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# ALMOST SURE CONVERGENCE OF A LINEAR COMBINATION OF U-STATISTICS

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Abstract. We derive the rate of almost sure convergence for a linear combination of U-statistics under the condition that the kernel is degenerate or not necessarily degenerate. These results also give the rate of almost sure convergence for V-statistic and LB-statistic as special cases.

**1 Introduction** Let $\theta(F)$ be an estimable parameter or a regular functional of a distribution $F$ and $g(x_1, ..., x_k)$ be its kernel of degree $k$. We assume that the kernel $g(x_1, ..., x_k)$ is symmetric. Let $X_1, ..., X_n$ be a random sample of size $n$ from the distribution $F$. U-statistic $U_n$ and V-statistic $V_n$ are well-known as estimators of $\theta(F)$. The U-statistic $U_n$ is given by

$$U_n = \binom{n}{k}^{-1} \sum_{1 \le j_1 < \cdots < j_k \le n} g(X_{j_1}, \ldots, X_{j_k}), \tag{1.1}$$

where $\sum_{1 \le j_1 < \cdots < j_k \le n}$ denotes the summation over all integers $j_1, \ldots, j_k$ satisfying $1 \le j_1 < \cdots < j_k \le n$. The V-statistic $V_n$ is given by (1.3) below. (See, for example, Lee (1990) .) For the U-statistic, the rate of almost sure convergence is derived by Sen (1974) for not necessarily degenerate kernel. For the rate under the moment condition with lower order, Giné and Zinn (1992) also derive it by the method different from Sen (1974) and show the almost sure convergence of the V-statistic as its application. They, furthermore, derive the rate of almost sure convergence for degenerate kernel. The purpose of this paper is to derive the rate of almost sure convergence for a linear combination of U-statistics as stated below.

As an estimator of $\theta(F)$, we consider a linear combination of U- statistics (see Toda and Yamato (2001), and Kondo and Yamato (2001)): Let $w(r_1, \ldots, r_j; k)$ be a nonnegative and symmetric function of positive integers $r_1, \ldots, r_j$ such that $j = 1, \ldots, k$ and $r_1 + \cdots + r_j = k$, where $k$ is the degree of the kernel $g$ and fixed. We assume that at least one of $w(r_1, \ldots, r_j; k)$'s is positive. We put

$$d(k, j) = \sum\nolimits^{+}_{r_1 + \cdots + r_j = k} w(r_1, \ldots, r_j; k)$$

for $j = 1, 2, ..., k$, where the summation $\sum^{+}_{r_1+\cdots+r_j=k}$ is taken over all positive integers $r_1, ..., r_j$ satisfying $r_1 + \cdots + r_j = k$ with $j$ and $k$ fixed. For $j = 1, \ldots, k$, let $g_{(j)}(x_1, ..., x_j)$ be the kernel given by

$$g_{(j)}(x_1, \ldots, x_j) = \frac{1}{d(k, j)} \sum\nolimits^{+}_{r_1 + \cdots + r_j = k} w(r_1, \ldots, r_j; k) g(\underbrace{x_1, \ldots, x_1}_{r_1}, \ldots, \underbrace{x_j, \ldots, x_j}_{r_j}).$$

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Let $U_n^{(j)}$ be the U-statistic associated with this kernel $g_{(j)}(x_1, \dots, x_j)$ for $j = 1, \dots, k$. The kernel $g_{(j)}(x_1, \dots, x_j)$ is symmetric because of the symmetry of $w(r_1, \dots, r_j; k)$. If $d(k, j)$ is equal to zero for some $j$, then the associated $w(r_1, \dots, r_j; k)$'s are equal to zero. In this case, we let the corresponding statistic $U_n^{(j)}$ be zero. We note that $g_{(k)} = g$ and so $U_n^{(k)} = U_n$.
Then the linear combination $Y_n$ of U-statistics is given by

$$Y_n = \frac{1}{D(n,k)} \sum_{j=1}^{k} d(k,j) \binom{n}{j} U_n^{(j)}, \tag{1.2}$$

where $D(n, k) = \sum_{j=1}^{k} d(k, j)\binom{n}{j}$. Since $w$'s are nonnegative and at least one of them is positive, $D(n, k)$ is positive.

The Y-statistic $Y_n$ includes U-statistic, V-statistic and LB-statistic of (1.4) below as special cases. If $w$ is the function given by $w(r_1, \dots, r_j; k) = k!/(r_1! \cdots r_j!)$ for positive integers $r_1, \dots, r_j$ such that $j = 1, \dots, k$ and $r_1 + \cdots + r_j = k$, then $d(k, j) = j!\mathcal{S}(k, j)$ $(j = 1, \dots, k)$ and $D(n, k) = n^k$ where $\mathcal{S}(k, j)$ are the Stirling number of the second kind. The corresponding statistic $Y_n$ is equal to V-statistic given by

$$V_n = \frac{1}{n^k} \sum_{j_1=1}^{n} \cdots \sum_{j_k=1}^{n} g(X_{j_1}, \dots, X_{j_k}). \tag{1.3}$$

If $w$ is the function given by $w(r_1, \dots, r_j; k) = 1$ for positive integers $r_1, \dots, r_j$ such that $j = 1, \dots, k$ and $r_1 + \cdots + r_j = k$, then $d(k, j) = \binom{k-1}{j-1}$ $(j = 1, \dots, k)$ and $D(n, k) = \binom{n+k-1}{k}$. The corresponding statistic $Y_n$ is equal to LB-statistic $B_n$ given by

$$B_n = \binom{n+k-1}{k}^{-1} \sum_{r_1+\cdots+r_n=k} g(\underbrace{X_1, \dots, X_1}_{r_1}, \dots, \underbrace{X_n, \dots, X_n}_{r_n}), \tag{1.4}$$

where $\sum_{r_1+\cdots+r_n=k}$ denote the summation over all non-negative integers $r_1, ..., r_n$ satisfying $r_1 + \cdots + r_n = k$ (see Toda and Yamato (2001)).

In Section 2 we derive the rate of almost sure convergence of the statistic $Y_n$ for the kernel $g$ which is not necessarily degenerate.
In Section 3 we derive the rate of almost sure convergence of the statistic $Y_n$ for the kernel $g$ which is degenerate.

## 2 Almost sure convergence (general case)

We consider almost sure convergence of $Y_n$ in this section without the assumption for degeneracy of the kernel $g$.

**Lemma 2.1** (Sen (1974)) *Suppose that $E \mid g \mid^p < \infty$ for $0 < p < 2$.*
(i) *If $0 < p < 1$, then $n^{k(1-p^{-1})} \mid U_n \mid$ converges to 0 as $n$ tend to $\infty$ almost surely* a.s. *That is,*

$$n^{k(1-p^{-1})} \mid U_n \mid \to 0 \quad \text{a.s.}$$

(ii) *If $1 \le p < 2$, then*

$$n^{(1-p^{-1})} \mid U_n - \theta \mid \to 0 \quad \text{a.s.}$$

The above (i) is also given by Giné and Zinn (1992). By using this Lemma, we show the following Theorems 2.2, Proposition 2.3 and Theorems 2.4.

**Theorem 2.2** *Suppose that for* $0 < p < 1$

$$E \mid g_{(j)} \mid^{\frac{j}{k}p} < \infty, \quad j = 1, 2, \dots, k. \tag{2.1}$$

*We assume that* $n^k/D(n,k)$ *converges as* $n \to \infty$. *Then for* $0 < p < 1$

$$n^{k(1-p^{-1})} \mid Y_n \mid \to 0 \quad \text{a.s.} \tag{2.2}$$

The above moment condition (2.1) may be replaced by

$$E \mid g(X_{i_1}, \dots, X_{i_k}) \mid^{\frac{j}{k}p} < \infty, \quad 1 \le i_1 \le \dots \le i_k \le k \ \text{ and } \ j = \sharp\{i_1, \dots, i_k\},$$

where $\sharp\{i_1, \dots, i_k\}$ is the number of distinct integers among $i_1, \dots, i_k$.
Since $\sum_{j=1}^{k} d(k,j)\binom{n}{j}/D(n,k) = 1$, the convergence of $n^k/D(n,k)$ is equivalent to the convergence $n^k/D(n,k) \to k!/d(k,k)$ as $n \to \infty$.

**Proof of Theorem 2.2**: For $1 \le j \le k$ and $0 < p < 1$, $0 < jp/k < 1$. Thus by (i) of Lemma 2.1, $n^{j(1-k/(jp))} \mid U_n^{(j)} \mid \to 0$ a.s. By the convergence of $n^k/D(n,k)$, for $1 \le j \le k$ we have

$$n^{k(1-p^{-1})} \frac{d(k,j)}{D(n,k)} \binom{n}{j} \mid U_n^{(j)} \mid \tag{2.3}$$

$$= \frac{d(k,j)}{j!} \frac{(n)_j}{n^j} \frac{n^k}{D(n,k)} n^{j(1-\frac{k}{jp})} \mid U_n^{(j)} \mid \to 0 \quad \text{a.s.},$$

where $(n)_j = n(n-1)\cdots(n-j+1)$.
Applying (2.3) to the right-hand side of

$$n^{k(1-p^{-1})} \mid Y_n \mid \le \sum_{j=1}^{k} \frac{d(k,j)}{D(n,k)} \binom{n}{j} n^{k(1-p^{-1})} \mid U_n^{(j)} \mid,$$

we can get (2.2). □

For V-statistic $V_n$, $n^k/D(n,k)$ converges to one as $n \to \infty$ because of $n^k/D(n,k) = 1$, and for LB-statistic $B_n$, $n^k/D(n,k)$ converges to $k!$ as $n \to \infty$ because of $n^k/D(n,k) = n^k/\binom{n+k-1}{k}$ (see Yamato et al. (2001), p.4). Thus we have the following: Under the condition (2.1), for $0 < p < 1$

$$n^{k(1-p^{-1})} \mid V_n \mid \to 0 \quad \text{a.s.},$$

$$n^{k(1-p^{-1})} \mid B_n \mid \to 0 \quad \text{a.s.}$$

Since $U_n \to \theta$ a.s. under the condition $E \mid g \mid < \infty$, by (i) of Lemma 2.1 we get the following Proposition.

**Proposition 2.3** *Suppose that*

$$E \mid g(X_{i_1}, \dots, X_{i_k}) \mid^{\sharp\{i_1,\dots,i_k\}/k} < \infty \ \text{ for } \ 1 \le i_1 \le \dots \le i_k \le k. \tag{2.4}$$

*We assume that* $n^k/D(n,k)$ *converges as* $n \to \infty$. *Then*

$$Y_n \to \theta \quad \text{a.s.}$$

This convergence is shown under the condition $E \mid g(X_{i_1}, \ldots, X_{i_k}) \mid < \infty$ for $1 \leq i_1 \leq \cdots \leq i_k \leq k$ by Proposition 3.1 of Toda and Yamato (2001). Proposition 2.3 shows it under weaker conditions. As special cases of Proposition 2.3, we have $V_n \to \theta$ a.s. and $B_n \to \theta$ a.s. The convergence $V_n \to \theta$ a.s. is given by Proposition of Giné and Zinn (1992) , p.274.

**Theorem 2.4** *Suppose that*

$$E \mid g(X_1, \ldots, X_k) \mid^p < \infty \quad \text{for} \quad 1 < p < 2$$

*and*

$$E \mid g(X_{i_1}, \ldots, X_{i_k}) \mid < \infty \quad \text{for} \quad 1 \leq i_1 \leq \cdots \leq i_k \leq k.$$

*We assume that* $n^k/D(n,k)$ *converges as* $n \to \infty$. *Then for* $1 < p < 2$

$$n^{(1-p^{-1})} \mid Y_n - \theta \mid \to 0 \quad \text{a.s.}$$

**Proof**: $Y_n - \theta$ can be written as follows:

$$(2.5) \qquad Y_n - \theta = \frac{d(k,k)}{D(n,k)} \binom{n}{k} [U_n - \theta] + \sum_{j=1}^{k-1} \frac{d(k,j)}{D(n,k)} \binom{n}{j} [U_n^{(j)} - Eg_{(j)}]$$
$$+ \sum_{j=1}^{k-1} \frac{d(k,j)}{D(n,k)} \binom{n}{j} [Eg_{(j)} - \theta].$$

By the assumptions and (ii) of Lemma 2.1,

$$(2.6) \qquad n^{(1-p^{-1})} \frac{d(k,k)}{D(n,k)} \binom{n}{k} \mid U_n - \theta \mid$$
$$= \frac{(n)_k}{D(n,k)} \frac{d(k,k)}{k!} \cdot n^{(1-p^{-1})} \mid U_n - \theta \mid \to 0 \ \text{ a.s.}$$

For $1 \leq j \leq k-1$, we have $j < k - 1 + p^{-1}$ and $U_n^{(j)} \to Eg_{(j)}$ a.s. because of $E \mid g_{(j)} \mid < \infty$ by the assumption. Thus

$$(2.7) \qquad n^{(1-p^{-1})} \frac{d(k,j)}{D(n,k)} \binom{n}{j} \mid U_n^{(j)} - Eg_{(j)} \mid$$
$$= \frac{n^k}{D(n,k)} \frac{d(k,j)}{j!} \frac{(n)_j}{n^{k-1+p^{-1}}} \mid U_n^{(j)} - Eg_{(j)} \mid \to 0 \ \text{ a.s.}$$

For $1 \leq j \leq k-1$, because of $j < k - 1 + p^{-1}$, we have

$$(2.8) \qquad n^{(1-p^{-1})} \frac{d(k,j)}{D(n,k)} \binom{n}{j} = \frac{n^k}{D(n,k)} \frac{d(k,j)}{j!} \frac{(n)_j}{n^{k-1+p^{-1}}} \to 0 \ \text{ as } \ n \to \infty.$$

Applying (2.6), (2.7) and (2.8) to the right-hand side of (2.5), we get $n^{(1-p^{-1})} \mid Y_n - \theta \mid \to 0$ a.s. □

As stated immediarely after Proof of Theorem 2.2, $D(n,k)$ satifies the condition of Theorem 2.4 for V-statistic and LB-statistic. Thus we have the following corollary.

**Corollary 2.5** *Under the moment conditions for the kernel* $g$ *of Theorem* 2.4, *for* $1 < p < 2$

$$n^{(1-p^{-1})} \mid V_n - \theta \mid \to 0 \quad \text{a.s.,}$$

$$n^{(1-p^{-1})} \mid B_n - \theta \mid \to 0 \quad \text{a.s.}$$

**3 Almost sure convergence (degenerate case)** In this section we assume degeneracy of the kernel $g$. We put

$$\psi_j(x_1,\dots,x_j) = E[g(X_1,\dots,X_k) \mid X_1 = x_1,\dots,X_j = x_j], \quad j = 1,...,k$$

$$\sigma_j^2 = Var[\psi_j(X_1,\dots,X_j)], \quad j = 1,...,k.$$

We assume that

$$\sigma_1^2 = \cdots = \sigma_{d-1}^2 = 0 \quad \text{and} \quad \sigma_d^2 > 0.$$

That is, the U-statistic and/or the kernel $g$ are/is degenerate of order $d-1$. Under this degeneracy, the following almost sure convergence of $U_n$ is shown.

**Lemma 3.1** (Giné and Zinn (1992)) *Let $s$ be $k - d/2 < s < k$ and assume that*

$$E \mid g(X_1,\dots,X_k) \mid^{\frac{d}{s+d-k}} < \infty.$$

*Then*

$$n^{k-s}(U_n - \theta) \to 0 \quad \text{a.s.}$$

Relating to the Y-statistic $Y_n$, the U-statistics $U_n^{(j)}$ $(j = 1,\dots,k)$ have the following properties.

**Lemma 3.2** (Yamato et al. (2001)) *If $d = 2l+1$ and $l$ is a positive integer, then $EU_n^{(k)} = EU_n^{(k-1)} = \cdots = EU_n^{(k-l+1)} = EU_n^{(k-l)} = \theta$. The orders of degeneracy of $U_n^{(k-1)}$,..., $U_n^{(k-l+1)}$, $U_n^{(k-l)}$ are at least $2(l-1),...,2,0$, respectively.*
*If $d = 2l$ and $l$ is a positive integer, then $EU_n^{(k)} = EU_n^{(k-1)} = \cdots = EU_n^{(k-l+2)} = EU_n^{(k-l+1)} = \theta$. The orders of degeneracy of $U_n^{(k-1)}$,...,$U_n^{(k-l+2)}$,$U_n^{(k-l+1)}$ are at least $2l-3,...,3,1$, respectively.*

We state the almost sure convergence of $Y_n$ in case that $d$ is even or odd, separately.

**Theorem 3.3** *Suppose that $n^k/D(n,k)$ converges as $n \to \infty$. We assume that $d = 2l$ $(l = 1,2,\dots)$ and the following moment conditions.*

$$E \mid g \mid^{\frac{d}{s+d-k}} < \infty, \;\; k-l < s < k. \tag{3.1}$$

*For $j = 1,2,\dots,l-1$*

$$E \mid g_{(k-j)} \mid^{\frac{d-2j}{s+d-k-j}} < \infty, \;\; k-l < s < k-j, \tag{3.2}$$

$$E \mid g_{(k-j)} \mid < \infty, \;\; k-j \le s < k. \tag{3.3}$$

*For $j = 1,2,\dots,k-l$*

$$E \mid g_{(j)} \mid < \infty. \tag{3.4}$$

*Then for $k-l < s < k$, that is, for $k-d/2 < s < k$,*

$$n^{k-s}(Y_n - \theta) \to 0 \quad \text{a.s.}$$

We note that the above moment condition such that $E \mid g_{(r)} \mid^q < \infty$ may be replaced by

$$E \mid g(X_{i_1}, \dots, X_{i_k}) \mid^q < \infty \text{ for } 1 \le i_1 \le \cdots \le i_k \le k \text{ and } \sharp\{i_1, \dots, i_k\} = r.$$

For the exponent of (3.1), $1 < d/(s+d-k) < 2$ $(k-l < s < k)$. For the exponent of (3.2), $1 < (d-2j)/(s+d-k-j) < d/(s+d-k) < 2$ $(j = 1, \dots, l-1,\ \ k-l < s < k-j)$.

**Proof**: (i) Since the U-statistic $U_n$ is assumed to have the order $d-1 (= 2l-1)$ of degeneracy, by Lemma 3.1 and (3.1) for $k-l < s < k$, $n^{k-s}(U_n - \theta) \to 0$ a.s. Thus by the convergence of $n^k/D(n,k)$ we have

$$\frac{d(k,k)}{D(n,k)}\binom{n}{k} n^{k-s}(U_n - \theta) = \frac{(n)_k}{D(n,k)}\frac{d(k,k)}{k!} n^{k-s}(U_n - \theta) \to 0 \quad \text{a.s.}$$

(ii) For $j = 1, 2, \dots, l-1$, the U-statistic $U_n^{(k-j)}$ has the order $2(l-j)-1$ of degeneracy at least by Lemma 3.2. Therefore by Lemma 3.1 and (3.2) for $k - l (= k-j-2(l-j)/2) < s < k-j$,

$$n^{k-j-s}(U_n^{(k-j)} - \theta) \to 0 \quad \text{a.s.} \tag{3.5}$$

For $k-j \le s < k$, because of $k-j-s \le 0$, (3.3) and Lemma 3.2, we have $n^{k-j-s}(U_n^{(k-j)} - \theta) \to 0$ a.s. Thus by the convergence of $n^k/D(n,k)$, for $j = 1, 2, \dots, l-1$ we have

$$\frac{d(k,k-j)}{D(n,k)}\binom{n}{k-j} n^{k-s}(U_n^{(k-j)} - \theta) \tag{3.6}$$

$$= \frac{n^k}{D(n,k)}\frac{d(k,k-j)}{(k-j)!}\frac{(n)_{(k-j)}}{n^{k-j}} n^{k-j-s}(U_n^{(k-j)} - \theta) \to 0 \quad \text{a.s.}$$

Note that if the order of degeneracy of $U_n^{(k-j)}$ $(j = 1, 2, \dots, l-1)$ is larger than $2(l-j)-1$ , then for example, we suppose it be $2(l-j)$. Then for $k-j-[2(l-j)+1]/2) < s < k-j$, we have (3.5). Therefore for $k - l (= k-j-2(l-j)/2) < s < k-j$ we have (3.5) and so (3.6).

(iii) We consider for the convergence of $U_n^{(j)}$ $(j = 1, \dots, k-l)$. By the assumption (3.4) $U_n^{(j)} \to Eg_{(j)}$ a.s. For $k-l < s < k$, by the inequality $1 \le j \le k-l$ we get $j - s \le -1$. Thus

$$\frac{d(k,j)}{D(n,k)}\binom{n}{j} n^{k-s}(U_n^{(j)} - \theta)$$

$$= \frac{n^k}{D(n,k)}\frac{d(k,j)}{j!}\frac{(n)_{(j)}}{n^j} n^{j-s}(U_n^{(j)} - \theta) \to 0 \quad \text{a.s.}$$

$Y_n - \theta$ can be written as follows:

$$Y_n - \theta = \frac{d(k,k)}{D(n,k)}\binom{n}{k}[U_n - \theta] + \sum_{j=1}^{l-1}\frac{d(k,k-j)}{D(n,k)}\binom{n}{k-j}[U_n^{(k-j)} - \theta]$$

$$+ \sum_{j=1}^{k-l}\frac{d(k,j)}{D(n,k)}\binom{n}{j}[U_n^{(j)} - \theta].$$

Multipling $n^{k-s}$ on the both sides of the above and using the convergence shown in (i), (ii) and (iii), we get or $k-l < s < k$, $n^{k-s}(Y_n - \theta) \to 0$ a.s. □

As stated in Section 2, for V-statistic and LB-statistic $D(n.k)$ satisfies the condition of Theorem 3.2. Thus we have the following Corollary.

**Corollary 3.4** *We assume that $d = 2l$ $(l = 1, 2, \dots)$ and the moment conditions given in Theorem 3.2. Then for $k - l < s < k$, that is, for $k - d/2 < s < k$,*

$$n^{k-s}(V_n - \theta) \to 0 \quad \text{a.s.}$$

$$n^{k-s}(B_n - \theta) \to 0 \quad \text{a.s.}$$

**Theorem 3.5** *Suppose that $n^k/D(n,k)$ converges as $n \to \infty$. We assume that $d = 2l + 1$ $(l = 1, 2, \dots)$ and the following moment conditions.*

$$E \mid g \mid^{\frac{d}{s+d-k}} < \infty, \quad k - \frac{d}{2} < s < k. \tag{3.7}$$

*For $j = 1, 2, \dots, l - 1$*

$$E \mid g_{(k-j)} \mid^{\frac{d-2j}{s+d-k-j}} < \infty, \quad k - \frac{d}{2} < s < k - j, \tag{3.8}$$

$$E \mid g_{(k-j)} \mid < \infty, \quad k - j \leq s < k. \tag{3.9}$$

$$E \mid g_{(k-l)} \mid^{\frac{k-l}{s}} < \infty, \quad k - \frac{d}{2} + \frac{1}{2} < s < k, \tag{3.10}$$

$$E \mid g_{(k-l)} \mid^{\frac{1}{l+s-k+1}} < \infty, \quad k - \frac{d}{2} < s \leq k - \frac{d}{2} + \frac{1}{2} (= k - l). \tag{3.11}$$

*For $j = 1, 2, \dots, k - l - 1$*

$$E \mid g_{(j)} \mid < \infty. \tag{3.12}$$

*Then for $k - d/2 < s < k$, that is, for $k - l - 1/2 < s < k$,*

$$n^{k-s}(Y_n - \theta) \to 0 \quad \text{a.s.}$$

For the exponent of (3.7), $1 < d/(s + d - k) < 2$ $(k - d/2 < s < k)$. For the exponent of (3.8), $1 < (d-2j)/(s+d-k-j) < d/(s+d-k) < 2$ $(j = 1, \dots, l-1, \ k-d/2 < s < k-j)$. For the exponent of (3.10), $0 < (k - l)/s < 1$ $(k - (d/2) + 1 < s < k)$. For the exponent of (3.11), $1 \leq 1/(l + s - k + 1) < 2$ $(k - d/2 < s \leq k - (d/2) + 1)$.

**Proof**: (i) The U-statistic $U_n$ has the order $d - 1 (= 2l)$ of degeneracy. Under the condition (3.7), by the same reason as stated in (i) of Proof of Thorem 3.3, for $k - d/2 < s < k$ we have

$$\frac{d(k,k)}{D(n,k)} \binom{n}{k} n^{k-s}(U_n - \theta) \to 0 \quad \text{a.s.}$$

(ii) For $j = 1, 2, \dots, l-1$, the U-statistic $U_n^{(k-j)}$ has the order $2(l-j)$ of degeneracy at least by Lemma 3.2. Therefore under the conditions (3.8) and (3.9), by same reason as stated in (ii) of Proof of Thorem 3.3, for $k - d/2 < s < k$ we have

$$\frac{d(k,k-j)}{D(n,k)} \binom{n}{k-j} n^{k-s}(U_n^{(k-j)} - \theta) \to 0 \quad \text{a.s.}$$

(iii) The U-statistic $U_n^{(k-l)}$ is not degenerate by Lemma 3.2. For $k-l<s<k$, we note $k-l-s<0$. Under the condition (3.10) by (i) of Lemma 2.1, for $k-l<s<k$, $n^{k-l-s}(U_n^{(k-l)}-\theta)=n^{k-l-s}U_n^{(k-l)}-n^{k-l-s}\theta \to 0$ a.s.
For $k-l-1/2(=k-d/2)<s\le k-l$, we have $0\le k-l-s<1/2$. If we put $p=1/(l+s-k+1)$, then we have $1\le p<2$. Therefore under the condition (3.11), by (ii) of Lemma 2.1, $n^{k-l-s}(U_n^{(k-l)}-\theta)=n^{1-p^{-1}}(U_n^{(k-l)}-\theta)\to 0$ a.s.
Hence by the convergence of $n^k/D(n,k)$, for $k-d/2<s<k$ we have

$$\frac{d(k,k-l)}{D(n,k)}\binom{n}{k-l}n^{k-s}(U_n^{(k-l)}-\theta)$$

$$=\frac{n^k}{D(n,k)}\frac{d(k,k-l)}{(k-l)!}\frac{(n)_{(k-l)}}{n^{k-l}}n^{k-l-s}(U_n^{(k-l)}-\theta)\to 0 \quad \text{a.s.}$$

(iv) We consider for the convergence of $U_n^{(j)}$ $(j=1,\dots,k-l-1)$. If $k-d/2(=k-l-1/2)<s<k$, then because of $1\le j\le k-l-1$ we have $j-s<-1/2$. Thus under the condition (3.12), by the same reason as (iii) of Proof of Theorem 3.3, for $k-d/2<s<k$ we have

$$\frac{d(k,j)}{D(n,k)}\binom{n}{j}n^{k-s}(U_n^{(j)}-\theta)\to 0 \quad \text{a.s.}$$

$Y_n-\theta$ can be written as follows:

$$Y_n-\theta=\frac{d(k,k)}{D(n,k)}\binom{n}{k}[U_n-\theta]+\sum_{j=1}^{l-1}\frac{d(k,k-j)}{D(n,k)}\binom{n}{k-j}[U_n^{(k-j)}-\theta]$$

$$+\frac{d(k,k-l)}{D(n,k)}\binom{n}{k-l}[U_n^{(k-l)}-\theta].$$

$$+\sum_{j=1}^{k-l-l}\frac{d(k,j)}{D(n,k)}\binom{n}{j}[U_n^{(j)}-\theta].$$

Multipling $n^{k-s}$ on the both sides of the above and using the convergence shown in (i), (ii), (iii) and (iv), we get for $k-d/2<s<k$, $n^{k-s}(Y_n-\theta)\to 0$ a.s. □

**Corollary 3.6** *We assume that $d=2l+1$ $(l=1,2,\dots)$ and the moment conditions given in Theorem 3.5. Then for $k-d/2<s<k$,*

$$n^{k-s}(V_n-\theta)\to 0 \quad \text{a.s.}$$

$$n^{k-s}(B_n-\theta)\to 0 \quad \text{a.s.}$$

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