SPECHT'S RATIO IN THE YOUNG INEQUALITY

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Abstract. The Young operator inequality is represented for $\lambda \in [0,1]$ as follows

$$A \bigtriangledown_{1-\lambda} B \geq A \sharp_{1-\lambda} B$$

for positive invertible operators A and B with $0 < m \leq A, B \leq M$, $m < M$. In this note we show the following converse inequality of the Young operator inequality on the ratio, independent of λ:

$$S(h) A \sharp_{1-\lambda} B \geq A \bigtriangledown_{1-\lambda} B \ (\geq A \sharp_{1-\lambda} B),$$

where the constant $S(h) = \frac{h^{\frac{1}{h-1}}}{e \log h^{\frac{1}{h-1}}}$ ($h = \frac{M}{m}$) is Specht's ratio. Moreover we show another converse inequality of it on the difference:

$$L(1,h) \log S(h) A \geq A \bigtriangledown_{1-\lambda} B - A \sharp_{1-\lambda} B \ (\geq 0),$$

where $L(m,M) = \frac{M-m}{\log M - \log m}$ is the logarithmic mean.

1. Introduction

We cite the Young inequality which is considered as the λ-weighted arithmetic-geometric mean inequality as follows:

The Young inequality . *Let a and b be positive numbers. Then the inequality*

$$(1.1) \qquad (1-\lambda)a + \lambda b \geq a^{1-\lambda} b^{\lambda}$$

holds for every $\lambda \in [0,1]$.

In this note, an operator means a bounded linear operator acting on a complex Hilbert space H. The inequality (1.1) is extended to an operator version by the following two means. Let A and B be positive invertible operators. For every $\lambda \in [0,1]$, we denote by $\bigtriangledown_\lambda$ the λ-weighted arithmetic mean as follows:

$$(1.2) \qquad A \bigtriangledown_\lambda B := (1-\lambda)A + \lambda B,$$

and by $\sharp_\lambda$ the λ-weighted geometric mean as follows:

$$(1.3) \qquad A \sharp_\lambda B := A^{\frac{1}{2}} (A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^{\lambda} A^{\frac{1}{2}}.$$

The λ-weighted geometric mean is introduced by F.Kubo and T.Ando in [3]. The following arithmetic mean - geometric mean inequality is regarded as an operator version of the Young inequality:

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The Young operator inequality . *Let A and B be positive invertible operators. Then the inequality*

$$A \bigtriangledown_{1-\lambda} B \geq A \sharp_{1-\lambda} B \tag{1.4}$$

holds for every $\lambda \in [0,1]$.

For the sake of convenience, we recall some constants as follows: Let m and M be real numbers with $0 < m < M$. Then the logarithmic mean $L(m,M)$ (cf. [2]) is defined by

$$L(m,M) = \frac{M-m}{\log M - \log m}.$$

Next the constant $S(h)$ defined by

$$S(h) = \frac{h^{\frac{1}{h-1}}}{e \log h^{\frac{1}{h-1}}} \quad (h > 1)$$

is called Specht's ratio [1], [4], which is the best upper bound of the arithmetic mean by the geometric one for positive numbers: For $x_i \in [m,M]$ with $M > m > 0$ $(i = 1, 2, \cdots, n)$, the following inequality holds

$$S(h) \sqrt[n]{x_1 x_2 \cdots x_n} \geq \frac{x_1 + x_2 + \cdots + x_n}{n} \ (\geq \sqrt[n]{x_1 \cdots x_n}), \tag{1.5}$$

where the constant $h = \frac{M}{m}$ is called a condition number in the sense of Turing [5].

In this note, we show converse inequalities of the Young operator inequality (1.4). First we show the following converse inequality of (1.4) on the ratio independent of $\lambda \in [0,1]$: For positive invertible operators A and B with $0 < m \leq A, B \leq M$ and $h = \frac{M}{m} (> 1)$

$$S(h) A \sharp_{1-\lambda} B \geq A \bigtriangledown_{1-\lambda} B \ (\geq A \sharp_{1-\lambda} B).$$

Moreover we show the following converse inequality of (1.4) on the difference:

$$L(1,h) \log S(h) A \geq A \bigtriangledown_{1-\lambda} B - A \sharp_{1-\lambda} B (\geq 0).$$

¿From two inequalities stated above, we can recognize that Specht's ratio plays the important role in the converse of the Young operator inequality (1.4).

2. Converse ratio inequality in the Young operator inequality

In this section, we show a converse ratio inequality of the Young operator inequality (1.4), i.e., a ratio inequality of $A \bigtriangledown_{1-\lambda} B$ by $A \sharp_{1-\lambda} B$ as follows:

Theorem 2.1. *Let A and B be positive invertible operators with $0 < m \leq A, B \leq M$ and $h = \frac{M}{m} > 1$. Then the inequality*

$$S(h) A \sharp_{1-\lambda} B \geq A \bigtriangledown_{1-\lambda} B \ (\geq A \sharp_{1-\lambda} B) \tag{2.1}$$

holds for every $\lambda \in [0,1]$.

As in (1.5) which is considered as a converse ratio inequality of the Young inequality (1.1), Specht's ratio is used in (2.1) which is a converse ratio inequality of the Young operator inequality (1.4). To prove Theorem 2.1, we need some results cited as Lemmas 2.2 and 2.3.

In our seminar talk, J.I. Fujii gave the following properties by considering Specht's ratio $S(t)$ as a function for $t > 0$:

Lemma 2.2. *A function $S(t)$ is strictly decreasing for $0 < t < 1$ and strictly increasing for $t > 1$. Furthermore the following equations hold*

$$S(1) = 1 \quad \text{and} \quad S(t) = S(\frac{1}{t}) \text{ for all } t > 0.$$

Proof. We have by L'Hospital's theorem

$$\lim_{t\to 1} \log S(t) = \lim_{t\to 1} \log \frac{t^{\frac{1}{t-1}}}{e \log t^{\frac{1}{t-1}}} = \lim_{t\to 1} \left(\frac{\log t}{t-1} - 1 - \log \frac{\log t}{t-1} \right)$$
$$= \lim_{t\to 1} \left(\frac{1}{t} - 1 - \log \frac{1}{t} \right) = 0,$$

and so $S(1) = 1$. Moreover the equation $(\frac{1}{t})^{\frac{1}{\frac{1}{t}-1}} = t^{\frac{t}{t-1}} = t \cdot t^{\frac{1}{t-1}}$ implies the equation

$$S(\frac{1}{t}) = \frac{(\frac{1}{t})^{\frac{1}{\frac{1}{t}-1}}}{e \log (\frac{1}{t})^{\frac{1}{\frac{1}{t}-1}}} = \frac{t \cdot t^{\frac{1}{t-1}}}{e \log t^{\frac{t}{t-1}}} = \frac{t^{\frac{1}{t-1}}}{e \log t^{\frac{1}{t-1}}} = S(t).$$

Furthermore we have by a differential calculation

$$\frac{d}{dt} \log S(t) = \frac{d}{dt} \left(\frac{\log t}{t-1} - 1 - \log \frac{\log t}{t-1} \right) = \frac{\frac{1}{t}(t-1) - \log t}{(t-1)^2} - \frac{t-1}{\log t} \frac{\frac{1}{t}(t-1) - \log t}{(t-1)^2}$$
$$= \frac{(\log t - t + 1)(1 - \frac{1}{t} - \log t)}{(t-1)^2 \log t}.$$

So for any $t > 1$, a function $S(t)$ is strictly increasing from $\frac{d}{dt} \log S(t) > 0$ by the Klein inequality ($1 - \frac{1}{t} \leq \log t \leq t - 1$ for any $t > 0$) and $\log t > 0$. On the other hand, for $0 < t < 1$ we see that a function $S(t)$ is strictly decreasing. □

In the following lemma, we show a converse ratio inequality of the Young inequality (1.1):

Lemma 2.3. *Let a be a positive number. Then the inequality*

$$S(a)a^{1-\lambda} \geq (1-\lambda)a + \lambda \; (\geq a^{1-\lambda}) \tag{2.2}$$

holds for every $\lambda \in [0, 1]$.

Consequently, for $a, b > 0$ the inequality

$$S(\frac{a}{b})a^{1-\lambda}b^{\lambda} \geq (1-\lambda)a + \lambda b \; (\geq a^{1-\lambda}b^{\lambda}) \tag{2.3}$$

holds for every $\lambda \in [0, 1]$.

Proof. Let $a \neq 1$. We put a function $f_a(\lambda)$ derived from the Young inequality (1.1) in the case $b = 1$ as follows:

$$f_a(\lambda) := \frac{(1-\lambda)a + \lambda}{a^{1-\lambda}} = \frac{(1-a)\lambda + a}{a^{1-\lambda}} = (\frac{1-a}{a}\lambda + 1)a^{\lambda}.$$

Then we obtain the constant $S(a) = \frac{a^{\frac{1}{a-1}}}{e \log a^{\frac{1}{a-1}}}$ as the maximum of $f_a(\lambda)$ for $\lambda \in [0, 1]$. Indeed, we have by an elementary differential calculation

$$f'_a(\lambda) = \left\{ \frac{1-a}{a} + (\frac{1-a}{a}\lambda + 1) \log a \right\} a^{\lambda},$$

and so the equation $f_a'(\lambda)=0$ has the following unique solution $\lambda=\lambda_a$:

$$\lambda_a=\frac{a}{1-a}\left(\frac{a-1}{a\log a}-1\right)=\frac{a}{a-1}-\frac{1}{\log a}\ (\in[0,1]).$$

In fact, the Klein inequality ensures $\lambda_a\in[0,1]$. Furthermore it is easily seen that

$$f_a'(\lambda)>0 \text{ for } \lambda<\lambda_a \quad\text{and}\quad f_a'(\lambda)<0 \text{ for } \lambda>\lambda_a.$$

Therefore a maximum of $f_a(\lambda)$ takes at $\lambda=\lambda_a$, and moreover we have

$$\max_{0\le\lambda\le1} f_a(\lambda)=f_a(\lambda_a)=\frac{\frac{a-1}{\log a}}{a^{-\frac{1}{a-1}+\frac{1}{\log a}}}=\frac{a^{\frac{1}{a-1}}}{e\log a^{\frac{1}{a-1}}}=S(a).$$

In the case of $a=1$, the inequality (2.2) is ensured by Lemma 2.2 ($S(1)=1$).

The desired inequality (2.3) is obtained by replacing a with $\frac{a}{b}$ in (2.2). Hence the proof of Lemma 2.3 is complete. □

We show Theorem 2.1 by considering the operator version of Lemma 2.3 as follows:

Proof of Theorem 2.1. Let C be a positive operator with $0<m\le C\le M$, $m<M$. Then by using the functional calculus in the inequality (2.2), the inequality

$$\max_{m\le t\le M} S(t)\ C^{1-\lambda}\ge(1-\lambda)C+\lambda\ (\ge C^{1-\lambda})$$

holds for every $\lambda\in[0,1]$. Moreover since the maximum of $S(t)$ in $t\in[m,M]$ is given by $\max\{S(m),S(M)\}$ from Lemma 2.2, we have

$$\max\{S(m),S(M)\}\ C^{1-\lambda}\ge(1-\lambda)C+\lambda\ (\ge C^{1-\lambda}). \tag{2.4}$$

Here we replace C with $A^{-\frac12}BA^{-\frac12}$ in (2.4). Then we obtain $\frac{m}{M}\le A^{-\frac12}BA^{-\frac12}\le\frac{M}{m}$, i.e., $\frac{1}{h}\le A^{-\frac12}BA^{-\frac12}\le h$. Hence we have for any $\lambda\in[0,1]$

$$S(h)(A^{-\frac12}BA^{-\frac12})^{1-\lambda}\ge(1-\lambda)A^{-\frac12}BA^{-\frac12}+\lambda$$

by $S(h)=S(\frac{1}{h})$ in Lemma 2.2. Multiplying both sides by $A^{\frac12}$, we have

$$S(h)A^{\frac12}(A^{-\frac12}BA^{-\frac12})^{1-\lambda}A^{\frac12}\ge(1-\lambda)B+\lambda A,$$

and hence we have the desired inequality (2.1). □

3. The converse difference inequality in the Young operator inequality

In this section, we show a converse difference inequality of the Young operator inequality (1.4), i.e., the upper bound of $A\bigtriangledown_{1-\lambda}B-A\sharp_{1-\lambda}B\ (\ge0)$ as follows:

Theorem 3.1. *Let A and B be positive invertible operators with $0<m\le A,B\le M$ and $h=\frac{M}{m}>1$. Then the inequality*

$$L(1,h)\log S(h)A\ge A\bigtriangledown_{1-\lambda}B-A\sharp_{1-\lambda}B\ (\ge0) \tag{3.1}$$

holds for every $\lambda\in[0,1]$.

The logarithmic mean and Specht's ratio are used in a converse difference inequality (3.1) of the Young operator inequality (1.4).

To prove Theorem 3.1, we show a converse difference inequality of the Young inequality (1.1) in the following way:

Lemma 3.2. *Let a be a positive number. Then the inequality*

$$L(1,a)\log S(a) \geq (1-\lambda)a + \lambda - a^{1-\lambda} \ (\geq 0) \tag{3.2}$$

holds for every $\lambda \in [0,1]$.

Consequently, for $a, b > 0$ the inequality

$$L(a,b)\log S(\frac{a}{b}) \geq (1-\lambda)a + \lambda b - a^{1-\lambda}b^{\lambda} \ (\geq 0) \tag{3.3}$$

holds for every $\lambda \in [0,1]$.

Proof. Let $a \neq 1$. We put a function $g_a(\lambda)$ derived from the Young inequality (1.1) in the case $b = 1$ as follows:

$$g_a(\lambda) := (1-\lambda)a + \lambda - a^{1-\lambda} = (1-a)\lambda + a - a^{1-\lambda}.$$

Then we want to determine the maximum of $g_a(\lambda)$. We have by an elementary differential calculation

$$g_a'(\lambda) = (1-a) + a^{1-\lambda}\log a \tag{3.4}$$

and so the equation $g_a'(\lambda) = 0$ has the following unique solution $\lambda = \lambda_a$:

$$\lambda_a = 1 - \frac{\log \frac{a-1}{\log a}}{\log a} = 1 - \log_a \frac{a-1}{\log a} = \log_a \frac{a\log a}{a-1} = \frac{\log \frac{a\log a}{a-1}}{\log a}.$$

By the Klein inequality, it follows that λ_a is included in $[0,1]$. Since we have $g_a''(\lambda) = -a^{1-\lambda}(\log a)^2 < 0$ by (3.4), a maximum of $g_a(\lambda)$ takes at $\lambda = \lambda_a$, and moreover we have

$$\begin{aligned}\max_{0\leq\lambda\leq 1} g_a(\lambda) &= g_a(\lambda_a)\\ &= \frac{1-a}{\log a}\log\frac{a\log a}{a-1} + a - \frac{(a-1)}{\log a} = \frac{a-1}{\log a}(-\log\frac{a\log a}{a-1} + \frac{a\log a}{a-1} - 1)\\ &= \frac{a-1}{\log a}\log\frac{a^{\frac{a}{a-1}}}{e\log a^{\frac{a}{a-1}}} = \frac{a-1}{\log a}\log\frac{a^{\frac{1}{a-1}}}{e\log a^{\frac{1}{a-1}}} = L(1,a)\log S(a).\end{aligned}$$

In the case of $a = 1$, the inequality (3.2) is ensured by Lemma 2.2 ($S(1) = 1$) and a property of mean ($\lim_{a\to 1} L(a) = 1$).

The desired inequalitiy (3.3) is obtained by replacing a with $\frac{a}{b}$ in (3.2). Hence the proof of Theorem 3.2 is complete. □

We show Theorem 3.1 by considering the operator version of Theorem 3.2 as follows:

Proof of Theorem 3.1. Let C be a positive operator with $0 < m \leq C \leq M$, $m < M$. Then by using the functional calculus in the inequality (3.2), the inequality

$$\max_{m\leq t\leq M} L(1,t)\log S(t) \geq (1-\lambda)C + \lambda - C^{1-\lambda} \ (\geq 0) \tag{3.5}$$

holds for every $\lambda \in [0,1]$. Here we replace C with $A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$ in (3.5). Then we have $\frac{m}{M} \leq A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \leq \frac{M}{m}$, i.e., $\frac{1}{h} \leq A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \leq h$. Hence we have for any $\lambda \in [0,1]$

$$L(1,h)\log S(h) \geq (1-\lambda)A^{-\frac{1}{2}}BA^{-\frac{1}{2}} + \lambda - (A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{1-\lambda}$$

by Lemma 2.2, that is,

$$L(1,h)\log S(h)A \geq (1-\lambda)B + \lambda A - A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{1-\lambda}A^{\frac{1}{2}}.$$

So we have the desired inequality (3.1). □

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