; in which $\pi_\mu$ correspond here to $\lambda_\mu$ in (14)″ below).

**Theorem 5.** *Let the functions $U(x, u)$ in $(1)'$ and $f^\mu(x, u)$ in $(2)'$ be $r$-th and first degree homogeneous with respect to $x^\mu$ and $u^\sigma$, respectively. Then there exists the following conserved quantity $\Omega$ for the maximizing problem of $(1)'$ under the constraints $(2)'$:*

$$(14)'' \qquad \Omega = (r-1)\lambda_\mu \dot{x}^\mu - (\dot{\lambda}_\mu + \rho\lambda_\mu)\, x^\mu .$$

**3. An application to generalized growth model of von Neumann type.** The theorems established in the preceding sections can be applied effectively for the derivation of new conservation laws in several economic growth models. Here is given an application within the $n$ capital goods $q^i = K^i$ and $n$ capital formations $\dot{q}^i = \dot{K}^i$. Accordingly in the theorem 1 through the theorem 4, $U(\dot{K}, K)$ and $F^a(\dot{K}, K)$ are regarded respectively as an $r$-th degree homogeneous utility function and a system of $s$-th degree homogeneous transformation functions with respect to $\dot{K}^i$ and $K^i$, and $\rho$ is a constant discount rate (see [2]-[5] for the application of the theorem 5). In the situation, the conserved quantity (14) leads immediately to

$$-\Omega = r\left(\lambda_a K^i \frac{\partial F^a}{\partial K^i} - e^{-\rho t}\left(\dot{K}^i \frac{\partial U}{\partial \dot{K}^i} - U\right)\right) + \rho K^i \left(\lambda_a \frac{\partial F^a}{\partial \dot{K}^i} + e^{-\rho t}\frac{\partial U}{\partial \dot{K}^i}\right).$$

Particularly for $U = c_i \dot{K}^i$ ($c_i$: const.; cf. [13], in which $U$ is of the form $U = \dot{K}^1$), this quantity $-\Omega$ is reduced to

$$-\Omega = \lambda_a K^i \frac{\partial F^a}{\partial K^i} + \rho K^i \left(\lambda_a \frac{\partial F^a}{\partial \dot{K}^i} + c_i e^{-\rho t}\right).$$

Here we assume that $U$ is a total time derivative of a first degree homogeneous function $G(K)$ with respect to $K^i$, i.e., $U = \dot{K}^i \partial G/\partial K^i$. Then $U$ is first degree homogeneous with respect to $\dot{K}^i$, i.e., $\dot{K}^i \partial U/\partial \dot{K}^i = U$; and is also of degree zero with respect to $K^i$, i.e., $K^i \partial U/\partial K^i = 0$. Accordingly it follows that

$$\begin{aligned}
\int e^{-\rho t} U\, dt &= \int e^{-\rho t} \dot{K}^i \frac{\partial U}{\partial \dot{K}^i} dt = e^{-\rho t} K^i \frac{\partial U}{\partial \dot{K}^i} - \int K^i \frac{d}{dt}\left(e^{-\rho t}\frac{\partial U}{\partial \dot{K}^i}\right) dt \\
&= e^{-\rho t} K^i \frac{\partial U}{\partial \dot{K}^i} + \rho \int e^{-\rho t} K^i \frac{\partial U}{\partial \dot{K}^i} dt - \int e^{-\rho t} K^i \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{K}^i}\right) dt \\
&= e^{-\rho t} K^i \frac{\partial U}{\partial \dot{K}^i} + \rho \int e^{-\rho t} K^i \frac{\partial U}{\partial \dot{K}^i} dt.
\end{aligned}$$

Therefore the conserved quantities (17) and (18) with first degree ($r = 1$) homogeneous function $U = \dot{K}^i \partial G(K)/\partial K^i$ have the appearances respectively in the following theorem.

**Theorem 6.** *Let $G = G(K)$ be first degree homogeneous function with respect to the $n$ capital goods $K^i$. Then for the maximizing problem of*

$$(1)'' \qquad \int_0^T e^{-\rho t} \dot{K}^i \frac{\partial G}{\partial K^i} dt,$$

*under constraints of $s$-th degree homogeneous transformation functions*

$$(2)'' \qquad F^a(\dot{K}, K) = 0 \quad (a = 1, ..., m),$$

*there exist the following two conserved quantities* $\Omega_1$ and $\Omega_2$:

$$\Omega_1 = \lambda_a K^i \frac{\partial F^a}{\partial K^i} + \rho \int e^{-\rho t} U \, dt, \tag{19}$$

$$\Omega_2 = \lambda_a K^i \frac{\partial F^a}{\partial \dot{K}^i} - \rho \int e^{-\rho t} K^i \frac{\partial U}{\partial \dot{K}^i} dt. \tag{20}$$

**Remark 1.** The conserved quantities (19) and (20) reduce respectively, if $\rho = 0$, to $\Omega_1 = \lambda_a K^i \partial F^a / \partial K^i$ and $\Omega_2 = \lambda_a K^i \partial F^a / \partial \dot{K}^i$; and moreover, if $n = 2$, $m = 1$ and $s = 1$, to the Samuelson's ones [13] $\Omega_1 = \lambda K^i \partial F / \partial K^i$ and $\Omega_2 = \lambda K^i \partial F / \partial \dot{K}^i$ in which $\lambda$, $Y \equiv K^i \partial F / \partial K^i$ and $W \equiv -K^i \partial F / \partial \dot{K}^i$ are regarded respectively as the implicit price, the national income and the national wealth.

**Remark 2.** What is important and interesting in the above theorem is that the Samuelson's conservation laws $\Omega_1 = \lambda K^i \partial F / \partial K^i$ and $\Omega_2 = \lambda K^i \partial F / \partial \dot{K}^i$ are valid for the utility of the form $U = dG/dt$ where $G = G(K)$ is a first degree homogeneous function with respect to $K^i$, and for the transformation function $F(\dot{K}, K)$ with arbitrary degree of homogeneity with respect to $K^i$ and $\dot{K}^i$ (in [13], $U = \dot{K}^1$ and $F(\dot{K}, K)$ is assumed to be of first degree homogeneous function).

**Remark 3.** As seen before, $(\xi^i, \eta_a) = (q^i, (r-s)\lambda_a)$ is solutions satisfying (6) and (7) on the optimal paths, which are written here respectively as $(\xi^i, \eta_a) = (K^i, (1-s)\lambda_a)$. And, in view of the equations from (5) with $U = \dot{K}^i \partial G(K) / \partial K^i$ and $\rho = 0$:

$$\frac{d}{dt}\left(\lambda_a \frac{\partial F^a}{\partial \dot{K}^i}\right) - \lambda_a \frac{\partial F^a}{\partial K^i} = 0,$$

we have immediately the other one $(\xi^i, \eta_a) = (0, \lambda_a)$. The above conserved quantities $-\Omega_1$ of (19) or $\Omega_2$ of (20) can be obtained also by substituting $(\xi^i, \eta_a) = (K^i, (1-s)\lambda_a)$ for (9), or $(\xi_1^i, \eta_a^1) = (K^i, (1-s)\lambda_a)$ and $(\xi_2^i, \eta_a^2) = (0, \lambda_a)$ for (8), respectively.

In what follows, let $0 < T < \infty$, $\rho = 0$, $n = 2$ and $m = 1$, and a transformation function of two capital goods $F = F(\dot{K}^1, \dot{K}^2, K^1, K^2)$ be a homogeneous second order polynomial of the form:

$$F = a_1(\dot{K}^1)^2 + a_2(\dot{K}^2)^2 + \mu(a_1(K^1)^2 + a_2(K^2)^2) \quad (a_1,\, a_2,\, \mu\text{: const.}; \ a_1 a_2 < 0,\ \mu \neq 0). \tag{2)'''}$$

Then (19) and (20) lead respectively to the following conserved quantities $\Xi_1 \equiv \frac{1}{2}\Omega_1/\mu$ and $\Xi_2 \equiv \frac{1}{2}\Omega_2$:

$$\Xi_1 = \lambda(a_1(K^1)^2 + a_2(K^2)^2),$$

$$\Xi_2 = \lambda(a_1 K_1 \dot{K}^1 + a_2 K_2 \dot{K}^2).$$

So, in the identity

$$\dot{\Xi}_1 = \dot{\lambda}(a_1(K^1)^2 + a_2(K^2)^2) + 2\Xi_2 = 0,$$

the following conserved quantity $\Xi_3$ is observed:

$$\Xi_3 = \dot{\lambda}(a_1(K^1)^2 + a_2(K^2)^2).$$

Accordingly, since $\Xi_3/\Xi_1 = \dot{\lambda}/\lambda = \alpha$ ($\alpha$: const.), $\lambda$ is determine as

$$\lambda = Ce^{\alpha t} \quad (C\text{: const.}).$$

Here note that the Lagrangian $L$ in the consideration is of the form

$$L = \dot{K}^1 \frac{\partial G}{\partial K^1} + \dot{K}^2 \frac{\partial G}{\partial K^2} + \lambda F,$$

where $G = G(K^1, K^2)$ is first degree homogeneous function with respect to $K^1$ and $K^2$. By substituting the path $\lambda = Ce^{\alpha t}$ for a part of relating Euler-Lagrange equations with the Lagrangian $L$:

$$\frac{d}{dt}\left(\lambda \frac{\partial F}{\partial \dot{K}^i}\right) - \lambda \frac{\partial F}{\partial K^i} = 0 \quad (i = 1, 2),$$

the following second order differential equations are obtained:

$$\ddot{K}^i + \alpha \dot{K}^i - \mu K^i = 0 \quad (i = 1, 2);$$

whose subsidiary equation has two distinct real solutions $-\frac{1}{2}(\alpha + \sqrt{D})$ and $-\frac{1}{2}(\alpha - \sqrt{D})$ if its discriminant $D \equiv \alpha^2 + 4\mu$ satisfies $D > 0$, or a coinside solution $-\frac{1}{2}\alpha$ if $D = 0$, or two complex solutions $-\frac{1}{2}\alpha \pm \frac{1}{2}i\sqrt{-D}$ if $D < 0$, which are used respectively to denote the solutions $K^i$. And then, together with $\lambda = Ce^{\alpha t}$, the solutions $K^i$ are substituted for $\Xi_1 =$ const. to complete the final appearances of $K^i$.

**Theorem 7.** *In the maximizing problem of* $(1)''$ $(0 < T < \infty, \rho = 0)$, *let* $G = G(K^1, K^2)$ *be first degree homogeneous function with respect to the two capital goods* $K^1$ *and* $K^2$ *and the transformation function* $F$ *be a homogeneous second order polynomial of the form* $(2)'''$. *Then the optimal paths* $\lambda$ *and* $K^i$ $(i = 1, 2)$ *are determined as* $\lambda = Ce^{\alpha t}$ ($C$: *const.*) *and as follows according as* $D = \alpha^2 + 4\mu$ *is positive, zero and negative*:

$$\text{(i)} \quad K^i = A^i e^{-\frac{1}{2}(\alpha + \sqrt{D})t} + B^i e^{-\frac{1}{2}(\alpha - \sqrt{D})t} \qquad \text{if } D > 0,$$

$$\text{(ii)} \quad K^i = e^{-\frac{\alpha}{2}t}(A^i t + B^i) \qquad \text{if } D = 0,$$

$$\text{(iii)} \quad K^i = e^{-\frac{\alpha}{2}t}\left(A^i \cos \tfrac{1}{2}\sqrt{-D}\, t + B^i \sin \tfrac{1}{2}\sqrt{-D}\, t\right) \qquad \text{if } D < 0,$$

*where* $A^1$, $B^1$ *are arbitrary constants and* $A^2 = \pm\sqrt{-a_1/a_2}\, A^1$, $B^2 = \pm\sqrt{-a_1/a_2}\, B^1$, *in the right hand sides of which, the signs* $\pm$ *correspond respectively for* (ii) *and* (iii).

The conserved quantity $\Xi_1$ is always zero (so that $\Xi_2$ is also) for the optimal paths $K^i$ $(i = 1, 2)$ of (ii) and (iii) in the theorem 7; and also for those of (i) with the constants $A^2 = \pm\sqrt{-a_1/a_2}\, A^1$ and $B^2 = \pm\sqrt{-a_1/a_2}\, B^1$ (the signs $\pm$ correspond respectively). But nonzero conserved quantity $\Xi_1 \neq 0$ and also $\Xi_2 \neq 0$ are given respectively as $\Xi_1 = 4a_1 CA^1B^1$ and $-\Xi_2 = 2\alpha a_1 CA^1B^1$ by the optimal path $\lambda = Ce^{\alpha t}$ ($C$: const., $C \neq 0$) and the optimal paths $K^i$ $(i = 1, 2)$ of (i) in the theorem 7 with the constants $A^2 = \pm\sqrt{-a_1/a_2}\, A^1 \neq 0$ and $B^2 = \mp\sqrt{-a_1/a_2}\, B^1 \neq 0$ (the signs $\pm$ and $\mp$ correspond respectively). So that the constant of the income-wealth (output-capital) ratio (Samuelson [13]) is $Y/W = -\mu\Xi_1/\Xi_2 = 2\mu/\alpha$. Therefore, in view of $\alpha = \dot{\lambda}/\lambda = d(\log\lambda)/dt$, it is concluded:

**Theorem 8.** *Together with the optimal path* $\lambda = Ce^{\alpha t}$ ($C$: *const.*, $C \neq 0$), *the only optimal paths* $K^i$ $(i = 1, 2)$ *of* (i) *in the theorem 7 with the constants* $A^2 = \pm\sqrt{-a_1/a_2}\, A^1 \neq 0$ *and* $B^2 = \mp\sqrt{-a_1/a_2}\, B^1 \neq 0$ (*the signs* $\pm$ *and* $\mp$ *correspond respectively*) *are provided with nonzero conserved quantities* $\mu\Xi_1 = 4a_1\mu CA^1B^1$ (*product of the implicit price* $\lambda$ *and the national income* $Y = K^i\partial F/\partial K^i$) *and* $-\Xi_2 = 2\alpha a_1 CA^1B^1$ (*product of the implicit price* $\lambda$

*and the national wealth* $W = -K^i \partial F / \partial \dot{K}^i$), *which give the constant of the income-wealth ratio* $Y/W = 2\mu/(d(\log\lambda)/dt)$:

$$\frac{\{\text{national income}\}}{\{\text{national wealth}\}} = \frac{2\mu}{\{\text{rate of logarithm of implicit price}\}} = \text{constant}.$$

## References


[1] E. Bessel-Hagen, *Über die Erhaltungssätze der Electrodynamik*, Math. Ann. **84** (1921), 258-276.

[2] F. Fujiwara, F. Mimura and T. Nôno, *New derivation of conservation laws and its application to more general neoclassical optimal growth models*, Tensor, N.S. **60** (1998), 63-73.

[3] F. Fujiwara, F. Mimura and T. Nôno, *New derivation of conservation laws in one and two sectors growth models*, to apper in Tensor, N.S. **62** (2000).

[4] F. Fujiwara, F. Mimura and T. Nôno, *New derivation of conservation laws in three-sector growth models*, Reports of the Japan Institute of Industrial Science **10** (2000), 1-10.

[5] F. Fujiwara, F. Mimura and T. Nôno, *Conservation laws and optimal paths in external two-sector growth model*, Economic theory, dynamics and markets (Essays in honor of R. Sato), ed., T. Negishi, R. V. Ramachandran and K. Mino, Kluwer Academic Publishers, Boston, (2001), 153-164.

[6] F. Mimura and T. Nôno, *A method for deriving new conservation laws*, Bull. Kyushu Inst. Tech. Math. Natur. Sci. **42** (1995), 1-17.

[7] F. Mimura and T. Nôno, *A method for the derivation of new conservation laws in economic growth models*, Bull. Kyushu Inst. Tech. Math. Natur. Sci. **44** (1997), 7-22.

[8] F. Mimura, F. Fujiwara and T. Nôno, *New derivation of conservation laws for optimal economic growths*, Tensor, N.S. **58** (1997), 195-207.

[9] F. Mimura, F. Fujiwara and T. Nôno, *New derivation of conservation laws for optimal control problem and its application to economic growth models*, Global competition and integration, ed., R. Sato, R. V. Ramachandran and K. Mino, Kluwer Academic Publishers, Boston, (1999), 240-266.

[10] M. Neamţu, *Dynamical systems with extreme integrals and conservation laws*, Differ. Geom. Dyn. Syst. **2** (2000), 63-69.

[11] E. Noether, *Invariante Variations Probleme*, Nachr. Ges. Wiss. Göttingen Math.-Phys. Kl. II **1918** (1918), 235-257.

[12] P. A. Samuelson, *Foundation of economic analysis*, Enlarged Edition, Harvard Economic Studies **80** (1983).

[13] P. A. Samuelson, *Law of conservation of the capital-output ratio*, Proc. Nat. Acad. Sci., Appl. Math. Sci. **67** (1970), 1477-1479.

[14] J. von Neumann, *Über ein ökonomisches Gleichungs-System unt eine Verallgemeinerung des Brouwerschen Fixpunktsatzes*, in *Ergebnisse eines Mathematischen Kolloquiums*, ed. by K. Menger, no. 8 (1973) (*A model of general economic equilibrium*, Rev. Econom. Stud. **14** (1945/46), 1-9.



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