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ARITHMETICO-GEOMETRIC AND GEOMETRICO-HARMONIC MEANS OF TWO CONVEX FUNCTIONALS

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ABSTRACT. In this paper, we extend the notions of arithmetico-geometric, arithmetico-harmonic and geometrico-harmonic operator means to convex functionals. We also prove that the arithmetico-harmonic functional mean coincides with the convex geometric mean constructed in [2]. In the quadratical case, we find again some results for operators discussed in [1] and [6].

0 Introduction The geometric mean (resp. the harmonic mean) of two positive operators A and B was defined at the first time by T. Ando, [1] :

$$AgB = \max \left\{ X; \begin{pmatrix} A & X \\ X & B \end{pmatrix} \geq 0 \right\}$$

$$\text{resp. } AhB = \max \left\{ X; \begin{pmatrix} 2A & 0 \\ 0 & 2B \end{pmatrix} \geq \begin{pmatrix} X & X \\ X & X \end{pmatrix} \right\}.$$

In [5], J. Fujii introduced the arithmetico-geometric and arithmetico-harmonic operator means from analogies of positive numbers as follows :
Let $A_1 = A, B_1 = B, A_{n+1} = A_n g B_n$ (resp.$A_{n+1} = A_n h B_n$) and $B_{n+1} = \frac{1}{2}(A_n + B_n)$ for all $n \geq 1$. Then the arithmetico-geometric mean AagB (resp. the arithmetico-harmonic mean AahB) of A and B is the same limit of A_n and B_n in the strong operator convergence.
The geometrico-harmonic mean $AghB$ of two positive operators A and B was defined by F. Kubo in [7].
J.Fujii and M.Fujii, [6], have showed, by using the Gelfand-representation, that the operator means mentioned above are reduced to the numerical means and thus $AahB = AgB$ was proved directly by T. Ando (cf. [7]).

In another part and very recently, M. Atteia and M. Raïssouli, [2], constructed the geometric mean of two convex functionals as follows :
Let $f, g : H \longrightarrow \mathbb{R} \cup \{+\infty\}$ (H is a real Hilbert space) be two convex functionals, the geometric functional mean $f \tau g$ of f and g is the pointwise limit of the sequence $(\varphi_n(f,g))_n$ defined by :

$$(AR) \begin{cases} \varphi_o(f,g) & = & \frac{1}{2}f + \frac{1}{2}g \\ \\ \varphi_{n+1}(f,g) & = & \frac{1}{2}\varphi_n(f,g) + \frac{1}{2}(\varphi_n(f^*,g^*))^* \quad (n \geq 0) \end{cases}$$

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where f^* denotes the conjugate functional of f defined by the formulae
$\forall x^* \in H \quad f^*(x^*) = \sup\{< x^*, x > -f(x), x \in H\}$ (cf. [7]).

The (AR) algorithm permits the authors of [2], under a functional angle, to deduce another definition of the geometric mean of two positive operators which, of course, coincides with the above one. Moreover, to illustrate the importance of the (AR) algorithm, the authors of [2] give a physical interpretation of the geometric operator mean as an equivalent resistor of an electrical trip with matrices elements.

In this paper, we extend the previous operator means to convex functionals. We define the arithmetico-geometric, arithmetico-harmonic and geometrico-harmonic means of two convex functionals f and g. We also prove that the arithmetico-harmonic mean of f and g coincides with their convex geometric mean introduced in [2]. In particular, when f and g are two quadratical functionals associated to the positive operators A and B, we find again the result $AahB = AgB$ which was proved in [6] and [7].

1 Preliminaries Throughout this paper ; H denotes a real (or complex) Hilbert space equipped with its inner product $< ., . >$ and the associate hilbertian norm $||.||$.
Let $\overline{\mathbb{R}}^H$ be the space of mappings from H into $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ endowed with the order relation :
$\forall\, f, g \in \overline{\mathbb{R}}^H \;;\; f \leq g \Longleftrightarrow \forall x \in H \;\; f(x) \leq g(x)$
where we prolong the structure of $\mathbb{R}$ to $\overline{\mathbb{R}}$ by setting :
$\forall x \in \overline{\mathbb{R}} \quad -\infty \leq x \leq +\infty$ and $x + (+\infty) = +\infty$.
Let us denote by $\Gamma_o(H)$ the cone of all convex, lower semicontinuous functionals not identically equal to $+\infty$ and not taking the value $-\infty$.
Consider a functional $f : H \longrightarrow \widetilde{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$, we denote by dom f the effective domain of f defined by : dom $f = \{x \in H; f(x) < +\infty\}$.
If H is a real space, the Legendre-Fenchel conjugate of f, denoted by f^*, is given by the formulae :
$\forall\, x^* \in H \quad f^*(x^*) = \sup\{< x^*, x > -f(x); x \in H\}$.
It is well known that for all $f : H \longrightarrow \widetilde{\mathbb{R}}$ such that $f \neq +\infty$, one has $f^* \in \Gamma_o(H)$, $f^{**} = (f^*)^* \leq f$ and $f^{***} = f^*$, moreover if $f \in \Gamma_o(H)$ then $f^{**} = f$ (cf .[4] for example).
Let $f, g : H \longrightarrow \widetilde{\mathbb{R}}$. It is easy to see that if $f \leq g$ then $g^* \leq f^*$ and
$\forall\;\; \alpha \in]0, 1[\;\; (\alpha f + (1-\alpha)g)^* \leq \alpha f^* + (1-\alpha)g^* \qquad (1.1)$

Remark 1.1. If H is a complex Hilbert space, the Legendre-Fenchel conjugation can be extended by :

$$\forall\, x^* \in H \quad f^*(x^*) = \sup\{\mathrm{Re} < x^*, x > -f(x); x \in H\}.$$

For a fixed $x \in H$, the function $\phi_x : x^* \longrightarrow \mathrm{Re} < x^*, x > -f(x)$ is convex and so the function $f^* : x^* \rightarrow \sup\{\phi_x(x^*); x \in H\}$ is also one.
We also can verify that the above mentionned properties related to the classical Legendre-Fenchel conjugation are verified for this extended one.

Example 1.1. Let H be a real Hilbert space. Given a symmetric positive operator A (with closed range) from H to H, we define

$$f_A(x) = \frac{1}{2} < Ax, x > \text{ for all } x \in H,$$

it is well known that [3]

$$f_A^*(x^*) = \begin{cases} \frac{1}{2} < A^+x^*, x^* > & \text{if } x^* \in \text{ ran } A \\ \\ +\infty & \text{else} \end{cases}$$

where $A^+ := \lim_{\epsilon \to o^+} (A + \epsilon I)^{-1}$ for the sake of convenience.
In particular, if moreover A is invertible then $f_A^* = f_{A^{-1}}$.

Proposition 1.1. If H is a complex Hilbert space, the relation $f_A^* = f_{A^{-1}}$ holds for all positive invertible operator A from H to H.

Proof . For all x^*, x in H, we can write

$$\text{Re } < x^*, x > \leq | \text{ Re } < x^*, x > | = | \text{ Re } < A^{-\frac{1}{2}}x^*, A^{\frac{1}{2}}x > |$$

where $A^{\frac{1}{2}}$ denotes the positive square root of A, i.e. the only positive invertible operator S such that $S^2 = A$.
Using the Schwartz's inequality, it comes that

$$\text{Re } < x^*, x > \leq \| A^{-\frac{1}{2}}x^* \| \| A^{\frac{1}{2}}x \| \leq \frac{1}{2} \| A^{-\frac{1}{2}}x^* \|^2 + \frac{1}{2} \| A^{\frac{1}{2}}x \|^2$$

Since A is symmetric then $\| A^{\frac{1}{2}}x \|^2 = < Ax, x >$ and we obtain
$\sup\{ \text{ Re } < x^*, x > - \frac{1}{2} < Ax, x >; x \in H\} \leq \frac{1}{2} < A^{-1}x^*, x^* >$
Remarking that the second member of this latter inequality is attained at $x = A^{-1}x^*$, this concludes the proof . ■

Let $f, g : H \longrightarrow \mathbb{R} \cup \{+\infty\}$ be two given functionals in $\Gamma_o(H)$, assume that $\text{dom } f \cap \text{ dom } g \neq \emptyset$ and consider the sequence $(\varphi_n(f,g))_n$ defined by, [2]:

$$\begin{cases} \varphi_o(f,g) & = & \frac{1}{2}f + \frac{1}{2}g \\ \\ \varphi_{n+1}(f,g) & = & \frac{1}{2}\varphi_n(f,g) + \frac{1}{2}(\varphi_n^*(f,g)) \quad (n \geq 0) \end{cases}$$

where, by definition, we put $\varphi_n^*(f,g) = (\varphi_n(f^*, g^*))^*$ for every $n \geq 0$.

Theorem 1.1 and **Definition 1.1, [2]**. Let f, g in $\Gamma_o(H)$ such that $\text{dom } f \cap \text{ dom } g \neq \emptyset$. Then the sequence $(\varphi_n(f,g))_n$ converges pointwise to a convex functional $f\tau g$ satisfying that $(f^*\tau g^*)^* \leq f\tau g$. If moreover, the condition $\text{dom } \varphi_o(f,g) = \text{ dom } \varphi_o^*(f,g)$ holds then one has $f\tau g = (f^*\tau g^*)^*$ and $\text{dom } (f\tau g) = \text{ dom } f \cap \text{ dom } g$. The functional $f\tau g$ is called the convex geometric mean of f and g.

The assumption $\text{dom } \varphi_o(f,g) = \text{ dom } \varphi_o^*(f,g)$ is not necessary for the pointwise convergence of $(\varphi_n(f,g))_n$ to define $f\tau g$. The example below explains its necessity to have the relationship $f\tau g = (f^*\tau g^*)^*$.

Example 1.2. Let E and F be two closed subspaces of H. Take $f = \Psi_E$ and $g = \Psi_F$ where Ψ_E is the indicator function of E, i.e. $\Psi_E(x) = 0$ if $x \in E$ and $\Psi_E(x) = +\infty$ else.

Recall that [8], $f^* = \Psi_{E^\perp}$ and $g^* = \Psi_{F^\perp}$ where $E^\perp$ denotes the orthogonal complement of E defined by : $E^\perp = \{x^* \in H; < x^*, x >= 0 \text{ for all } x \in E\}$.
By induction on $n \in \mathbb{N}$, it is easy to see that $\varphi_n(f,g) = \Psi_{E\cap F}$ for all $n \in \mathbb{N}$ and so $(\varphi_n(f,g))_n$ converges pointwise to $\Psi_{E\cap F}$. It follows that $f\tau g = \Psi_{E\cap F}$ and $(f^*\tau g^*)^* = \Psi_{(E^\perp \cap F^\perp)^\perp}$, and consequently the relation $f\tau g = (f^*\tau g^*)^*$ is not needed, since in general $E \cap F \neq (E^\perp \cap F^\perp)^\perp$.

Remark 1.2. Let $f, g \in \Gamma_o(H)$ such that dom $f \cap$ dom $g \neq \emptyset$, we recall the following properties [2]:

i) $f \leq g \Longrightarrow f\tau h \leq g\tau h$ for all $h \in \Gamma_o(H)$
ii) $f\tau g = g\tau f$ and $f\tau f = f$
iii) $(\frac{1}{2}f^* + \frac{1}{2}g^*)^* \leq (f^*\tau g^*)^* \leq f\tau g \leq \frac{1}{2}f + \frac{1}{2}g$
iv) $f\tau g = (\frac{1}{2}f + \frac{1}{2}g)\tau(\frac{1}{2}f^* + \frac{1}{2}g^*)^*$

Remark 1.3. Let f, g satisfying that dom $f \cap$ dom $g \neq \emptyset$. The condition dom $\varphi_o(f,g) =$ dom $\varphi_o^*(f,g)$ which equivalent to dom $\varphi_o^*(f,g) =$ dom $f \cap$ dom g is verified when f and g are with finite values, i.e. $f, g : H \longrightarrow \mathbb{R}$.

Definition 1.2. For f, g such that dom $f \cap$ dom $g \neq \emptyset$, the functionals $\varphi_o(f,g) = \frac{1}{2}f + \frac{1}{2}g$ and $\varphi_o^*(f,g) = (\frac{1}{2}f^* + \frac{1}{2}g^*)^*$ are called, respectively, the arithmetic and harmonic means of f and g.

When f and g are two quadratical functionals, i.e. $\forall x \in H, f(x) = \frac{1}{2} < Ax, x >$ and $g(x) = \frac{1}{2} < Bx, x >$ with A and B are two symmetric invertible positive operators, then $\forall x \in H, \varphi_o(f,g)(x) = \frac{1}{2} < (AaB)x, x >$ and $\varphi_o^*(f,g)(x) = \frac{1}{2} < (AhB)x, x >$ where $AaB = \frac{1}{2}(A+B)$ and $AhB = (\frac{1}{2}A^{-1} + \frac{1}{2}B^{-1})^{-1}$ are, respectively, the arithmetic and harmonic means of A and B.

Corollary 1.1, [2]. Let $f = \frac{1}{2} < A.,. >$ and $g = \frac{1}{2} < B.,. >$, where A and B are two symmetric invertible positive linear operators from H to H. Then $f\tau g = \frac{1}{2} < (AgB).,. >$ where $AgB = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{1}{2}}A^{\frac{1}{2}}$ is the geometric operator mean of A and B.

2 Arithmetico-geometric functional mean

Let f, g in $\Gamma_o(H)$ such that dom $f \cap$ dom $g \neq \emptyset$ and define the following algorithm

$$\left\{\begin{array}{lcl} f_o & = & f \\ & & \\ g_o & = & g \end{array}\right. \quad \text{and} \quad \left\{\begin{array}{lcl} f_{n+1} & = & \frac{1}{2}(f_n + g_n) \\ & & \\ g_{n+1} & = & f_n\tau g_n \end{array}\right. \quad (n \geq 0).$$

By induction, it is easy to verify that , $f_n \in \Gamma_o(H)$ and $g_n \in \Gamma_o(H)$, for every $n \geq 0$.

Theorem 2.1 and **Definition 2.1**. Let $f, g \in \Gamma_o(H)$ such that dom $f \cap$ dom $g \neq \emptyset$, suppose that dom $\varphi_o(f,g) =$ dom $\varphi_o^*(f,g)$. Then the sequences (f_n) and (g_n) both converge pointwise to the same convex functional $f \overset{+}{\tau} g$, called the arithmetico-geometric mean of f and g, and satisfying that $(\frac{1}{2}f^* + \frac{1}{2}g^*)^* \leq f\tau g \leq f \overset{+}{\tau} g \leq \frac{1}{2}f + \frac{1}{2}g$.

Proof. Using Theorem 1.1 and Remark 1.2, iii), we have
$g_{n+1} = f_n\tau g_n \leq \frac{1}{2}f_n + \frac{1}{2}g_n = f_{n+1}$ for all $n \geq 0$
from which we deduce $f_{n+1} \leq f_n$ for every $n \geq 1$.
Knowing that $g_{n+1} = f_n\tau g_n$ and $f_n \geq g_n$ for each $n \geq 1$, we obtain by Remark 1.2, i) and ii) : $g_{n+1} \geq g_n\tau g_n = g_n$.
In summary, we have proved :

$$f\tau g = g_1 \leq ... \leq g_n \leq g_{n+1} \leq f_{n+1} \leq f_n \leq ... \leq f_1 = \frac{1}{2}(f+g) \qquad (2.1)$$

We conclude that (f_n) (resp.(g_n)) is decreasing and lower bounded by g_1 (resp. increasing upper bounded by f_1), then (f_n) and (g_n) both converge pointwise in $\widetilde{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$. Put that $\varphi = \lim f_n$ and $\psi = \lim g_n$, using dom $(f\tau g) =$ dom $(\frac{1}{2}f + \frac{1}{2}g)$; we deduce from (2.1) that dom $\varphi =$ dom ψ.
Letting $n \to +\infty$ in relation $f_{n+1} = \frac{1}{2}f_n + \frac{1}{2}g_n$ to obtain $\varphi = \frac{1}{2}\varphi + \frac{1}{2}\psi$ which, combined with dom $\varphi =$ dom ψ, yields that $\varphi = \psi := f \overset{+}{\tau} g$. ■

Corollary 2.1. Let A and B be two symmetric invertible positive operators from H to H, we define the sequences:

$$\begin{cases} A_o = A \\ \\ B_o = B \end{cases} \quad \text{and} \quad \begin{cases} A_{n+1} = \frac{1}{2}(A_n + B_n) \\ \\ B_{n+1} = A_n g B_n \end{cases} \quad (n \geq 0).$$

Then (A_n) and (B_n) both converge, in the strong operator convergence, to the arithmetico-geometric operator mean $AagB$ of A and B.

Proof . Let us take $f(x) = \frac{1}{2} < Ax, x >$ and $g(x) = \frac{1}{2} < Bx, x >$ for $x \in H$. By induction on $n \in \mathbb{N}$, we prove easily that : for all $x \in H, f_n(x) = \frac{1}{2} < A_n x, x >$ and $g_n(x) = \frac{1}{2} < B_n x, x >$ where f_n and g_n are defined in Theorem 2.1 and A_n, B_n are in Corollary 2.1.
By virtue of Theorem 2.1, for all $x \in H$ the sequences $(< A_n x, x >)_n$ and $(< B_n x, x >)_n$ converge to the same limit which is clearly a quadratical functional $< (AagB)x, x >$, that is to say (A_n) and (B_n) converge strongly by boundedness of sequences of positive operators.
■

3 Arithmetico-harmonic functional mean

For f, g in $\Gamma_o(H)$ satisfying dom $f \cap$ dom $g \neq \emptyset$, we consider the sequences (f_n) and (g_n) defined as follows :

$$\begin{cases} f_o = f \\ \\ g_o = g \end{cases} \quad \text{and} \quad \begin{cases} f_{n+1} = \frac{1}{2}(f_n + g_n) \\ \\ g_{n+1} = (\frac{1}{2}f_n^* + \frac{1}{2}g_n^*)^* \end{cases} \quad (n \geq 0).$$

Clearly $f_n \in \Gamma_o(H)$ and $g_n \in \Gamma_o(H)$ for all $n \geq 0$.

Theorem 3.1 and **Definition 3.1**. Let f and g in $\Gamma_o(H)$ satisfying that dom $f\cap$ dom $g \neq \emptyset$ and dom $\varphi_o(f, g) =$ dom $\varphi_o^*(f, g)$. Then (f_n) and (g_n) converge pointwise to the same convex functional denoted $f \overset{+}{\nabla} g$. $f \overset{+}{\nabla} g$ is said the arithmetico-harmonic functional mean of f and g and furthermore $f \overset{+}{\nabla} g = f\tau g$.

Proof. Take $\alpha = \frac{1}{2}$ in inequality (1.1) to write
$\forall n \geq 0 \quad g_{n+1} = (\dfrac{1}{2}f_n^* + \dfrac{1}{2}g_n^*)^* \leq \dfrac{1}{2}f_n + \dfrac{1}{2}g_n = f_{n+1}.$
We deduce $f_{n+1} \leq f_n$ and $g_{n+1} \geq g_n$ for each $n \geq 1$.
Consequently, one has shown
$(\dfrac{1}{2}f^* + \dfrac{1}{2}g^*)^* = g_1 \leq ... \leq g_n \leq g_{n+1} \leq f_{n+1} \leq f_n \leq ... \leq f_1 = \dfrac{1}{2}(f+g).$
Similarly to the proof of Theorem 2.1, we conclude that the sequences (f_n) and (g_n) both converge pointwise to the same convex functional $f \overset{+}{\nabla} g$.
In another hand, by virtue of Remark 1.2, iv), we have
$\forall n \geq 0 \;\; f_{n+1}\tau g_{n+1} = (\dfrac{1}{2}f_n + \dfrac{1}{2}g_n)\tau(\dfrac{1}{2}f_n^* + \dfrac{1}{2}g_n^*)^* = f_n\tau g_n$
so $\forall n \geq 0 \;\; f_n\tau g_n = f\tau g$, and due to Remark 1.2, iii), one has
$g_{n+1} = (\dfrac{1}{2}f_n^* + \dfrac{1}{2}g_n^*)^* \leq f_n\tau g_n = f\tau g \leq \dfrac{1}{2}(f_n + g_n) = f_{n+1},$
from which we deduce $f \overset{+}{\nabla} g = f\tau g$. ∎

Corollary 3.1 . Let us define the algorithm

$$\begin{cases} A_o &= A \\ B_o &= B \end{cases} \quad \text{and} \quad \begin{cases} A_{n+1} &= \frac{1}{2}(A_n + B_n) \\ B_{n+1} &= (\frac{1}{2}A_n^{-1} + \frac{1}{2}B_n^{-1})^{-1} \end{cases}$$

where A and B are two symmetric invertible positive operators of H. Then (A_n) and (B_n) converge strongly to the same operator $AahB$ known by the arithmetico-harmonic mean of A and B and furthermore $AahB = AgB$.

Proof . Similar to the demonstration of Corollary 2.1. ∎

4 Geometrico-harmonic functional mean Given f, g in $\Gamma_o(H)$ such that dom $f \cap$ dom $g \neq \emptyset$, let us define the following sequences :

$$\begin{cases} f_o &= f \\ g_o &= g \end{cases} \quad \text{and} \quad \begin{cases} f_{n+1} &= f_n\tau g_n \\ g_{n+1} &= (\frac{1}{2}f_n^* + \frac{1}{2}g_n^*)^* \end{cases} \quad (n \geq 0).$$

Let us note that $f_n \in \Gamma_o(H)$ and $g_n \in \Gamma_o(H)$ for each $n \geq 0$.

Theorem 4.1 and **Definition 4.1.** Letting f, g in $\Gamma_o(H)$ satisfying dom $f \cap$ dom $g \neq \emptyset$ and dom $\varphi_o(f,g) =$ dom $\varphi_o^*(f,g)$. Then the both sequences (f_n) and (g_n) converge pointwise to the same convex functional $f \overset{\tau}{\nabla} g$ called the geometrico-harmonic mean of f and g, and there hold :

$$(\frac{1}{2}f^* + \frac{1}{2}g^*)^* \leq f \overset{\tau}{\nabla} g \leq f\tau g \leq f \overset{+}{\tau} g \leq \frac{1}{2}f + \frac{1}{2}g.$$

Proof. From Remark 1.2, iii), we have $g_{n+1} \leq f_{n+1}$ for all $n \geq 0$ from which we deduce with the definition of f_n and g_n : $f_{n+1} \leq f_n$ and $g_{n+1} \geq g_n$ for each $n \geq 1$. It follows that :

$$(\frac{1}{2}f^* + \frac{1}{2}g^*)^* = g_1 \leq ... \leq g_n \leq g_{n+1} \leq f_{n+1} \leq f_n \leq ... \leq f_1 = (f^* \tau g^*)^* = f \tau g$$

We conclude that (f_n) and (g_n) converge pointwise. Call their limits $\varphi = \lim f_n$ and $\psi = \lim g_n$, clearly we have $\psi \leq \varphi$; and since $\operatorname{dom}(f\tau g) = \operatorname{dom}(\frac{1}{2}f^* + \frac{1}{2}g^*)^*$ thus $\operatorname{dom}\varphi = \operatorname{dom}\psi$. Thanks to Remark 1.2, i) and iii), we obtain : $f_{n+1} = f_n \tau g_n \leq f_n \tau \psi \leq \frac{1}{2}f_n + \frac{1}{2}\psi$
and we deduce $\varphi \leq \frac{1}{2}\varphi + \frac{1}{2}\psi$ which, with $\operatorname{dom}\varphi = \operatorname{dom}\psi$, implies that $\varphi \leq \psi$. This concludes the proof. ■

Similarly to corollaries 2.1 and 3.1, we have the :

Corollary 4.1. Let us consider the sequences $(A_n), (B_n)$:

$$\left\{ \begin{array}{lcl} A_o & = & A \\ & & \\ B_o & = & B \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{lcl} A_{n+1} & = & A_n g B_n \\ & & \\ B_{n+1} & = & (\frac{1}{2}A_n^{-1} + \frac{1}{2}B_n^{-1})^{-1} \end{array} \right. \quad (n \geq 0)$$

where A and B are two symmetric invertible positive operators from H to H. Then (A_n) and (B_n) converge strongly to the same operator $AghB$ geometrico-harmonic mean of A and B.

As in the first section , the assumption $\operatorname{dom}\varphi_0(f,g) = \operatorname{dom}\varphi_0^*(f,g)$ is not necessary for the pointwise convergence of (f_n) and (g_n) defined in Theorems 2.1, 3.1 and 4.1, but its necessity is to prove that $\lim_n f_n = \lim_n g_n$. Example 1.2 explains this situation : in fact, for the arithmetico-harmonic functional mean, for example, we verify that $f_n = \Psi_{E\cap F}$ and $g_n = \Psi_{(E^\perp \cap F^\perp)^\perp}$ for all $n \in \mathbb{N}$, the sequences (f_n) and (g_n) converge pointwise but not to the same limit.

Remark 4.1. The invertiblity for A or B is not necessary in the above results by the following convention

$$\left(\frac{A^{-1} + B^{-1}}{2}\right)^{-1} = \lim_{\epsilon \downarrow 0} \left(\frac{(A+\epsilon)^{-1} + (B+\epsilon)^{-1}}{2}\right)^{-1},$$

see [1].

Remark 4.2. For two symmetric positive operators A and B, let us take :
$\forall x \in H \quad f_A(x) = \frac{1}{2} < Ax, x >$ and $f_B(x) = \frac{1}{2} < Bx, x >$.
We can summarize the corollaries 1.1, 1.2, 1.3 and 1.4 by :

$$f_A \tau f_B = f_{AgB}, f_A \overset{+}{\tau} f_B = f_{AagB}, f_A \overset{+}{\nabla} f_B = f_{AahB} = f_{AgB}, f_A \overset{\tau}{\nabla} f_B = f_{AghB}.$$

It is clear that $f_A \leq f_B \iff A \leq B$ (where $A \leq B \iff B - A$ is positive).
And combining the above results, one has

$$(\frac{1}{2}A^{-1} + \frac{1}{2}B^{-1})^{-1} = AhB \leq AghB \leq AgB = AahB \leq AagB \leq AaB = \frac{1}{2}(A+B).$$

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