BONSALL'S CONSTRUCTIONS OF SPECTRAL MEASURES AND SUCCESSIVE APPROXIMATION IN WAZAN

MASATOSHI FUJII AND MASAHIRO NAKAMURA

Received September 18, 2001

Abstract. Bonsall gave two constructions of the resolution of the identity of a self-adjoint operator, which are successive approximations by a polynomial $x(1-(x-1)^2)$ and a rational function $\frac{2x^2}{1+x^2}$. On the other hand, about 300 years ago, Murase gave a successive approximation to obtain a root of an algebraic equation of order 3. In this note, we observe that, under the light of Murase's method, two constructions due to Bonsall are essentially same.

1. In 1673, Yoshimasu Murase, a researcher of the traditional Japanese mathematics Wazan, published a book [4], which contains an original method to solve a numerical equation of order 3:

$$4x^2(14-x) = 192.$$

He deformed it to

$$x^2 = \frac{48}{14-x},$$

and gave a successive approximation method by

$$x_{n+1}^2 = \frac{48}{14-x_n} \quad (n = 1, 2, \cdots).$$

For example, put $x_0 = 0$ according to [5], then

$$x_1 = 1.84,\ x_2 = 1.976,\ x_3 = 1.9989,\ x_4 = 1.999907,$$

which converge rapidly to a solution $x = 2$.

2. In 1960, F.F.Bonsall presented an ingenious construction of the spectral family of a positive operator A by an iterative method:

$$A_{n+1} = 2A_n^2(1+A_n^2)^{-1} \qquad (n = 0, 1, 2, \cdots), \tag{1}$$

which converges to the projection $E(1)^{\perp}$, where $A_0 = A$.

In the previous note [3], we have observed that (1) is nothing but

$$A_{n+1} = 1 \; ! \; A_n^2 \qquad (n = 0, 1, 2, \cdots), \tag{2}$$

where ! is the harmonic mean in the Kubo-Ando theory on operator means. The observation in [3] visibly enlights the Bonsall construction.

2000 *Mathematics Subject Classification.* 47A10, 47A64, 47B15 and 01A27.

Key words and phrases. Successive approximation in Wazan, polynomial order 3, resolution of the identity, Kubo-Ando theory and harmonic mean.

3. In 1963, Bonsall came back to the construction of the spectral family again. Probably, he was unsatisfied by (1) since the function of the right hand side is a rational function:

$$f(x) = \frac{2x^2}{1+x^2}. \tag{3}$$

He proposed an iteration by the polynomial

$$g(x) = x(1-(x-1)^2). \tag{4}$$

Namely the successive approximation is given by

$$A_{n+1} = A_n(1-(A_n-1)^2) \quad (n = 0, 1, 2, \cdots) \tag{5}$$

with $A_0 = A$ (and $\| A \| = 1$).

In the present note, we apply Murase's method to Bonsall's constructions of the spectral family in the following sense: Murase used an approximation method to solve a given algebraic equation. On the other hand, Bonsall did it to construct the spectral family. So we pay our attention to the *original* algebraic equations in (2) and (5). Thus the algebraic equations corresponding to (5) and (2) are given by

$$x = x(1-(x-1)^2) \tag{6}$$

and

$$x = \frac{2x^2}{1+x^2} \tag{7}$$

respectively. They are just the same as algebraic equations: We might say that *Bonsall's two constructions for spectral family are essentially same.*

References

[1] F.F.Bonsall, *A formula for the spectral family of an operator*, London Math. Soc.,**35**(1960), 321-333.

[2] F.F.Bonsall, *A polynomial iteration for the spectral family of an operator*, Proc. Glasgow Math. Soc., **6**(1963), 65-69.

[3] M.Fujii and M.Nakamura, *Bonsall's construction of spectral measures*, Sci. Math., **2**(1999), 145-146.

[4] Y.Murase, *Sanpo-Kotsuzenkai* (in Japanese), Edo, 1673.

[5] T.Suzuki, *Wazan no Seiritsu* (in Japanese), private ed., 1997.

Department of Mathematics, Osaka Kyoiku University, Asahigaoka, Kashiwara, Osaka 582-8582, Japan.

E-mail address: `mfujii@@cc.osaka-kyoiku.ac.jp (M.Fujii)`