ON THE SPAN OF STARLIKE CURVES

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Abstract. We prove that the dual monotone span of a starlike curve X is not smaller than the infimum, $\varepsilon(X)$, of the set of positive numbers m such that a chain with mesh m covers X.

1 Introduction. We review the definitions introduced by *A*. Lelek in [2] and [3]. Let X be a nonempty connected metric space. The span $\sigma(X)$ of X is the least upper bound of the set of real numbers r, $r \geq 0$, that satisfy the following condition.

There exists a connected space Y and a pair of continuous functions $f, g : Y \to X$ such that

$$f(Y) = g(Y) \tag{1}$$

and $\mathrm{dist}[f(y), g(y)] \geq r$ for every $y \in Y$.

Relaxing the requirement posed by equality (1) to the inclusion $f(Y) \subseteq g(Y)$ produces the definition of the semispan $\sigma_0(X)$ of X. Requiring that g be onto gives the definitions of the surjective span $\sigma^*(X)$ and the surjective semispan $\sigma_0^*(X)$.

It was pointed out in [3] that

$$0 \leq \sigma(X) \leq \sigma_0(X) \leq \mathrm{diam}(X).$$

It follows from a more general result of Lelek [3, Th.2.1, p39] that when X is a continuum then $\sigma_0 \leq \varepsilon(X)$. A different, direct, proof can be found in [1].

In this paper we concentrate on the case when X is a simple closed curve in the plane. Notice that in this case $\sigma^*(X) = \sigma(X)$ and $\sigma_0^*(X) = \sigma_0(X)$. Next, we review the definitions introduced in [1], starting with the monotone span $\sigma_m(X)$ of X.

Definition 1. If X is a simple closed curve then

$$\sigma_m(X) = \sup_{f,g} \inf_{t \in [0,1]} \|f(t) - g(t)\|,$$

where $f, g : [0, 1] \to X$ are continuous on [0, 1], monotone on [0, 1), and $f([0, 1]) = X = g([0, 1])$.

Next we define the dual monotone span $\overline{\sigma}_m(X)$ of X.

Definition 2. If X is a simple closed curve then

$$\overline{\sigma}_m(X) = \inf_{h,k} \sup_{t \in [0,1]} \|h(t) - k(t)\|,$$

where $h, k : [0, 1] \to X$ are continuous on $[0, 1]$, monotone on $[0, 1)$, $h([0, 1]) = X = k([0, 1])$, $h(0) = k(0)$, there exists a point $t' \in (0, 1)$ such that $h([0, t']) \cap k([0, t']) = \{h(0)\}$ and neither $h([0, t'])$ nor $k([0, t'])$ is a singleton.

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Finally, we define the dual effectively monotone span $\overline{\sigma}_{em}(X)$.

Definition 3. If X is a simple closed curve then

$$\overline{\sigma}_{em}(X) = \inf_{h,k} \sup_{t\in[0,1]} \|h(t) - k(t)\|,$$

where $h, k : [0,1] \to X$ are continuous, $h([0,1]) = X = k([0,1])$, $h(0) = k(0)$, there exists a point $t_o \in (0,1)$ such that $h(t_o) = k(t_o) \neq h(0)$ and $h([0,t_o]) \cap k([0,t_o]) = \{h(0), h(t_o)\}$.

It was proven in [1] that $\overline{\sigma}_{em}(X) \leq \varepsilon(X)$.

The span of X is equal to $\varepsilon(X)$ when X is the boundary of a convex region (see [4]).

2 Dual monotone span and starlike curves. Recall that a starlike curve is a simple closed curve whose every point can be seen from a fixed point in the bounded component of its complement. Thus, X is starlike if there is a point 0 in the bounded component D of $\mathbb{C}\backslash X$ such that $D \supset 0P\backslash\{P\}$ for each point $P \in X$.

For a pair of two distinct points $A, B \in X$ we denote the counterclockwise are on X, with the endpoints A and B, by $AB^{\sim}$.

We shall prove that $\varepsilon(X) \leq \overline{\sigma}_m(X)$ for any starlike curve X. First, we need another definition and several auxiliary lemmas.

Definition 4. Let X be a starlike polygon, let $h, k : [0,1] \to X$ be the functions described in Definition 3, and let $B = h(0) = k(0)$, $E = h(t_o) = k(t_o)$, where $t_o \in (0,1)$ meats the conditions of Definition 3 as well. Let $BE^{\sim} = h([0,t_o])$, $EB^{\sim} = k([0,t_o])$. For each point $P \in BE^{\sim}(EB^{\sim})$, let t_p be the number in $(0,t_o)$ such that $h(t_p) = P(k(t_p) = P)$ and $\forall t \in (t_p, t_o)$ $h(t) \neq P(k(t) \neq P)$. For any pair of points $P, Q \in X\backslash\{B,E\}$ we say that P precedes Q (written $P \ll Q$) if and only if $t_p \leq t_Q$. For any $P \in X\backslash\{B,E\}$, $B \ll P \ll E$.

Lemma A. *Let X be a polygon, let h and k be as described in Definition 2, and let B, E and t_o be as described in Definitions 3 and 4. There exists a sequence $\{W_jW_j{}'\}_{j=1}^{N}$ of line segments with the following properties:*

1(A) $\forall 1 \leq j \leq N_0 \quad W_j \in BE^{\sim},\ W_j{}' \in EB^{\sim}$ and
$\exists t \in (0,t_o) \ni W_j = h(t),\ W_j{}' = k(t)$

2(A) $\forall 1 \leq j < N_0 \quad W_j \ll W_{j+1},\ W_j{}' \ll W_{j+1}{}'$

3(A) If $V \in X\backslash\{B,E\}$ is a vertex then $\exists j, 1 \leq j \leq N_o \ni V = W_j$ or $V = W_j{}'$.

Proof. Let $\{V_m\}_{m=1}^{M}$ be the increasing sequence in the order $\ll$ of all vertices on $X\backslash\{B,E\}$. Let $\{t_m\}_{m=1}^{M}$ be the corresponding non-decreasing sequence of numbers in $(0,t_o)$ such that $\forall 1 \leq m \leq M$ $t_m = t_{V_m}$ (see Definition 4). We use induction to define $\{W_j, W_j{}'\}_{j=1}^{N_0}$ as follows. Put $W_1 = h(_1)$, $W_1{}' = k(t_1)$. Soppose $W_{j-1}, W_{j-1}{}'$ are defined and suppose $W_{j-1} = h(t_{m-1})$ or $W_{j-1} = k(t_{m-1})$ for some m, $1 < m < M$. If $V_m = h(t_{m-1})$ or $V_m = k(t_{m-1})$ then put $W_j = h(t_{m+1})$ and $W_j{}' = k(t_{m+1})$; otherwise put $W_j = h(t_m)$ and $W_j{}' = k(t_m)$. □

Lemma B. If $QPCB$ is a convex quadrilateral with diagonals QC and PB then $\|BQ\| + \|PC\| < \|QC\| + \|PB\|$.

Proof follows from the law of cosines or the triangle inequality.

Lemma C. Suppose P is a simple closed polygonal line with at least four vertices, and let A, B, C and D be vertices of P such that $P \supset AB$, $P \supset CD$, $AB \cap CD = \emptyset$. Let $AC^{\sim}$, $DB^{\sim}$ be polygonal lines such that $AB \cup AC^{\sim} \cup CD \cup DB^{\sim} = P$, $AC^{\sim} \cap DB^{\sim} = \emptyset$, and let $d > \delta > 0$. If $\operatorname{diam} AC^{\sim} \leq d$ and $\operatorname{diam} DB^{\sim} \leq d$ then there exists a sequence $\{A_k C_k\}_{k=0}^{K}$ of pairwise disjoint polygonal lines such that

1(c) $\forall k = 0, \dots, K \quad A_k \in AB,\ C_k \in CD,$
$A_o = A,\ C_o = C,\ A_K = B,\ C_K = D$

2(c) $\forall 0 \leq k < K \quad \operatorname{diam} A_k C_k^{\sim} \leq d$

3(c) $\forall 0 < k < K$ $\operatorname{diam} P_k \leq d + \delta$, where P_k is the polygon whose boundary consists of $A_{k-1}C_{k-1}{}^{\sim}$, $C_{k-1}C_k$, $C_k A_k{}^{\sim}$ and $A_k A_{k-1}{}^{\sim}$, $k = 1, \dots, K$.

Theorem 1. *If X is a starlike curve then $\varepsilon(X) \leq \overline{\sigma}_m(X)$.*

Proof. Suppose X is a starlike polygon, B and E are two distinct points on X. Let $h, k : [0,1] \to X$ be continuous on $[0,1]$, monotone on $[0,1)$, $h(0) = k(0) = B$, $h(t_o) = k(t_o) = E$ for some $t_o \in (0,1)$, $h([0,t_o]) = BE^{\sim}$, $k([0,t_o]) = EB^{\sim}$. Suppose $\{W_j W_j{}'\}_{j=0}^{N_o}$ is a sequence of line segments described in Lemma A. Let $\delta > 0$. We shall define a chain of closed polygons $\{R_j\}_{j=1}^{N}$ that satisfy the following conditions:

1(1) $\displaystyle\bigcup_{j=1}^{N} R_j \supset X$

2(1) $\forall 1 \leq j < N \quad R_j \cap R_{j+1} \subseteq \partial R_j \cap \partial R_{j+1} \neq \emptyset$
$\forall j, k \quad |k - j| > 1 \Rightarrow R_j \cap R_k = \emptyset$

3(1) $\forall 1 \leq j \leq N$ $\operatorname{diam} R_j \leq \overline{\sigma}_m(X) + \delta.$

Without loss of generality, we can assume that B is a vertex. It follows from Lemma A that $W_1 \neq B$, $W_1{}' \neq B$, $W_1 \in BE^{\sim}$, $W_1{}' \in EB^{\sim}$. We choose two points W_0 and $W_o{}'$ so that $B \in W_o W_o{}'$, $W_o W_o{}' \cap W_1 W_1{}' = \emptyset$ and $\|W_o W_o{}'\| \leq \overline{\sigma}_m(X)$. Without loss of generality, we shall assume that $\forall 1 < j \leq N_o$ $\|W_{j-1}{}' W_j{}'\| \leq \overline{\sigma}_m(X)$, $\|W_{j-1} W_j\| \leq \overline{\sigma}_m(X)$. To make the notation shorter, we give the working name Q_j to the polygon with vertices $W_{j-1}, W_j, W_j{}', W_{j-1}{}'$, $j = 1, \dots, N_o$, after eliminating $W_{j-1}W_{j-1}{}'$ from the sequence whenever $W_j W_j{}' \supset W_{j-1} W_{j-1}{}'$. These polygons will be modified in the course of our construction.

Let n be the largest natural number such that $Q_n \supset \displaystyle\bigcup_{j=1}^{n} Q_j$.

If $n > 1$ then we put $Q_1 = Q_n$, relabel the remaining line segments in $\{W_j W_j{}'\}$ in the consecutive manner as $W_2 W_2{}', W_3 W_3{}', \dots$, and let Q_j be the polygon with the new vertices $W_{j-1}, W_j, W_j{}', W_{j-1}{}'$ for each j. Suppose the line segments $W_j W_j{}'$, $j = 1, \dots, N_o$ are not pairwise disjoint and let m be the natural number such that

4(1) $\displaystyle W_m W_m{}' \cap \bigcup_{j=1}^{m-1} Q_j \neq \emptyset$ and $\displaystyle\forall j = 2, \dots, m-1 \ \ W_j W_j{}' \cap \bigcup_{i=1}^{j-1} Q_i = \emptyset.$

Consider the following cases:

I. $W_m, W_m{}' \in \bigcup_{j=1}^{m-1} Q_j$

II. $W_m, W_m{}' \notin \bigcup_{j=1}^{m-1} Q_j$

IIII. $W_m \in \bigcup_{j=1}^{m-1} Q_j$ and $W_m{}' \notin \bigcup_{j=1}^{m-1} Q_j$.

Note that the argument in the case when $W_m{}' \in \bigcup_{j=1}^{m-1} Q_j$ and $W_m \notin \bigcup_{j=1}^{m-1} Q_j$ would be symmentric to the one we will offer in III.

In case I, let m_o be the largest number, $m_o \geq m$, such that $\forall j = m, \ldots, m_o \;\; W_jW_j{}' \in \bigcup_{i=1}^{m-1} Q_i$. Relabel the line segments $W_jW_j{}'$ for $j > m_o$ as follows. Put $W_m = W_{m_o+1}$, $W_m{}' = W_{m_o+1}{}', \ldots, W_{m+i} = W_{m_o+i+1}$, $W_{m+i}{}' = W_{m_o+i+1}, \ldots$, and consider case II or III if necessary.

Case II calls for the following distinction:

IIa $W_mW_m{}' \cap \bigcup_{j=1}^{m-1} W_jW_j{}' = \emptyset$

IIb $W_mW_m{}' \cap \bigcup_{j=1}^{m-1} W_jW_j{}' \neq \emptyset$

In case IIa, there is exactly one t, $1 \leq t \leq m-1$, such that $Q_t \cap W_mW_m{}' = \emptyset$. Let $m = t$, and let Q_m be the polygon with vertices $W_{t-1}, W_m, W_m{}', W_{t-1}{}'$.

In case IIb, the procedure is as follows. Let i be the smallest number such that $W_iW_i{}' \cap W_mW_m{}' \neq \emptyset$. If $i < m-1$ then eliminate $W_jW_j{}'$ for all $j, i < j < m$, for which $W_jW_j{}' \cap W_mW_m{}' = \emptyset$ so that it can be assumed, without loss of generality, that either $W_iW_i{}' \cap W_jW_m{}' \neq \emptyset$ or $W_iW_i{}' \cap W_j{}'W_m \neq \emptyset$ for $j = i+1, \ldots, m-1$. Recall that, by Lemma A, $\|W_iW_j{}'\| = \|h(t_j) - k(t_j)\| \leq \overline{\sigma}_m(X)$ for all j.

Suppose first that $\|W_{m-1}W_m{}'\| \leq \overline{\sigma}_m(X)$. Let n be the largest number such that $n < i$ and $W_nW_n{}' \cap W_{n+1}W_m{}' = \emptyset$.

If $\forall j, n < j \leq m$, $\|W_jW_m{}'\| \leq \overline{\sigma}_m(X)$ then define the new Q_{n+1} to be the polygon with vertices $W_n, W_{n+1}, W_m{}'$ and $W_n{}'$, the new Q_{n+2} to be the triangle $W_{n+1}W_{n+2}W_m{}', \ldots$, the new Q_m to be the triangle $W_{m-1}W_mW_m{}'$. Otherwise, let j be the largest number, $n < j < m-1$, such that $\|W_jW_m{}'\| > \overline{\sigma}_m(X)$. Then, by Lemma B, $\|W_{j+1}W_j{}'\| \leq \overline{\sigma}_m(X).,\ldots, \|W_mW_j{}'\| \leq \overline{\sigma}_m(X)$, and we define the new Q_{j+1} to be the triangle $W_jW_{j+1}W_j{}'$, the new Q_{j+2} to be the triangle $W_{j+1}W_{j+2}W_j{}', \ldots$, the new Q_m to be the triangle $W_{m-1}W_mW_j{}'$.

Suppose now that $\|W_{m-1}W_m{}'\| > \overline{\sigma}_m(X)$. Then, by Lemma B, $\|W_mW_{m-1}{}'\| \leq \overline{\sigma}_m(X)$. If $\operatorname{int}(W_mW_{m-1}{}') \cap BE^{\sim} = \emptyset$ then we define our new Q_m to be the triangle $W_{m-1}W_mW_{m-1}{}'$. Otherwise, we define a new W_{m-1} by chooseing a point on $BE^{\sim}$, arbitrarily chose to $\operatorname{int}(W_mW_{m-1}{}') \cap BE^{\sim}$ and preceding it, and if necessary, redefine each W_j that succeeds it in the order $\ll$ on X, arbitrarily close to, and preceding, the new W_{m-1}. Here, Q_m is the quadrilateral $W_{m-1}'W_{m-1}W_mW_m'$.

We now turn to case III. First, we claim that $W_mW_m' \cap W_{m-1}W_{m-1}' \neq \emptyset$. This is clearly true when $W_m \in W_{m-1}W_{m-1}'$. Suppose then that $W_m \notin W_{m-1}W_{m-1}'$ and let D be the bounded component of the complement of X and let $0 \in D$ be the point with respect to which X is starlike. Let L be the line passing thorough W_m and W_{m-1}' and let V_1 and V_2 be the open half planes such that $V_1 \cup L \cup V_2 = \mathbb{C}$, $W_{m-1} \in V_2, V_1 \cap V_2 = \emptyset$. Since $D \supset 0W_{m-1} \setminus \{W_{m-1}\}$ and $D \supset 0W_{m-1}' \setminus \{W_{m-1}'\}$, we have $0 \in V_1$, and hence $W_m' \in V_2$. It follows from the latter that $W_mW_m' \cap W_{m-1}W_{m-1}' \neq \emptyset$. If $W_m \in W_{m-1}W_{m-1}'$ we define the new Q_m to be the triangle $W_mW_m'W_{m-1}'$.

If $\|W_{m-1}W_m'\| \leq \overline{\sigma}_m(X)$ then define the new Q_m to be the triangle $W_{m-1}W_m'W_{m-1}'$.

Suppose now that $\|W_{m-1}W_m'\| > \overline{\sigma}_m(X)$. Assume , without loss of generality, that $W_m \cap \bigcup_{j=1}^{m-1} W_jW_j' = \emptyset$. If the angle at W_m in the triangle $W_m'W_mW_{m-1}$ is smaller than $\pi/2$ then let C be the perpendicular projection of W_m' onto W_mW_{m-1}. Otherwise, put $C = W_m$. Note that $\|W_m'C\| \leq \overline{\sigma}_m(X)$. Also, $\|W_{m-1}'C\| \leq \overline{\sigma}_m(X)$ since the angle at C in the triangle $W_{m-1}'CW_{m-1}$ exceeds $\pi/2$ and $\|W_{m-1}W_{m-1}'\| \leq \overline{\sigma}_m(X)$. Let i be the largest number, $i < m-1$, such that $W_iW_i' \cap CW_m' = \emptyset$. Choose a sequence of points $\{C_j\}_{j=i+1}^{m-2}$ such that the polygonal lines connecting W_j, C_j and W_j', $j = i+1, \ldots, m-2$, are pairwise disjoint and do not intersect the polygonal line $W_{m-1}'CW_{m-1}$, $C_j \neq C$, $\|C_jC\| < \delta$, $j = i+1, \ldots, m-2$. Notice now that since for each $j = i+1, \ldots, m-2$ the angle at C_j in the triangle $W_j'C_jW_j$ exceeds $\pi/2$ and $\|W_jW_j'\| \leq \overline{\sigma}_m(X)$ we have $\|C_jW_j'\| < \overline{\sigma}_m(X)$ and $\|C_jW_j\| < \overline{\sigma}_m(X)$. We define the new Q_{i+1} to be the ploygon with vertices W_i, W_{i+1}, C_{i+1}, W_{i+1}' and W_i', the new Q_j to be the polygon with vertices W_{j-1}, W_j, C_j, W_j', W_{j-1}' and C_{j-1} for $j = i+2, \ldots, m-2$, the new Q_{m-1} to be the polygon with vertices W_{m-2}, W_{m-1}, C, W_{m-1}', W_{m-2}' and C_{m-2}, and the new Q_m to be the triangle $W_{m-1}'CW_m'$.

This concludes the description of the procedures applied in cases I, II and III.

We now put $I_j = Q_j \cap Q_{j+1}$, and let W_j and W_j' be the endpoints of I_j lying on $BE^{\sim}$ and $EB^{\sim}$, respectively, $1 \leq j < m$, except for I_{m-1} in case III when $\|W_{m-1}W_m'\| > \overline{\sigma}_m(X)$. We define I_m to be the line segment W_mW_m' in cases I and IIa. We define I_m to be the polygonal line connecting W_m, W_{m-1}' and W_m' in case IIb when $\|W_mW_{m-1}'\| \leq \overline{\sigma}_m(X) < \|W_{m-1}W_m'\|$ and $\mathrm{int}(W_mW_{m-1}') \cap X = \emptyset$. If $\mathrm{int}(W_mW_{m-1}') \cap X \neq \emptyset$ while $\|W_mW_{m-1}'\| \leq \overline{\sigma}_m(X) < \|W_{m-1}W_m'\|$ then $I_m = W_mW_m'$.

In the case when $\|W_{m-1}W_m'\| \leq \overline{\sigma}_m(X)$ in IIb, we put $I_m = W_mW_m'$ provided that for each j $\|W_{m-1}W_m'\| \leq \overline{\sigma}_m(X)$, $n < j \leq m$, and define I_m to be the polygonal line connecting W_m, W_j' and W_m' otherwise, where j is the largest number, $n < j < m-1$, such that $\|W_jW_m'\| > \overline{\sigma}_m(X)$.

Finally, in case III we define I_m to be the line segment $W_{m-1}W_m'$ when $\|W_{m-1}W_m'\| \leq \overline{\sigma}_m(X)$, and to be the line segment CW_m' when $\|W_{m-1}W_m'\| > \overline{\sigma}_m(X)$. In the latter case I_{m-1} is defined to be the polygonal line connecting W_{m-1}, C and W_{m-1}'.

Suppose that $\{n : n > m,\ W_nW_n' \cap \bigcup_{j=1}^{n-1} Q_j \neq \emptyset\} \neq \emptyset$.

Let m_1, $m_1 > m$, be the smallest number such that

5(1) $W_m W_m{}' \cap \bigcup_{j=1}^{m_1-1} Q_j \neq \emptyset$

We consider cases I, II, III, where m is replaced by m_1, with the following modification of case II.

IIa $W_{m_1} W_{m_1}{}' \cap \left[\bigcup_{j=1}^{m} I_j \cup \bigcup_{j=m+1}^{m_1-1} W_j W_j{}' \right] = \emptyset$

The procedure in this case dose not change.

IIb$_1$ $W_{m_1} W_{m_1}{}' \cap I_m = \emptyset$ and

$$W_{m_1} W_{m_1}{}' \cap \left[\bigcup_{j=1}^{m-1} I_j \cup \bigcup_{j=m+1}^{m_1-1} W_j W_j{}' \right] \neq \emptyset$$

The procedure we follow in this case was described previously for case IIb.

IIb$_2$ $W_{m_1} W_{m_1}{}' \cap I_m \neq \emptyset$ and $W_{m_1} W_{m_1}{}' \cap W_m W_m{}' = \emptyset$

IIb$_3$ $W_{m_1} W_{m_1}{}' \cap W_m W_m{}' \neq \emptyset$ and $W_{m_1} W_{m_1}{}' \cap W_{m-1} W_{m-1}{}' \neq \emptyset$

IIb$_4$ $W_{m_1} W_{m_1}{}' \cap W_m W_m{}' \neq \emptyset$ and $W_{m_1} W_{m_1}{}' \cap W_{m-1} W_{m-1}{}' = \emptyset$

Clearly, IIb$_2$ can only happen if I_m is the polygonal line $W_m W_{m-1}{}' W_m{}'$ (resulting from the case IIb for m). Recall that then $\|W_m W_{m-1}{}'\| \leq \overline{\sigma}_m(X) < \|W_{m-1} W_m{}'\|$ and $\mathrm{int}(W_m W_{m-1}{}') \cap BE^{\sim} = \emptyset$. Hence, the angle at $W_m{}'$ in the quadrilateral $W_{m-1} W_m W_{m-1}{}' W_m{}'$ must be smaller than $\pi/2$, for otherwise $W_{m-1} W_{m-1}{}'$ would constitute the longest side of the triangle $W_{m-1} W_{m-1}{}' W_m{}'$ and consequently, $\|W_{m-1} W_m{}'\| < \|W_{m-1} W_{m-1}{}'\| \leq \overline{\sigma}_m(X)$. Similarly, the angle at W_{m-1} in the same quadrilateral must be smaller that $\pi/2$ for otherwise $\|W_{m-1} W_m{}'\| < \|W_m W_m{}'\| \leq \overline{\sigma}_m(X)$. In the even when the angle at $W_{m-1}{}'$ is also smaller than $\pi/2$, we let W be the orthogonal projection of W_{m-1} onto $W_m{}' W_{m-1}{}'$ and let V be the intersection point of $W_{m-1} W$ and the ray with the endpoint $W_{m-1}{}'$ containing $W_{m-2}{}'$. We replace $W_{m-1}{}'$ with V. Notice that the angle at V in the quadrilateral $W_{m-1} W_m V W_m{}'$ is larger than $\pi/2$ and $\|W_{m-1} V\| < \|W_m W_{m-1}{}'\| \leq \overline{\sigma}_m(X)$. It follows from the above argument that we can assume without loss of generality that the angle at $W_{m-1}{}'$ in the quadrilateral $W_{m-1} W_m W_{m-1}{}' W_m{}'$ exceeds $\pi/2$, as dose the angle at $W_{m-1}{}'$ in the triangle $W_m W_{m-1}{}' W_m{}'$.

Therefore, in case IIb$_2$ we choose a point Z such that $\|W_{m-1}{}' Z\| < \delta$, $Z \notin \bigcup_{j=1}^{m} Q_j$, and define our new Q_{m+1} to be the polygon $W_m W_{m_1} Z W_{m_1}{}' W_m{}' W_{m-1}{}'$ while I_{m+1} is the polygonal line $W_{m_1} Z W_{m_1}{}'$. Note that for sufficiently small δ $\mathrm{diam} I_{m+1} \leq \|W_{m_1} W_{m_1}{}'\| \leq \overline{\sigma}_m(X)$. Put $m_1 = m+1$ and delete all $W_j W_j{}'$ for $m < j < m_1$.

The case IIb$_3$ with $I_m \neq W_m W_m{}'$ is handled similarly to IIb$_2$. In addition to the point Z described in the latter, we choose a point Y such that $\|W_m Y\| < \delta$, $Y \notin \bigcup_{j=1}^{m} Q_j$, and define our new Q_{m+1} to be the polygon $W_mW_{m_1}YZW_{m_1}{}'W_m{}'W_{m-1}{}'$ while I_{m+1} is the polygonal line $W_{m_1}YZW_{m_1}{}'$. Note that for sufficiently small δ $\mathrm{diam} I_{m+1} \leq \|W_{m_1}W_{m_1}{}'\| \leq \overline{\sigma}_m(X)$. Put $m_1 = m+1$ and delete all $W_jW_j{}'$ for $m < j < m_1$.

If $I_m = W_mW_m{}'$ then case IIb$_3$ is handled the same way as case IIb$_1$.

Consider the case IIb$_4$. Delete all $W_jW_j{}'$ from the sequence $\{W_jW_j{}'\}_{j<m}$ such that $W_jW_j{}' \cap W_mW_m{}' \neq \emptyset$ and $W_jW_j{}' \cap W_{m_1}W_{m_1}{}' = \emptyset$. Either for all remaining $j,\ j < m,\ W_jW_j{}' \cap W_mW_m{}' = \emptyset$ or there exists $j,\ j < m-1$ such that $W_jW_j{}' \cap W_mW_m{}' \neq \emptyset$ and $W_jW_j{}' \cap W_{m_1}W_{m_1}{}' \neq \emptyset$.

If the latter occurs, let $m-i$ be the largest number, $m-i < m-1$, such that $W_{m-i}W_{m-i}{}' \cap W_mW_m{}' \neq \emptyset$ and $W_{m-i}W_{m-i}{}' \cap W_mW_m{}' \neq \emptyset$ and apply case IIb (described previously for m) to m woth $m-i$ in place of $m-1$, deleting $W_jW_j{}'$ for $m-i < j \leq m-1$. Then, apply case IIb$_3$.

If the former occurs, we simply have case IIb and deal with it accordingly.

We now define $I_j,\ W_j,\ W_j{}'$ for $j \leq m_1$ in the same way as described before for $j \leq m$, and if $\{n : n > m_1,\ W_nW_n{}' \cap \bigcup_{j=1}^{n-1} Q_j \neq \emptyset\} \neq \emptyset$ then we let $m_2,\ m_2 > m_1$, be the smallest number such taht $W_{m_2}W_{m_2}{}' \cap \bigcup_{j=1}^{m_2-1} Q_j \neq \emptyset$.

We thus construct a sequence $m,\ m_1,\ m_2,\ \ldots,\ m_M$, where m_M is the smallest number, $m_M > m_{M-1}$, such that $W_{m_M}W_{m_M}{}' \cap \bigcup_{j=1}^{m_M-1} Q_j \neq \emptyset$ and $\{n : n > m_M,\ W_nW_n{}' \cap \bigcup_{j=1}^{n-1} Q_j \neq \emptyset\} = \emptyset$. An application of one of the cases I, IIa, IIb$_1$, IIb$_2$, IIb$_3$, IIb$_4$, III for each $m_n,\ n = 1, \ldots, M$, results in the construction of a sequence $\{Q_j\}_{j=1}^M$ of closed polygons with the following properties :

(i) $\bigcup_{j=1}^{M} Q_j \supset X$

(ii) $\mathrm{diam} I_j \leq \overline{\sigma}_m(X),\ I_j = Q_j \cap Q_{j+1},\ j = 1, \ldots, M-1$

(iii) $\forall 1 \leq j < M$ either $I_j \cap I_{j+1} = \emptyset$ or $I_j \cap I_{j+1}$ is a singleton and $X \supset I_j \cap I_{j+1}$

(iv) $\forall 1 \leq j < k < M \quad Q_k \cap Q_j \neq \emptyset \Rightarrow X \supset \bigcap_{i=j}^{k} Q_i = Q_k \cap Q_j$ and $\bigcap_{i=j}^{k} Q_i$ is a singleton

(v) $\forall 1 \leq j < M \quad Q_j \cap BE^{\sim}\ (Q_j \cap EB^{\sim})$ is either a line segment or a point.

Our goal is to define a sequence of closed polygons that satisfy conditions 1(1), 2(1) and 3(1). To this end, we modify I_j or I_{j+1} for each $j,\ 0 < j < M$, for which $I_j \cap I_{j+1} \neq \emptyset$.

If $W_j \in I_{j+1}$ ($W_j{}' \in I_{j+1}$) then choose a point $Z \in X$ such that $\mathrm{dist}(Z, W_j) < \delta$ ($\mathrm{dist}(Z, W_j{}') < \delta$), $\|ZW_j{}'\| \leq \overline{\sigma}_m(X) + \delta$ ($\|W_jZ\| \leq \overline{\sigma}_m(X) + \delta$) and $ZW_j{}' \cap I_{j+1} = \emptyset$ ($W_jZ \cap I_{j+1} = \emptyset$). Then, define the new I_j to be $ZW_j{}'(W_jZ)$.

If $W_{j+1} \in I_j$ ($W_{j+1}{}' \in I_j$) then choose a point $Z \in X$ such that $\mathrm{dist}(Z, W_{j+1}) < \delta$) ($\mathrm{dist}(Z, W_{j+1}{}') < \delta$), $\|ZW_{j+1}{}'\| \leq \overline{\sigma}_m(X) + \delta$ ($\|W_{j+1}Z\| \leq \overline{\sigma}_m(X) + \delta$) and $ZW_{j+1}{}' \cap I_j = \emptyset$ ($W_{j+1}Z \cap I_j = \emptyset$). Then, define the new I_{j+1} to be $ZW_{j+1}{}'(W_{j+1}Z)$.

In order to satisfy 3(1) we apply lemma C to each Q_j with diameter exceeding $\overline{\sigma}_m(X) + \delta$. We put $AB = W_{j-1}W_j,\ CD = W_{j-1}{}'W_j{}',\ CA^{\sim} = I_{j-1},\ BD^{\sim} = I_j$. By Lemma C, Q_j is partitioned into a finite number of closed polygons $R_{j1},\ R_{j2},\ \dots,\ R_{jm(j)}$ with diameters not exceeding $\overline{\sigma}_m(X)+\delta$. Furthermore, ordering the indices of all polygons obtained this way and including all non partitioned Q_j's, results in a chain $\{R_j\}_{j=1}^N$ of closed polygons that satisfy 1(1)–3(1). Since $\delta>0$ was arbitrary and X has been chained with a chain whose mesh does not exceed $\overline{\sigma}_m(X)+\delta$, this concludes the proof of Theorem 1.

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