FUZZIFICATIONS OF $a\&\mathcal{I}$-IDEALS IN IS-ALGEBRAS

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Abstract. The fuzzification of $a\&\mathcal{I}$-ideals in **IS**-algebras is considered. Relations between fuzzy $p\&\mathcal{I}$-ideals and fuzzy $a\&\mathcal{I}$-ideals are stated. Characterizations of fuzzy $a\&\mathcal{I}$-ideals are given. Extension property for fuzzy $a\&\mathcal{I}$-ideal is established.

1. Introduction

The notion of BCK-algebras was proposed by Y. Imai and K. Iséki in 1966. In the same year, K. Iséki [2] introduced the notion of a BCI-algebra which is a generalization of a BCK-algebra. In 1993, Y. B. Jun et al. [5] introduced a new class of algebras related to BCI-algebras and semigroups, called a BCI-semigroup/BCI-monoid/BCI-group. In 1998, for the convenience of study, Y. B. Jun et al. [8] renamed the BCI-semigroup (resp, BCI-monoid and BCI-group) as the **IS**-algebra (resp. **IM**-algebra and **IG**-algebra) and studied further properties of these algebras (see [7] and [8]). In [9], E. H. Roh et al. introduced the concept of a $p\&\mathcal{I}$-ideal in an **IS**-algebra, and gave necessary and sufficient conditions for an $\mathcal{I}$-ideal to be a $p\&\mathcal{I}$-ideal, and also stated a characterization of **PS**-algebras by $p\&\mathcal{I}$-ideals. Y. B. Jun and E. H. Roh [6] considered the fuzzification of a $p\&\mathcal{I}$-ideal in an **IS**-algebra, and investigated some of their properties. E. H. Roh et al. [10] discussed the notion of $a\&\mathcal{I}$-ideals in **IS**-algebras. In this paper, we consider the fuzzification of $a\&\mathcal{I}$-ideals in **IS**-algebras. We give relations between fuzzy $p\&\mathcal{I}$-ideals and fuzzy $a\&\mathcal{I}$-ideals. We state characterizations of fuzzy $a\&\mathcal{I}$-ideals. We finally establish the extension property for fuzzy $a\&\mathcal{I}$-ideals.

2. Preliminaries

By a *BCI-algebra* we mean an algebra $(X, *, 0)$ of type (2,0) satisfying the following conditions:

- $((x * y) * (x * z)) * (z * y) = 0$,
- $(x * (x * y)) * y = 0$,
- $x * x = 0$,
- $x * y = 0$ and $y * x = 0$ imply $x = y$

for all $x, y, z \in X$. A BCI-algebra X satisfying $0 \leq x$ for all $x \in X$ is called a *BCK-algebra.* In any BCK/BCI-algebra X one can define a partial order "$\leq$" by putting $x \leq y$ if and only if $x * y = 0$.

A BCI-algbera X has the following properties:

(P1) $x * 0 = x$,

(P2) $(x * y) * z = (x * z) * y$,

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(P3) $(x * z) * (y * z) \leq x * y$
for all $x, y, z \in X$. A nonempty subset I of a BCK/BCI-algebra X is called an *ideal* of X if it satisfies
(i) $0 \in I$,
(ii) $x * y \in I$ and $y \in I$ imply $x \in I$ for all $x, y \in X$.

Definition 2.1. (Jun et al. [8]) An **IS**-*algebra* is a non-empty set X with two binary operations "$*$" and "$\cdot$" and constant 0 satisfying the axioms

- $I(X) := (X, *, 0)$ is a BCI-algebra.
- $S(X) := (X, \cdot)$ is a semigroup.
- the operation "$\cdot$" is distributive (on both sides) over the operation "$*$", that is,

$$x \cdot (y * z) = (x \cdot y) * (x \cdot z) \text{ and } (x * y) \cdot z = (x \cdot z) * (y \cdot z), \ \forall x, y, z \in X.$$

Example 2.2. Let $X = \{0, a, b, c\}$ be a set with the following Cayley tables:

$*$	0	a	b	c
0	0	0	c	b
a	a	0	c	b
b	b	b	0	c
c	c	c	b	0

$\cdot$	0	a	b	c
0	0	0	0	0
a	0	0	0	0
b	0	0	b	c
c	0	0	c	b

Then X is an **IS**-algebra (see [8]).

Definition 2.3. [1, Definition 2.3] A nonempty subset A of an **IS**-algebra X is said to be *left* (resp. *right*) *stable* if $x \cdot a \in A$ (resp. $a \cdot x \in A$) whenever $x \in S(X)$ and $a \in A$.

In what follows, the terminology "stable" means "left stable", and let X denote an **IS**-algebra unless otherwise specified.

Definition 2.4. [8, Definition 3] A nonempty subset A of X is called an $\mathcal{I}$-*ideal* of X if it satisfies
(i) A is a stable subset of $S(X)$,
(ii) for any $x, y \in I(X)$, $x * y \in A$ and $y \in A$ imply that $x \in A$.

Definition 2.5. [9, Definition 3.1] A nonempty subset A of X is called a *p*&$\mathcal{I}$-*ideal* of X if it satisfies
(i) A is a stable subset of $S(X)$,
(ii) for any $x, y, z \in I(X)$, $(x * z) * (y * z) \in A$ and $y \in A$ imply that $x \in A$.

We place a bar over a symbol to denote a fuzzy set so $\bar{\mu}$, $\bar{A}$, $\bar{X}$, $\cdots$ all represent fuzzy set in a set.

Definition 2.6. [3, Definition 4] A fuzzy set $\bar{A}$ in X is called a *fuzzy* $\mathcal{I}$-*ideal* of X if it satisfies
(i) $\bar{A}$ is a fuzzy ideal of a BCI-algebra X,
(ii) $\bar{A}(xy) \geq \bar{A}(y) \ \ \forall x, y \in X$.

Definition 2.7. [6, Definition 3.2] A fuzzy set $\bar{A}$ in X is called a *fuzzy p*&$\mathcal{I}$-*ideal* of X if it satisfies
(i) $\bar{A}$ is a fuzzy stable set in $S(X)$,
(ii) $\bar{A}(x) \geq \min\{\bar{A}((x * z) * (y * z)), \bar{A}(y)\} \ \ \forall x, y, z \in X$.

Note that every fuzzy $p\&\mathcal{I}$-ideal is a fuzzy $\mathcal{I}$-ideal, but the converse is not true (see [6, Theorem 3.4 and Example 3.5]).

3. Fuzzy $a\&\mathcal{I}$-ideals

Definition 3.1. [10, Definition 3.1] A non-empty subset A of an **IS**-algebra X is called a *$a\&\mathcal{I}$-ideal* of X if it satisfies

(i) A is a stable subset of $S(X)$,

(ii) for any $x, y, z \in I(X)$, $(x * z) * (0 * y) \in A$ and $z \in A$ imply that $y * x \in A$.

Definition 3.2. A fuzzy set $\bar{A}$ in X is called a *fuzzy $a\&\mathcal{I}$-ideal* of X if it satisfies

(i) $\bar{A}$ is a fuzzy stable set in $S(X)$,

(ii) $\bar{A}(y * x) \geq \min\left\{\bar{A}\Big((x * z) * (0 * y)\Big),\ \bar{A}(z)\right\}\ \ \forall x, y, z \in I(X)$.

Example 3.3. Let $X = \{0, a, b, c\}$ be a set with the following Cayley tables:

$*$	0	a	b	c
0	0	0	b	b
a	a	0	c	b
b	b	b	0	0
c	c	b	a	0

$\cdot$	0	a	b	c
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
c	0	a	b	c

Then X is an **IS**-algebra (see [8]). We can easily check that a fuzzy set $\bar{A}$ in X given by $\bar{A}(0) = \bar{A}(a) = 0.6$ and $\bar{A}(b) = \bar{A}(c) = 0.2$ is a fuzzy $a\&\mathcal{I}$-ideal of X.

Proposition 3.4. *If $\bar{A}$ is a fuzzy $a\&\mathcal{I}$-ideal of X, then $\bar{A}(0 * x) \geq \bar{A}(x) \geq \bar{A}(0 * (0 * x))$ for all $x \in X$.*

Proof. Since $\bar{A}$ is fuzzy stable, it follows that $\bar{A}(0) = \bar{A}(0x) \geq \bar{A}(x)$ for all $x \in X$. Taking $z = y = 0$ in Definition 3.2(ii), we have

$$\bar{A}(0 * x) \geq \min\left\{\bar{A}\Big((x * 0) * (0 * 0)\Big),\ \bar{A}(0)\right\} = \bar{A}(x)$$

for all $x \in X$. Now putting $y = x$ and $x = z = 0$ in Definition 3.2(ii), we have

$$\bar{A}(x) \geq \min\left\{\bar{A}\Big((0 * 0) * (0 * x)\Big),\ \bar{A}(0)\right\} = \bar{A}\Big(0 * (0 * x)\Big)$$

for all $x \in X$. □

Theorem 3.5. *Every fuzzy $a\&\mathcal{I}$-ideal is a fuzzy $\mathcal{I}$-ideal.*

Proof. Let $\bar{A}$ be a fuzzy $a\&\mathcal{I}$-ideal of X. Taking $y = 0$ in Definition 3.2(ii) and using Proposition 3.4, we get

$$\begin{aligned}\bar{A}(x) &\geq \bar{A}\Big(0 * (0 * x)\Big) \geq \bar{A}(0 * x)\\ &\geq \min\left\{\bar{A}\Big((x * z) * (0 * 0)\Big),\ \bar{A}(z)\right\}\\ &= \min\left\{\bar{A}(x * z),\ \bar{A}(z)\right\}.\end{aligned}$$

for all $x, z \in X$. Hence $\bar{A}$ is a fuzzy $\mathcal{I}$-ideal of X. □

The following example shows that the converse of Theorem 3.5 may not be true.

Example 3.6. Let X be an **IS**-algebra in Example 3.3 and let $\bar{B}$ be a fuzzy set in X defined by $\bar{B}(0) = \bar{B}(b) = 0.8$ and $\bar{B}(a) = \bar{B}(c) = 0.5$. Then $\bar{B}$ is a fuzzy $\mathcal{I}$-ideal of X, but it is not a fuzzy $a\&\mathcal{I}$-ideal of X since

$$\bar{B}(a * b) = 0.5 < 0.8 = \min\Big\{\bar{B}\Big((b * b) * (0 * a)\Big),\ \bar{B}(b)\Big\}.$$

We provide conditions for a fuzzy $\mathcal{I}$-ideal to be a fuzzy $a\&\mathcal{I}$-ideal.

Theorem 3.7. *Let $\bar{A}$ be a fuzzy $\mathcal{I}$-ideal of X. Then the following are equivalent.*

(i) *$\bar{A}$ is a fuzzy $a\&\mathcal{I}$-ideal of X.*

(ii) $\bar{A}\Big(y * (x * z)\Big) \geq \bar{A}\Big((x * z) * (0 * y)\Big),\ \forall x, y, z \in I(X)$,

(iii) $\bar{A}(y * x) \geq \bar{A}\Big(x * (0 * y)\Big),\ \forall x, y \in I(X)$.

Proof. Assume that $\bar{A}$ is a fuzzy $a\&\mathcal{I}$-ideal of X. For every $x, y, z \in I(X)$, we have

$$\begin{aligned}
\bar{A}\Big(y * (x * z)\Big) &\geq \min\Big\{\bar{A}\Big(((x * z) * ((x * z) * (0 * y))) * (0 * y)\Big),\ \bar{A}\Big((x * z) * (0 * y)\Big)\Big\} \\
&= \min\Big\{\bar{A}\Big(((x * z) * (0 * y)) * ((x * z) * (0 * y))\Big),\ \bar{A}\Big((x * z) * (0 * y)\Big)\Big\} \\
&= \min\Big\{\bar{A}(0),\ \bar{A}\Big((x * z) * (0 * y)\Big)\Big\} \\
&= \bar{A}\Big((x * z) * (0 * y)\Big).
\end{aligned}$$

(iii) is by taking $z = 0$ in (ii) and using (P1). Suppose that (iii) holds. Note that

$$\Big(x * (0 * y)\Big) * \Big((x * z) * (0 * y)\Big) \leq x * (x * z) \leq z$$

for all $x, y, z \in I(X)$. Since $\bar{A}$ is order reversing, it follows that

$$\bar{A}\Big((x * (0 * y)) * ((x * z) * (0 * y))\Big) \geq \bar{A}(z).$$

Hence

$$\begin{aligned}
\bar{A}(y * x) &\geq \bar{A}\Big(x * (0 * y)\Big) \\
&\geq \min\Big\{\bar{A}\Big((x * (0 * y)) * ((x * z) * (0 * y))\Big),\ \bar{A}\Big((x * z) * (0 * y)\Big)\Big\} \\
&\geq \min\Big\{\bar{A}\Big((x * z) * (0 * y)\Big),\ \bar{A}(z)\Big\},
\end{aligned}$$

and so $\bar{A}$ is a fuzzy $a\&\mathcal{I}$-ideal of X. □

Lemma 3.8. [6, Theorem 3.9] *Let $\bar{A}$ be a fuzzy $\mathcal{I}$-ideal of X. Then $\bar{A}$ is a fuzzy $p\&\mathcal{I}$-ideal of X if and only if it satisfies*

$$\bar{A}(x) \geq \bar{A}\Big(0 * (0 * x)\Big),\ \ \forall x \in I(X).$$

Combining Proposition 3.4 and Lemma 3.8, we have the following theorem.

Theorem 3.9. *Every fuzzy $a\&\mathcal{I}$-ideal is a fuzzy $p\&\mathcal{I}$-ideal.*

The converse of Theorem 3.9 is false, as is shown in the following example.

Example 3.10. Let X be an **IS**-algebra in Example 2.2 and let $\bar{A}$ be a fuzzy set in X defined by $\bar{A}(0) = \bar{A}(a) = 0.7$ and $\bar{A}(b) = \bar{A}(c) = 0.5$. Then $\bar{A}$ is a fuzzy $p\&\mathcal{I}$-ideal of X, but it is not a fuzzy $a\&\mathcal{I}$-ideal of X because

$$\bar{A}(b * c) = 0.5 < 0.7 = \min\Big\{\bar{A}\Big((c * a) * (0 * b)\Big),\ \bar{A}(a)\Big\}.$$

Theorem 3.11. *In an associative* **IS**-*algebra X, that is, the identity $(x*y)*z = x*(y*z)$ holds in X, every fuzzy $\mathcal{I}$-ideal is a fuzzy $a\&\mathcal{I}$-ideal.*

Proof. Let $\bar{A}$ be a fuzzy $\mathcal{I}$-ideal of X. Note that

$$\begin{aligned}(y * x) * \Big(x * (0 * y)\Big) &= (y * x) * \Big((x * 0) * y\Big) = (y * x) * (x * y)\\ &= \Big(y * (x * y)\Big) * x = \Big((y * x) * y\Big) * x\\ &= \Big((y * y) * x\Big) * x = (0 * x) * x\\ &= 0 * (x * x) = 0 * 0 = 0\end{aligned}$$

for all $x, y \in X$. Hence

$$\begin{aligned}\bar{A}(y * x) &\geq \min\Big\{\bar{A}\Big((y * x) * (x * (0 * y))\Big),\ \bar{A}\Big(x * (0 * y)\Big)\Big\}\\ &= \min\Big\{\bar{A}(0),\ \bar{A}\Big(x * (0 * y)\Big)\Big\}\\ &= \bar{A}\Big(x * (0 * y)\Big).\end{aligned}$$

It follows from Theorem 3.7 that $\bar{A}$ is a fuzzy $a\&\mathcal{I}$-ideal of X. □

Theorem 3.12. (Extension property for fuzzy $a\&\mathcal{I}$-ideals) *Let $\bar{A}$ and $\bar{B}$ be fuzzy $\mathcal{I}$-ideal of X such that $\bar{A}(0) = \bar{B}(0)$ and $\bar{A} \subseteq \bar{B}$, that is, $\bar{A}(x) \leq \bar{B}(x)$ for all $x \in X$. If $\bar{A}$ is a fuzzy $a\&\mathcal{I}$-ideal of X, then so is $\bar{B}$.*

Proof. Let $x, y \in X$. Using Theorem 3.7(ii) and (P2), we have

$$\begin{aligned}\bar{B}\Big(y * (x * (x * (0 * y)))\Big) &\geq \bar{A}\Big(y * (x * (x * (0 * y)))\Big)\\ &\geq \bar{A}\Big((x * (x * (0 * y))) * (0 * y)\Big)\\ &= \bar{A}\Big((x * (0 * y)) * (x * (0 * y))\Big)\\ &= \bar{A}(0) = \bar{B}(0).\end{aligned}$$

Note that

$$\begin{aligned}&\Big((y * x) * (x * (0 * y))\Big) * \Big(y * (x * (x * (0 * y)))\Big)\\ = \ &\Big((y * x) * (y * (x * (x * (0 * y))))\Big) * \Big(x * (0 * y)\Big)\\ \leq \ &\Big(x * (x * (x * (0 * y)))\Big) * \Big(x * (0 * y)\Big)\\ = \ &\Big(x * (0 * y)\Big) * \Big(x * (0 * y)\Big) = 0,\end{aligned}$$

and so $\bar{B}\Big(((y*x)*(x*(0*y)))*(y*(x*(x*(0*y))))\Big) \geq \bar{B}(0)$. It follows that

$$\begin{aligned}
&\bar{B}\Big((y*x)*(x*(0*y))\Big)\\
\geq\ & \min\Big\{\bar{B}\Big(((y*x)*(x*(0*y)))*(y*(x*(x*(0*y))))\Big),\ \bar{B}\Big(y*(x*(x*(0*y)))\Big)\Big\}\\
\geq\ & \min\Big\{\bar{B}\Big(((y*x)*(x*(0*y)))*(y*(x*(x*(0*y))))\Big),\ \bar{B}(0)\Big\}\\
=\ & \bar{B}\Big(((y*x)*(x*(0*y)))*(y*(x*(x*(0*y))))\Big)\\
\geq\ & \bar{B}(0),
\end{aligned}$$

so that

$$\begin{aligned}
\bar{B}(y*x) &\geq \min\Big\{\bar{B}\Big((y*x)*(x*(0*y))\Big),\ \bar{B}\Big(x*(0*y)\Big)\Big\}\\
&\geq \min\Big\{\bar{B}(0),\ \bar{B}\Big(x*(0*y)\Big)\Big\}\\
&= \bar{B}\Big(x*(0*y)\Big).
\end{aligned}$$

Using Theorem 3.7, we know that $\bar{B}$ is a fuzzy $a\&\mathcal{I}$-ideal of X. □

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