ON IDEALS IN BCK-ALGEBRAS (II)

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Abstract. This paper is the continuation of the author's paper entitled the same name. We first introduce the notion of multiply commutative ideals in BCK-algebras, and investigate a number of their properties similar to n-fold commutative ideals. And then we clarify the essential attribute and some other properties of normal ideals (which were called n-fold weak commutative ideals) in BCK-algebras. Finally, we discuss various relations among multiply implicative ideals, multiply commutative ideals and normal ideals.

0. Introduction and preliminaries In [6] we introduced and studied the following four kinds of ideals in BCK-algebras: n-fold positive implicative ideals, n-fold commutative and n-fold weak commutative ideals, and multiply implicative ideals. This paper continues our discussion of [6]. Based on [3], we in section 1 introduce the notion of multiply commutative ideals in BCK-algebras, which is a generalization of the notion of n-fold commutative ideals, and investigate a number of their properties similar to n-fold commutative ideals. In section 2, we discuss certain further properties of n-fold weak commutative ideals in BCK-algebras, and we will find that the normality of a BCK-algebra can be characterized by the n-fold weak commutativity of its zero ideal. From this reason, an n-fold weak commutative ideal will be renamed as a normal ideal in this paper. In section 3, we will consider various relationships among multiply implicative ideals, multiply commutative ideals and normal ideals.

Throughout this paper we will freely use the symbols and terminologies of [8] or [7]. And we will denote X for a BCK-algebra, N for the set $\{1, 2, 3, \dots\}$ of all natural numbers, $x * y^n$ for $(\cdots((x * y) * y) * \cdots) * y$ in which y occurs n times, R_a for $\{x \in X \mid x * a = x\}$, $n(x, y)$ for a natural number relative to x and y.

As preliminaries, we enumerate some notions and results concerned as follows.

Definition 0.1. Let X be a BCK-algebra. Then

(1) X is called *n-fold commutative* if there is a fixed $n \in N$ such that for any $x, y \in X$, $x * y = x * (y * (y * x^n))$ (see [10]);

(2) X is called *multiply commutative* if for any $x, y \in X$, there is $n = n(x, y) \in N$ such that $x * y = x * (y * (y * x^n))$ (see [3]);

(3) X is called *multiply implicative* if for any $x, y \in X$, there is $n = n(x, y) \in N$ such that $x * (y * x^n) = x$ (see [5]);

(4) X is called *multiply positive implicative* if for any $x, y \in X$, there is $n = n(x, y) \in N$ such that $x * y^{n+1} = x * y^n$ (see [2]);

(5) X is called *normal* if for any $a \in X$, the set R_a is an ideal of X (see [4]);

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(6) X is called *J-semisimple* if for any nonzero element $a \in X$, there is a maximal ideal M of X such that $a \notin M$ (see [1], the authors of [1] called it *semisimple*).

Definition 0.2 (see [6]). Let A be a subset of a BCK-algebra X. Then

(1) A is called *an n-fold commutative ideal of* X if (i) $0 \in A$; (ii) there is a fixed $n \in N$ such that for any x, y, $z \in X$, $(x * y) * z \in A$ and $z \in A$ imply $x * (y * (y * x^n)) \in A$;

(2) A is called *an n-fold weak commutative ideal of* X if (i) $0 \in A$; (ii) there is a fixed $n \in N$ such that for any x, y, $z \in X$, $(x * (x * y^n)) * z \in A$ and $z \in A$ imply $y * (y * x) \in A$;

(3) A is called *a multiply implicative ideal of* X if (i) $0 \in A$; (ii) for any x, $y \in X$ and $z \in A$, if there is $n = n(x, y) \in N$ such that for all $m \geq n$, $(x * (y * x^m)) * z \in A$, it implies $x \in A$.

Theorem 0.1. *Let X be a BCK-algebra. Then*

(1) *X is normal if and only if $x * y = x$ implies $y * x = y$ for all x, $y \in X$ ([4], Theorem 2);*

(2) *any multiply implicative ideal of X is an ideal of X ([6], Proposition 4.4);*

(3) *an ideal A of X is multiply implicative if and only if for any x, $y \in X$, so long as $x * (y * x^n) \in A$ whenever n is great enough, it implies $x \in A$ ([6], Theorem 4.5).*

Theorem 0.2. *Let X be a multiply positive implicative BCK-algebra. Then*

(1) *the multiply implicativity, multiply commutativity and normality of X coincide ([3], Theorem 6);*

(2) *an ideal A of X is multiply implicative if and only if the quotient algebra X/A is multiply implicative ([6], Theorem 4.11).*

1. Multiply commutative ideals

Definition 1.1. A subset A of a BCK-algebra X is called a *multiply commutative ideal* of X if (1) $0 \in A$; (2) for any x, y, $z \in X$, $(x * y) * z \in A$ and $z \in A$ imply that there exists $n = n(x, y) \in N$ such that $x * (y * (y * x^n)) \in A$.

There indeed exist multiply commutative ideals of X, e.g., X itself is just one. Putting $y = 0$ in Definition 1.1(2), we obtain that $x * z \in A$ and $z \in A$ imply $x \in A$ for all x, $z \in X$, and we have got a fact as follows:

Proposition 1.1. Any multiply commutative ideal of a BCK-algebra X is an ideal of X.

However, an ideal may not be multiply commutative as shown in the following.

Example 1.1. The set $X = \{0, 1, 2\}$ together with the operation $*$ on X defined by $x * y = 0$ if $x \leq y$ and $x * y = x$ if $x > y$ forms a BCK-algebra. Its zero ideal $\{0\}$ is not multiply commutative, for $(1 * 2) * 0 = 0 \in \{0\}$ and $0 \in \{0\}$, but $1 * (2 * (2 * 1^n)) = 1 \notin \{0\}$ for all $n \in N$.

Nevertheless, if X is multiply commutative, then any ideal A of X must be multiply commutative. In fact, letting $(x*y)*z \in A$ and $z \in A$, by A being an ideal, we have $x*y \in A$.

Since $x * y = x * (y * (y * x^n))$ for some $n = n(x, y) \in N$, it follows $x * (y * (y * x^n)) \in A$, as required. Combining Proposition 1.1, the following holds.

Proposition 1.2. *In a multiply commutative BCK-algebra, the notion of multiply commutative ideals coincides with that of ideals.*

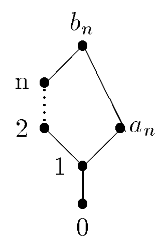
Similar discussion will give the next proposition.

Proposition 1.3. *An ideal A of a BCK-algebra X is multiply commutative if and only if for any $x, y \in X$, $x * y \in A$ implies $x * (y * (y * x^n)) \in A$ for some $n = n(x, y) \in N$.*

If the $n(x, y)$ in Definition 1.1 is identical with a fixed natural number n, A is clearly an n-fold commutative ideal, and so each of n-fold commutative ideals of X is multiply commutative, but a multiply commutative ideal of X may not be n-fold commutative for any natural number n.

Example 1.2. Let $X_n = \{0, 1, 2, \dots, n\} \cup \{a_n, b_n\}$, $n \in N$. Define an operation $*$ on X_n as follows: for any $u, v \in \{0, 1, 2, \dots, n\}$,

$$
\begin{aligned}
&u * v = \max\{0, u - v\},\\
&u * a_n = \max\{0, u - 1\},\\
&u * b_n = 0,\\
&a_n * u = \begin{cases} a_n, & \text{if } u = 0,\\ 1, & \text{if } u \neq 0,\end{cases}\\
&b_n * u = \begin{cases} b_n, & \text{if } u = 0,\\ n - u + 1, & \text{if } u \neq 0,\end{cases}\\
&a_n * a_n = a_n * b_n = b_n * b_n = 0,\\
&b_n * a_n = n.
\end{aligned}
$$



Then $(X_n; *, 0)$ is an $(n+1)$-fold commutative BCK-algebra, but not n-fold commutative (see, [10], Theorem 7). Now, immediately calculating gives

$$
\begin{cases} a_n * b_n = 0 \neq 1 = a_n * (b_n * (b_n * a_n^k)), & \text{if } k = 1, 2, \dots, n,\\ a_n * b_n = 0 = a_n * (b_n * (b_n * a_n^k)), & \text{if } k \geq n. \end{cases} \tag{I}
$$

Denote X for the set $\{(x_1, x_2, \dots) \mid x_i \in X_i$ and $x_i = 0$ whenever i is sufficiently large$\}$. Define an operation $*$ on X by

$$x * y = (x_1 * y_1, x_2 * y_2, \dots) \text{ for any } x = (x_1, x_2, \dots) \text{ and } y = (y_1, y_2, \dots) \in X.$$

Then $(X; *, 0)$ is a multiply commutative BCK-algebra where $0 = (0, 0, \dots)$ (see, [3], Example 1), and so its zero ideal is multiply commutative by Proposition 1.2. On the other hand, for any $n \in N$, putting $x = (0, \dots, 0, a_n, 0, \dots)$ and $y = (0, \dots, 0, b_n, 0, \dots) \in X$, we have $x * y = 0$, but $x * (y * (y * x^n)) \neq 0$, for $a_n * (b_n * (b_n * a_n^n)) = 1 \neq 0$ by (I). Therefore the zero ideal of X is not n-fold commutative for any $n \in N$.

We now see the notion of multiply commutative ideals is well-defined and a generalization of the notion of n-fold commutative ideals. And we will see from the next theorem that the multiply commutativity of a BCK-algebra can be characterized by the multiply commutativity of its zero ideal.

Theorem 1.4. *A BCK-algebra X is multiply commutative if and only if its zero ideal is multiply commutative.*

Proof. From Proposition 1.2, it suffices to prove the part "if". The following inequality naturally holds for any x, $y \in X$ and $n \in N$:

$$x * y \leq x * (y * (y * x^n)). \tag{I}$$

Put $u = x * (x * y)$, then $u \leq x$ and $u * y = 0$. Because the zero ideal of X is multiply commutative, by $u * y = 0$, there is $n = n(u, y) \in N$ such that $u * (y * (y * u^n)) = 0$, that is,

$$u \leq y * (y * u^n). \tag{II}$$

Also, left $*$ multiplying both sides of the inequality $u \leq x$ by y, and then right $*$ multiplying the left side by x and the right side by u $n - 1$ times, we obtain $y * x^n \leq y * u^n$, and so

$$y * (y * u^n) \leq y * (y * x^n). \tag{III}$$

Now, right $*$ multiplying both sides of (II) and of (III) by $y * (y * x^n)$, we get

$$\begin{aligned} u * (y * (y * x^n)) &\leq (y * (y * u^n)) * (y * (y * x^n)) \\ &\leq (y * (y * x^n)) * (y * (y * x^n)) = 0, \end{aligned}$$

namely, $u \leq y * (y * x^n)$, thus $x * (y * (y * x^n)) \leq x * u$. As $x * u = x * (x * (x * y)) = x * y$, it follows that $x * (y * (y * x^n)) \leq x * y$. Comparison with (I) gives $x * y = x * (y * (y * x^n))$, as required.

The following describes the situation of distribution of multiply commutative ideals.

Theorem 1.5. *Suppose that A, B are ideals of a BCK-algebra X with $A \subseteq B$. If A is multiply commutative, so is B.*

Proof. Let x, $y \in X$ such that $x * y \in B$. Put $u = x * (x * y)$, then $u \leq x$ and $u * y = 0$. Since $u \leq x$, by (III) of the last theorem, we have $y * (y * u^n) \leq y * (y * x^n)$, thus

$$x * (y * (y * x^n)) \leq x * (y * (y * u^n)). \tag{I}$$

Also, because A is multiply commutative, by $u * y = 0 \in A$, there exists $n = n(u, y) \in N$ such that $u * (y * (y * u^n)) \in A$, that is, $(x * (x * y)) * (y * (y * u^n)) \in A$, in other words, $(x * (y * (y * u^n))) * (x * y) \in A$. Since B is an ideal of X and $A \subseteq B$, by $x * y \in B$, it follows

$$x * (y * (y * u^n)) \in B. \tag{II}$$

Now, by (I) and (II), we obtain from B being an ideal of X that $x * (y * (y * x^n)) \in B$. Therefore Proposition 1.3 states that B is multiply commutative.

Corollary 1.6. *A BCK-algebra X is multiply commutative if and only if all of its ideals are multiply commutative.*

There is a close contact between a multiply commutative ideal A of X and the quotient algebra X/A.

Theorem 1.7. *Given an ideal A of a BCK-algebra X, A is multiply commutative if and only if the quotient algebra X/A is a multiply commutative BCK-algebra.*

Proof. We denote A_x for the congruence class in the quotient set X/A, containing x. Clearly, $A_0 = A$. Assume that A is multiply commutative and let A_x, $A_y \in X/A$ with $A_x * A_y = A_0$. Since $A_{x*y} = A_x * A_y = A_0$, we have $x * y \in A_0 = A$, then there is $n = n(x, y) \in N$ such that $x * (y * (y * x^n)) \in A$, and so $A_x * (A_y * (A_y * A_x^n)) = A_{x*(y*(y*x^n))} = A_0$, which means from

Proposition 1.3 that the zero ideal $\{A_0\}$ of X/A is multiply commutative. By Theorem 1.4, X/A is a multiply commutative BCK-algebra.

Conversely, our assumption of sufficiency together with Theorem 1.4 gives that $\{A_0\}$ is a multiply commutative ideal of X/A. Letting $x*y \in A$, since $A = A_0$, we have $A_x * A_y = A_0$, then there is some $n = n(x, y) \in N$ such that $A_x * (A_y * (A_y * A_x^n)) = A_0$. From this we get $x * (y * (y * x^n)) \in A_0 = A$. Hence A is multiply commutative of X by Proposition 1.3.

Proposition 1.8. *Any maximal ideal M of a BCK-algebra X is multiply commutative.*

Proof. Assume that $x * y \in M$. If $x \in M$, of course, $x * (y * (y * x^n)) \in M$ for any $n \in N$. If $x \notin M$, since M is maximal, there is $n = n(x, y) \in N$ such that $y * x^n \in M$. Also, we have

$$(x * (y * (y * x^n))) * (x * y) \leq y * (y * (y * x^n)) \leq y * x^n.$$

Now, because $x * y \in M$ and $y * x^n \in M$, by M being an ideal, $x * (y * (y * x^n)) \in M$. Therefore Proposition 1.3 implies that M is multiply commutative.

Proposition 1.9. *If $A_1, A_2, \ldots, A_n$ are multiply commutative ideals of a BCK-algebra X, so is the intersection $\bigcap_{i=1}^n A_i$.*

The proof of Proposition 1.9 is easy and omitted. It is worth attending that there is an unusual phenomenon that unlike the case of n-fold commutative ideals, the conclusion can not be extended to the intersection of an infinite number of multiply commutative ideals.

Example 1.3. Let $X_1, X_2, \ldots$ be as in Example 1.2. Denote $X = \prod_{n=1}^\infty X_n$, the direct product of $X_1, X_2, \ldots$. Then X with respect to the binary operation $*$ given by

$$(x_1, x_2, \ldots) * (y_1, y_2, \ldots) = (x_1 * y_1, x_2 * y_2, \ldots)$$

forms a BCK-algebra, but not multiply commutative (for details, see, [3], Example 3), thus its zero ideal is not multiply commutative by Theorem 1.4. It is easily seen that every X_n is a simple BCK-algebra, then the set $M_n = \{(x_1, \ldots, x_n, \ldots) \in X \mid x_n = 0\}$ is a maximal ideal of X. By Proposition 1.8, M_n is multiply commutative. Obviously, the intersection $\bigcap_{n=1}^\infty M_n$ is the zero ideal of X. However, as we have seen, it is not multiply commutative.

2. Normal ideals

Lemma 2.1. *Let A be an ideal of a BCK-algebra X. Then for any $x, y \in X$ and $n \in N$, $x * (x * y) \in A$ if and only if $x * (x * y^n) \in A$.*

Proof. Repeatedly using the fact that $x * (x * (x * y)) = x * y$, we have

$$\begin{aligned} x * (x * (x * y))^n &= (x * y) * (x * (x * y))^{n-1} \\ &= (x * (x * (x * y))^{n-1}) * y = \cdots = x * y^n. \end{aligned}$$

Now, if $x * (x * y) \in A$, since A is an ideal of X, $x * (x * y^n) \in A$ is got by

$$\begin{aligned} (x * (x * y^n)) * (x * (x * y))^n &= (x * (x * (x * y))^n) * (x * y^n) \\ &= (x * y^n) * (x * y^n) = 0. \end{aligned}$$

Conversely, if $x * (x * y^n) \in A$, since A is an ideal of X, by $x * (x * y) \leq x * (x * y^n)$, it follows $x * (x * y) \in A$.

We now see from which an n-fold weak commutative ideal of X must be an ideal of X (see, [6], Proposition 3.2) that the expression $x * (x * y^n)$ in Definition 0.2(2) can be replaced by $x*(x*y)$. Moreover, we will see from Theorem 2.3 below that the normality of a BCK-algebra can be characterized by the n-fold weak commutativity of its zero ideal. These lead us to give an equivalent definition of n-fold weak commutative ideals as follows.

Definition 2.1. A subset A of a BCK-algebra X is called a *normal ideal* of X if

(1) $0 \in A$;

(2) for any $x, y, z \in X$, $(x*(x*y))*z \in A$ and $z \in A$ imply $y*(y*x) \in A$.

Thus all statement in [6] relative to n-fold weak commutative idealscan be rewritten as the language of normal ideals, for example, we have

Proposition 2.2. *Let X be a BCK-algebra. Then*

(1) *any normal ideal of X is an ideal ([6], Proposition 3.2).*

(2) *an ideal A of X is normal if and only if for any $x, y \in X$, $x*(x*y) \in A$ implies $y*(y*x) \in A$ ([6], Theorem 3.4(2)).*

Let's investigate some further properties of normal ideals.

Theorem 2.3. *A BCK-algebra X is normal if and only if its zero ideal is normal.*

Proof. By Theorem 0.1(1), X is normal if and only if $x*y=x$ implies $y*x=y$ for all $x, y \in X$. By Proposition 2.2(2), the zero ideal of X is normal if and only if $x*(x*y)=0$ implies $y*(y*x)=0$ for any $x, y \in X$. Note that $x*y=x$ is equivalent to $x*(x*y)=0$ and $y*x=y$ to $y*(y*x)=0$, the assertion that $x*y=x$ implies $y*x=y$ is the same as $x*(x*y)=0$ implies $y*(y*x)=0$. Therefore X is normal if and only if the zero ideal of X is normal.

Theorem 2.4. *An ideal A of a BCK-algebra X is normal if and only if the quotient algebra X/A is a normal BCK-algebra.*

Proof. Assume that A is normal. For any $A_x, A_y \in X/A$, if $A_x*(A_x*A_y)=A_0$, since $A_0=A$, we have $x*(x*y) \in A$, then the normality of A gives that $y*(y*x) \in A$, which means that $A_y*(A_y*A_x)=A_0$. Now, by Proposition 2.2(2), the zero ideal $\{A_0\}$ of X/A is normal. By Theorem 2.3, X/A is normal.

Conversely, assume that X/A is normal, then by Theorem 2.3, the zero ideal $\{A_0\}$ of X/A is normal. For any $x, y \in X$, if $x*(x*y) \in A$, since $A=A_0$, we obtain $A_x*(A_x*A_y)=A_0$, then $A_y*(A_y*A_x)=A_0$ by the normality of $\{A_0\}$. Hence $y*(y*x) \in A_0=A$, and A is normal.

Theorem 2.5. *Let X be a multiply positive implicative BCK-algebra and let A, B be ideals of X with $A \subseteq B$. If A is normal, so is B.*

Proof. For any $x, y \in X$, since X is multiply positive implicative, there is $n=n(x,y) \in N$ such that $(x*y^n)*y=x*y^n$, then $(x*y^n)*((x*y^n)*y)=0 \in A$. By the normality of A, $y*(y*(x*y^n)) \in A$. By $A \subseteq B$, $y*(y*(x*y^n)) \in B$. Now, if $x*(x*y) \in B$, since B is an ideal of X, Lemma 2.1 implies $x*(x*y^n) \in B$. Hence the fact that

$$(y*(y*x))*(y*(y*(x*y^n))) \leq x*(x*y^n)$$

gives $y*(y*x) \in B$, which means from Proposition 2.2(2) that B is normal.

Theorems 2.3, 2.5 and Proposition 2.2(1) together imply the following corollary.

Corollary 2.6. *Let X be a multiply positive implicative BCK-algebra.*

(1) *X is normal if and only if any ideal of X is normal.*

(2) *If X is normal, the notion of ideals of X coincides with that of normal ideals of X.*

Proposition 2.7. *Each maximal ideal M of a BCK-algebra X is normal.*

Proof. Let $x*(x*y) \in M$. If $y \in M$, of course, $y*(y*x) \in M$. If $y \notin M$, by the maximality of M, $x*y^n \in M$ for some $n = n(x, y) \in N$. Since $x*(x*y) \in M$, Lemma 2.1 implies that $x*(x*y^n) \in M$. Now, by $x*y^n \in M$, $x \in M$. Hence the fact that $y*(y*x) \leq x$ gives that $y*(y*x) \in M$. Therefore the ideal M is normal by Proposition 2.2(2).

Proposition 2.8. *If $\{A_i\}_{i\in I}$ is the family of certain normal ideals of a BCK-algebra X, then the intersection $\bigcap_{i\in I} A_i$ is a normal ideal of X.*

The proof is obvious and omitted. Note that the maximal ideals of a nonzero J-semisimple BCK-algebra exist and the intersection of all of its maximal ideals is the zero ideal, we obtain

Corollary 2.9 ([4], Theorem 6)**.** *Any J-semisimple BCK-algebra is normal.*

3. Relations among three kinds of ideals We now consider the relationships among multiply implicative, multiply commutative and normal ideals.

Theorem 3.1. *Let A be a multiply implicative ideal of a BCK-algebra X. Then A is normal, but the inverse is false.*

Proof. Assume that A is multiply implicative, then A is an ideal of X by Theorem 0.1(2). If $x*(x*y) \in A$, by Lemma 2.1, $x*(x*y^n) \in A$ for all $n \in N$. Putting $u = y*(y*x)$, we have $u \leq x$ and $u \leq y$. Because $u \leq y$, we obtain $x*y^n \leq x*u^n$, then $u*(x*u^n) \leq u*(x*y^n)$. Also, by $u \leq x$, we have $u*(x*y^n) \leq x*(x*y^n)$. Hence $u*(x*u^n) \leq x*(x*y^n)$. Since A is an ideal of X and $x*(x*y^n) \in A$, it follows $u*(x*u^n) \in A$ for all $n \in N$. Now, Theorem 0.1(3) implies that $u \in A$, that is, $y*(y*x) \in A$, proving A is normal.

Next, let $X = \overline{N} \cup A \cup B$ where $\overline{N} = N \cup \{0\}$, $A = \{a_n \mid n \in \overline{N}\}$, $B = \{b_n \mid n \in \overline{N}\}$. Define an operation $*$ on X as follows: for any $m, n \in \overline{N}$,

$$\begin{aligned}
&m*n = \max\{0, m-n\},\\
&m*a_n = m*b_n = 0,\\
&a_m*n = a_{m+n},\\
&b_m*n = b_{m+n},\\
&a_m*a_n = b_m*b_n = \max\{0, n-m\},\\
&a_m*b_n = b_m*a_n = \max\{0, n-m+1\}.
\end{aligned}$$

Then $(X; *, 0)$ is a BCK-algebra (see [9]). For any $a \in X$, by routine verification, we obtain that $R_a = X$ if $a = 0$ and $R_a = \{0\}$ if $a \neq 0$, thus X is normal. By Theorem 2.3, the zero ideal of X is normal. On the other hand, we have $1*(a_0*1^n) = 1*a_n = 0$ for any $n \in N$, but $1 \neq 0$. Hence the zero ideal of X is not multiply implicative by Theorem 0.1(3).

It is easily seen that the zero ideal $\{0\}$ of X is multiply implicative if X is a multiply implicative BCK-algebra, thus Theorems 3.1 and 2.3 give

Corollary 3.2 ([5], Theorem 5)**.** *Any multiply implicative BCK-algebra is normal.*

Theorem 3.3. *Every multiply commutative ideal A of a BCK-algebra X is normal, but the inverse is not true.*

Proof. By Proposition 1.1, A is an ideal of X. Let $x, y \in X$ such that $x*(x*y) \in A$. Denote $u = y*(y*x)$, then $u*x = 0$ and $u \leq y$. By $u \leq y$, we have $x*y^n \leq x*u^n$, thus

$$u*(x*(x*y^n)) \leq u*(x*(x*u^n)).$$

Because A is multiply commutative, by $u * x = 0 \in A$, we obtain $u * (x * (x * u^n)) \in A$ for some $n = n(u, x) \in N$, which means from which A is an ideal that $u * (x * (x * y^n)) \in A$. Also, since $x * (x * y) \in A$, by Lemma 2.1, $x * (x * y^n) \in A$. Now, using the fact that A is an ideal once more, it follows $u \in A$, that is, $y * (y * x) \in A$. Therefore A is a normal ideal by Proposition 2.2(2).

The second half part can be seen from the following counter example.

Example 3.1. As we have known, the algebra $X = \prod_{n=1}^{\infty} X_n$ in Example 1.3 is not multiply commutative and the set $M_n = \{(x_1, \ldots, x_n, \ldots) \in X \mid x_n = 0\}$ is a maximal ideal of X. By Theorem 1.4, the zero ideal of X is not multiply commutative. On the other hand, by Proposition 2.7, M_n is normal, then so is the intersection $\bigcap_{n=1}^{\infty} M_n$ by Proposition 2.8. Note that the zero ideal of X is exactly equal to $\bigcap_{n=1}^{\infty} M_n$, we see it is normal.

Putting Theorems 1.4, 3.3 and 2.3 together, we obtain

Corollary 3.4 ([3], Theorem 5)**.** *Each multiply commutative BCK-algebra is normal.*

The following two examples show that a multiply implicative ideal may not be multiply commutative and the inverse is not true either.

Example 3.2. Let X be as in Example 3.1, then its zero ideal is not multiply commutative, but it is multiply implicative, in fact, for any $x = (x_1, x_2, \ldots)$ and $y = (y_1, y_2, \ldots)$ in X, if $x * (y * x^n) = 0$ whenever n is sufficiently large, then $x_i * (y_i * x_i^n) = 0$ for any $i \in N$. Note that any X_i is a simple BCK-algebra, if there exists some $x_i \neq 0$, one has $y_i * x_i^n = 0$ whenever n is sufficiently large, hence $x_i * (y_i * x_i^n) = x_i \neq 0$, a contradiction. Therefore $x_i = 0$, $i = 1, 2, \ldots$, namely, $x = (0, 0, \ldots)$, the zero element of X.

Example 3.3. The set $\overline{N} = N \cup \{0\}$ with the operation $*$ defined by $m * n = \max\{0, m - n\}$ clearly forms a commutative BCK-algebra. Let X be the set consisting of all mappings from N to $\overline{N}$ and let the operation $*$ on X be given by $(x * y)(n) = x(n) * y(n)$ for all $n \in N$. Then $(X; *, \theta)$ is also a commutative BCK-algebra where θ is the zero mapping: $\theta(n) = 0$, $n \in N$. As any commutative BCK-algebra is multiply commutative, by Proposition 1.2, each ideal of X is multiply commutative. On the other hand, put $a \in X$ such that $a(n) = 1$ for any $n \in N$, then the ideal I_a of X generated by $\{a\}$ is not multiply implicative (for details, see, [6], Example 4.14).

However, for a multiply positive implicative BCK-algebra, we have a nice result as follows.

Theorem 3.5. *Let X be a multiply positive implicative BCK-algebra and A an ideal of X. Then the following are equivalent:*

(1) *A is multiply implicative;*

(2) *A is multiply commutative;*

(3) *A is normal.*

Proof. We only need to prove that (3) implies (1) and (2). As X is multiply positive implicative, so is the quotient algebra X/A by routine verification. Assume that (3) holds, then by Theorem 2.4, X/A is normal, thus X/A is multiply implicative and multiply commutative by Theorem 0.2(1). Now, by Theorem 0.2(2), A is multiply implicative, (1) holding; by Theorem 1.7, A is multiply commutative, (2) holding. The proof is complete.

Note that a finite BCK-algebra must be multiply positive implicative, the following holds.

Corollary 3.6. In a finite BCK-algebra, the multiply implicativity, multiply commutativity and normality of an ideal coincide.

Before concluding our discussion, we summarize the relations among the three kinds of ideals as follows: let X be a BCK-algebra and A an ideal of X, then

$$\boxed{A \text{ is multiply implicative}} \Rightarrow \boxed{A \text{ is normal}} \Leftarrow \boxed{A \text{ is multiply commutative}}$$

Especially, if X is multiply positive implicative, then

$$\boxed{A \text{ is multiply implicative}} \Leftrightarrow \boxed{A \text{ is normal}} \Leftrightarrow \boxed{A \text{ is multiply commutative}}$$

Finally, we give an open problem: can we define so-called multiply positive implicative ideals in a BCK-algebra, which are similar to positive implicative ideals (refer to [8], pages 64-68), or to n-fold positive implicative ideals (refer to [6], section 1)?

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