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THE CONTINUITY OF OPERATORS IN PATTERN MORPHOLOGY *

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Abstract. In pattern analysis and image management, the information of an objective image can be recovered from a sequence approximate images. In mathematical form of expression we need to consider some types of continuity. Many researchers defined the limit and the upper limit of a sequence, using the concepts, characterized continuity in the space consisting of images. In the present paper, we shall give firstly some examples to show that there are some theoretical shortcomings in those results, then we shall give some corresponding correct results.

1 Preliminaries The theory of mathematical morphology was developed for binary images and extensively uses set-theoretic operations such as union, intersection, and set complement, because an image under consideration is always considered as a set in mathematical morphology. Since many algorithms for pattern analysis, which process noisy data, critically depend on an accurate geometrical and topological image description. We have to provide a precise mathematical description of an image X under consideration, and have to provide its crucial geometrical and topological structure. In the present paper, we shall eliminate some theoretical shortcoming in [1,3], and give some corresponding correct results.

Let R^n be the n-dimensional Euclidean space and N the set of natural numbers. For a space $(X, \mathcal{T})$, if a sequence $\{x_n : n \in N\}$ converges to x^* in X, then we write $x_n \xrightarrow{X} x^*$. If y is a point in R^n and A, B are subsets of R^n, let $A[y]$ be its translation by the point y, i.e., $A[y] = \{a + y : a \in A\}$, and $\check{A}$ the symmetric set of A with respect to the origin, i.e., $\check{A} = \{-a : a \in A\}$. $A \oplus B = \{a + b : a \in A, b \in B\}$ is called the dilation of set A by set B, and $A \ominus \check{B} = \cap_{b \in \check{B}} A[b]$ is the erosion of set A by set B. It is clear that $A \oplus (B \oplus C) = (A \oplus B) \oplus C$, $A \oplus B = B \oplus A$ and $A \ominus \check{B} = \{x : B[x] \subset A\}$. And $A \circ B = (A \ominus \check{B}) \oplus B$, $A \bullet B = (A \oplus \check{B}) \ominus B$ are called the opening operator, and closing operator of A and B respectively.

Let $E \subset R^n$ be a bounded closed set whose interior contains the origin. Set $\Gamma = \{F \subset R^n : F \text{ is closed}\}$, $\Omega = \{F \subset R^n : F \text{ is a nonempty compact set}\}$. $\Lambda = \{F \subset E : F \neq \emptyset, F \text{ is closed}\}$.

Definition 1.1 ([1,3]). *Let $G_1, G_2, \cdots, G_m$ be finite non-empty open sets, and $K_1, K_2, \cdots, K_p$ finite compact sets in R^n. Set*

$N(\{G_i\}, \{K_j\}) = \{F \in \Gamma : F \cap G_i \neq \emptyset$ *for* $i = 1, 2, \cdots, m; F \cap K_j = \emptyset$ *for* $j = 1, 2, \cdots, p\}$. *It is possible that $m = 0$ or $p = 0$, then $N(\{G_i\}, \{K_j\})$ is denoted by $N(, \{K_j\})$ or $N(\{G_i\},)$.*

$\mathcal{B} = \{N(\{G_i\}, \{K_j\}) : \{G_i\}$ is a finite family of non-empty open sets in R^n, and $\{K_j\}$ is a finite family of compact sets in $R^n\}$, *and* $\mathcal{T} = \{\cup \mathcal{B}^* : \mathcal{B}^* \subset \mathcal{B}\}$.

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It can be easily proved that $(\Gamma, \mathcal{T})$ is a topological space, $\mathcal{T}$ is called the HM topology. Hence, Ω and Λ, as subspaces of $(\Gamma, \mathcal{T})$, are topological spaces.

Lemma 1.2 ([3]). *A space $(\Gamma, \mathcal{T})$ and $(\Omega,\ \Lambda)$ are Hausdorff spaces with countable bases.*

From Lemma 1.2 we know that the convergency can be characterized by sequence in above spaces.

Lemma 1.3 ([3]). *For a sequence $\{F_i : i \in N\} \subset \Gamma$, then $F_i \xrightarrow{\Gamma} F$ if and only if the following two conditions are satisfied:*

1). If G is an open set in R^n and $G \cap F \neq \emptyset$, then G intersects eventually $\{F_i : i \in N\}$ (that is, there is an $n \in N$, such that $G \cap F_i \neq \emptyset$ for $i > n$).

2). If K is a compact set in R^n with $K \cap F = \emptyset$, then K does not eventually intersect $\{F_i : i \in N\}$ (that is, there is an $n \in N$, such that $K \cap F_i = \emptyset$ for $i > n$).

Lemma 1.4 ([4]). *If A_1 and A_2 are compact subspaces of R^n, then for every open subset W of the Cartesian product $R^n \times R^n$ which contains the set $A_1 \times A_2$ there exist open sets U_1 and U_2 in R^n such that $c(U_1) \times c(U_2)$ is compact in $R^n \times R^n$ and $A_1 \times A_2 \subset U_1 \times U_2 \subset c(U_1) \times c(U_2) \subset W$, where $c(-)$ is the closure operator in R^n.*

Definition 1.5 ([3]). *Let $\{F_i : i \in N\}$ be a sequence of the space Γ, then the upper limit $\overline{\lim} F_i$ of $\{F_i : i \in N\}$ is defined as $\overline{\lim} F_i = \bigcap_{i>0} \overline{\bigcup_{j>i} F_j}$.*

Definition 1.6 ([3]). *Suppose E is a separable space, and $\Psi : E \longrightarrow \Gamma$ is a map. Let $\{x_i : i \in N\}$ be a sequence of E with $x_i \xrightarrow{E} x$. Then we call Ψ to be upper semi-continuous if $\Psi(x) \supset \overline{\lim} \Psi(x_i)$.*

2. Some counterexamples

In [1,3] the authors discussed the continuity and the upper semi-continuity of operators. In this section we shall give some examples to show that a few results in [1,3] are not true. The following Theorems 2.1 and 2.3 were set up in [1] and cited in [3] without proofs.

Theorem 2.1 [1]. *The map $\Psi : \Gamma \times \Omega \longrightarrow \Gamma$ with $\Psi(F, K) = F \oplus K$ is continuous, and $\Phi : \Omega \times \Omega \longrightarrow \Omega$ with $\Phi(B, K) = B \oplus K$ is continuous , where $F \in \Gamma$, K and B are in Ω.*

The following example implies that Theorem 2.1 is not true.

Example 2.2. Let $n = 1$, that is, $R^n = R$, set $F_i = \{0, i\}$, $K_i = \{0, -i + 1\}$, and $F = K = \{0\}$. Then we have that $F_i \xrightarrow{\Omega} F$ and $K_i \xrightarrow{\Omega} K$ by Lemma 1.3. In this case, $F_i \oplus K_i = \{0, 1, -i + 1, i\}$, and $F \oplus K = \{0\}$. From Lemma 1.3 we get that $F_i \oplus K_i \xrightarrow{\Omega} \{0, 1\} \in \Omega$. But $F \oplus K = \{0\} \neq \{0, 1\}$. Hence the dilation is not a continuous map from $\Omega \times \Omega$ to Ω.

Theorem 2.3 [3]. *The map $\Psi : \Gamma \times \Omega \longrightarrow \Gamma$ with $\Psi(F, K) = F \circ K$ is upper semi-continuous, and $\Phi : \Omega \times \Omega \longrightarrow \Omega$ with $\Phi(B, K) = B \circ K$ is upper semi-continuous. The map $\Psi : \Gamma \times \Omega \longrightarrow \Gamma$ with $\Psi(F, K) = F \bullet K$ is upper semi-continuous, and $\Phi : \Omega \times \Omega \longrightarrow \Omega$ with $\Phi(B, K) = B \bullet K$ is upper semi-continuous, where $F \in \Gamma$, K and B are in Ω.*

The following Examples 2.4 and 2.5 imply that Theorem 2.3 is not true.

Example 2.4. Let $n = 1$, that is, $R^n = R$, set $F_i = \{0, 3, 4, -i, 1-i\}$, $K_i = \{0, 1, i\}$, $F = \{0, 3, 4\}$ and $K = \{0, 1\}$. Then we have that $F_i \xrightarrow{\Omega} F$ and $K_i \xrightarrow{\Omega} K$ by Lemma 1.3. In this case, $F_i \circ K_i = \{0, -i, -i+1\}$ for $i > 5$, and $F \circ K = \{3, 4\}$. From Lemma 1.3 we get that $F_i \circ K_i \xrightarrow{\Omega} \{0\} \in \Omega$. But $F \circ K = \{3, 4\} \not\supset \{0\}$. Hence the opening operator is not an upper semi-continuous map from $\Omega \times \Omega$ to Ω.

Example 2.5. Let $n = 1$, that is, $R^n = R$, set $F_i = \{0, 1+i, 1-i\}$, $K_i = \{0, i\}$, $F = \{0\}$ and $K = \{0\}$. Then we have that $F_i \xrightarrow{\Omega} F$ and $K_i \xrightarrow{\Omega} K$ by Lemma 1.3. In this case, $F_i \bullet K_i = \{0, 1, -i+1, 1+i\}$, and $F \bullet K = \{0\}$. From Lemma 1.3 we get that $F_i \bullet K_i \xrightarrow{\Omega} \{0, 1\} \in \Omega$. But $F \bullet K = \{0\} \not\supset \{0, 1\}$. Hence the closing operator is not an upper semi-continuous map from $\Omega \times \Omega$ to Ω.

3. The continuity of operators.

In this section we shall give new forms of Theorems 2.1 and 2.3.

Theorem 3.1. *The dilation operator* $\oplus : \Gamma \times \Lambda \longrightarrow \Gamma$ *is continuous.*

Proof At first, we note that $\oplus : \Gamma \times \Lambda \longrightarrow \Gamma$ is well-defined, that is, for every closed A and every compact B, $A \oplus B$ is closed. In fact, if $x \notin A \oplus B$, then for every $b \in B$, we have $x - b \notin A$. It follows from the continuity of the minus operator that there exist two open sets $U(b), V(b)$ in R^n such that $x \in U(b), b \in V(b)$ and $(U(k) - V(k)) \cap A = \emptyset$, where $U(b) - V(b) = \{y - z : y \in U(b), z \in V(b)\}$. Since B is compact there exists a finite subset $\{b_1, b_2, ..., b_n\} \subset B$ such that $V(b_1) \cup V(b_2) \cup ... \cup V(b_n) \supset B$. Let $U = U(b_1) \cap U(b_2) \cap ... \cap U(b_n)$. Then it is trivial that $x \in U$ and $U \cap (A \oplus B) = \emptyset$. Hence $A \oplus B$ is closed.

To show the continuity of $\oplus$ it suffices to check, that for every set $U \subset R^n$ and every compact subset $C \subset R^n$, the sets

$$\{(F, K) \in \Gamma \times \Lambda : (F \oplus K) \cap U \neq \emptyset\}$$

and

$$\{(F, K) \in \Gamma \times \Lambda : (F \oplus K) \cap C = \emptyset\}$$

are open in $\Gamma \times \Lambda$. The former is trivial. We have to show the later. Suppose $F \in \Gamma$ and $K \in \Lambda$ satisfying $(F \oplus K) \cap C = \emptyset$. Define $\phi : R^n \times R^n \longrightarrow R^n$ as $\phi(c, k) = c - k$. Then ϕ is continuous and $C \times K \subset \phi^{-1}(R^n \setminus F)$. It follows from Lemma 1.4 that there exist open sets U, V such that $c(U) \times c(V)$ is compact in $R^n \times R^n$ and $C \times K \subset U \times V \subset c(U) \times c(V) \subset \phi^{-1}(R^n \setminus F)$. Then $\phi(c(U) \times c(V)) \subset R^n$ is compact and $\phi(c(U) \times c(V)) \cap F = \emptyset$. That is, $F \in N(, \phi(c(U) \times c(V)))$ and $K \in N(, E \setminus V)$. Hence $(F, K) \in N(, \phi(c(U) \times c(V))) \times N(, E \setminus V)$. The remainder is to check that $(G \oplus L) \cap C = \emptyset$ for any $G \in N(, \phi(c(U) \times c(V)))$ and $L \in N(, E \setminus V)$ with $L \subset E$. In fact, for any $g \in G$ and $l \in L$ we have $g + l \notin C$ since, otherwise, $g = g + l - l \in \phi(C \times \mathrm{L}) \subset \phi(c(U) \times c(V))$ by $L \subset V$, which contradicts to $G \cap \phi(c(U) \times c(V)) = \emptyset$.

Theorem 3.2. *The erosion operator* $\ominus : \Gamma \times \Gamma \longrightarrow \Gamma$ *is upper semi-continuous.*

Proof. At first, it follows from the definition that $\ominus : \Gamma \times \Gamma \longrightarrow \Gamma$ is well-defined. Let $\{(F_i, K_i)\}$ be a sequence in $\Gamma \times \Gamma$ and $(F_i, K_i) \longrightarrow (F, K)$. Now suppose $x \notin F \ominus \check{K}$. Then there exists $k \in K$ such that $x + k \notin F = \bigcap_{i>0} c(\bigcup_{j>i} F_j)$. Thus there exists i_1 such that

$x + k \notin c(\bigcup_{j>i_1} F_j)$. It follows from the continuity of $+ : R^n \times R^n \longrightarrow R^n$ that there exist two open sets $U \ni x$ and $V \ni k$ such that

$$(U \oplus V) \cap c(\bigcup_{j>i_1} F_j) = \emptyset. \tag{1}$$

Since $V \cap K \neq \emptyset$ and $K_i \longrightarrow K$ in Γ we have some $i > i_1$ such that $V \cap K_j \neq \emptyset$ for all $j > i$. It follows that

$$U \cap c(\bigcup_{j>i} F_j \ominus \check{K}_j) = \emptyset. \tag{2}$$

In fact, for any $j > i$, (1) implies that $(U \oplus V) \cap F_j = \emptyset$. Moreover, for any $z \in U$, Choose $k_j \in V \cap K_j$. Then $z + k_j \in U \oplus V$ and hence $z + k_j \notin F_j$. So $z \notin F_j \ominus \check{K}_j$. Thus $U \cap (\bigcup_{j>i} F_j \ominus \check{K}_j) = \emptyset$. Because U is open (2) holds. At last, $U \ni x$ implies $x \notin c(\bigcup_{j>i} F_j \ominus \check{K}_j) \supset \bigcap_{i>0} c(\bigcup_{j>i} F_j \ominus \check{K}_j)$. This shows that $F \ominus \check{K} \supset \bigcap_{i>0} c(\bigcup_{j>i} F_j \ominus \check{K}_j)$, that is $\ominus : \Gamma \times \Gamma \longrightarrow \Gamma$ is upper semi-continuous.

Theorem 3.3. *The opening operator* $\circ : (\Gamma \times \Lambda) \cup (\Lambda \times \Gamma) \longrightarrow \Gamma$ *and the closing operator* $\bullet : (\Gamma \times \Lambda) \cup (\Lambda \times \Gamma) \longrightarrow \Gamma$ *are upper semi-continuous.*

Proof. It follows from the definitions and Theorems 3.1 and 3.2.

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