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THE RADIUS OF β-CONVEXITY FOR THE CLASSES OF λ-SPIRALLIKE ORDER α FUNCTIONS

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ABSTRACT. Some subclasses of analytic functions in the open unit disk $\mathbb{D}$ are considered. The object of the present paper is to derive sharp bounds for the radius of β-convexity for the classes of λ-spirallike of order α functions and p-fold λ-spirallike of order α functions.

1. INTRODUCTION

Let $\mathcal{A}$ denote the class of functions of the form

$$s(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. And let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of analytic and univalent functions $s(z)$ in $\mathbb{D}$.

A function $s(z)$ in $\mathcal{S}$ is said to be starlike if it satisfies

$$\operatorname{Re}\left\{\frac{zs'(z)}{s(z)}\right\} > 0 \quad (z \in \mathbb{D}). \tag{1.2}$$

We denote by $\mathcal{S}^*$ the class of all starlike functions. A function $s(z)$ in $\mathcal{S}$ is said to be convex if it satisfies

$$\operatorname{Re}\left\{1 + \frac{zs''(z)}{s'(z)}\right\} > 0 \quad (z \in \mathbb{D}). \tag{1.3}$$

Also we denote by $\mathcal{K}$ the class of all convex functions.

Definition 1.1. *A function $s(z)$ in $\mathcal{S}$ is said to be λ-spirallike if*

$$\operatorname{Re}\left\{e^{i\lambda} z \frac{s'(z)}{s(z)}\right\} > 0 \quad (z \in \mathbb{D}) \tag{1.4}$$

for some real λ $\left(|\lambda| < \frac{\pi}{2}\right)$. The class of these functions is denoted by $\mathcal{S}^*_\lambda$.

Definition 1.2. *A function $s(z)$ in $\mathcal{S}$ is said to be λ-spirallike of order α if*

$$\operatorname{Re}\left\{e^{i\lambda} z \frac{s'(z)}{s(z)}\right\} > \alpha \cos\lambda \quad (z \in \mathbb{D}) \tag{1.5}$$

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for some real λ $\left(|\lambda| < \frac{\pi}{2}\right)$ *and* α $(0 \le \alpha < 1)$*. We denote by* $\mathcal{S}^*_\lambda(\alpha)$ *all such functions which satisfy* (1.5)*.*

The above classes were introduced by Spacek ([11]). For $\lambda = 0$ in (1.4), a function $s(z)$ in the class $\mathcal{S}^*_0(\alpha)$ is a starlike function of order α.

Definition 1.3. *Let* F *denote a non-empty collection of functions* $s(z)$ *each of which is univalent in* $\mathbb{D}$*, and let* β *be given* $0 \le \beta \le 1$*. Then the real number*

$$R_\alpha(F) = \sup\{R|\mathrm{Re}\{J(\beta, s(z))\} > 0, |z| < R, s(z) \in F\} \tag{1.6}$$

is called the radius of β*-convexity of* F*, where* $J(\beta, s(z))$ *is defined by the relation,*

$$J(\beta, s(z)) = (1-\beta)z\frac{s'(z)}{s(z)} + \beta\left(1 + z\frac{s''(z)}{s'(z)}\right). \tag{1.7}$$

The radius of β-convexity was introduced by Miller, Mocanu and Reade ([4]). For $\beta = 0$ and $\beta = 1$ in (1.7), we define a starlike function (1.2) and a convex function (1.3), respectively.

Definition 1.4. *Consider a function* $s(z) = z + a_2z^2 + a_3z^3 + \cdots$ *which is univalent in* $\mathbb{D}$*. Then the function defined by the relation.*

$$f(z) = (s(z^p))^{\frac{1}{p}} = z + \sum_{n=1}^{\infty} a_{np+1}z^{np+1} \tag{1.8}$$

is also univalent in $\mathbb{D}$*, and* $f(z)$ *is called a* p*-fold univalent function. If the function* $f(z)$ *defined by the relation* (1.8) *satisfies the condition*

$$\mathrm{Re}\left\{e^{i\lambda}z\frac{f'(z)}{f(z)}\right\} > 0 \quad (z \in \mathbb{D}), \tag{1.9}$$

then the function $f(z)$ *is called a* p*-fold* λ*-spirallike function in* $\mathbb{D}$*, for some real* λ $\left(|\lambda| < \frac{\pi}{2}\right)$ ([1])*, and the class of these functions is denoted by* $\mathcal{S}^*_{\lambda p}$*. And also we can define the class of* p*-fold* λ*-spirallike functions of order* α *in* $\mathbb{D}$*, denoted by* $\mathcal{S}^*_{\lambda p}(\alpha)$*.*

The radius of β-convexity was introduced by Miller, Mocanu and Reade ([4]). There are many open problems about the radius of starlikeness, convexity and β-convexity for the classes $\mathcal{S}$ (cf. [1]). So, we derive sharp bounds for the radius of β-convexity for the classes of λ-spirallike of order α and p-fold λ-spirllike of order α functions.

2. The radius of β-convexity

To discuss our problems, we need the following lemmas.

Lemma 2.1. ([5]) *If $s(z) \in \mathcal{S}^*_\lambda(\alpha)$, then, for $|z| = r < 1$,*

$$\left| z\frac{s'(z)}{s(z)} - \frac{1 + \{2(1-\alpha)\cos\lambda e^{-i\lambda} - 1\}r^2}{1-r^2} \right| \leq \frac{2(1-\alpha)r\cos\lambda}{1-r^2}. \tag{2.1}$$

Lemma 2.2. ([10]) *If $p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots$ is analytic in $\mathbb{D}$, and satisfies* $\operatorname{Re} p(z) > 0$ *and $p(0) = 1$. Then, for $|z| = r < 1$,*

$$\left| z\frac{p'(z)}{p(z)} \right| \leq \frac{2r}{1-r^2}. \tag{2.2}$$

Lemma 2.3. ([7]) *If $p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots$ is analytic in $\mathbb{D}$, and satisfies* $\operatorname{Re} p(z) > 0$*, then, for $|z| = r < 1$,*

(i) $$|p_n| \leq 2 \quad \textit{for} \quad n \geq 1,$$

(ii) $$\frac{1-|z|}{1+|z|} \leq \operatorname{Re} p(z) \leq |p(z)| \leq \frac{1+|z|}{1-|z|}.$$

Lemma 2.4. *If $s(z) \in \mathcal{S}^*_\lambda(\alpha)$, then, for $|z| = r < 1$,*
(i) *for $\lambda \neq 0$,*

$$\left| 1 + z\frac{s''(z)}{s'(z)} - \frac{1 + \{2(1-\alpha)\cos\lambda e^{-i\lambda} - 1\}r^2}{1-r^2} \right| \leq \frac{2(1-\alpha)r\{1 + r + (1-r)|\sin\lambda|\}\cos\lambda}{(1-r)^2(1+r)|\sin\lambda|} \tag{2.3}$$

and
(ii) *for $\lambda = 0$,*

$$\left| 1 + z\frac{s''(z)}{s'(z)} - \frac{1 + (1-2\alpha)r^2}{1-r^2} \right| \leq \frac{4r(1-\alpha)\{1 + (1-\alpha)r\}}{(1-r^2)\{(1-\alpha)(1+r) + \alpha(1-r)\}}.$$

Proof. (i) For $\lambda \neq 0$, since $s(z) \in \mathcal{S}^*_\lambda(\alpha)$, then we can write

$$\frac{e^{i\lambda}\dfrac{zs'(z)}{s(z)} - \alpha\cos\lambda - i\sin\lambda}{(1-\alpha)\cos\lambda} = p(z), \tag{2.4}$$

where $p(z)$ is analytic in $\mathbb{D}$, and satisfies Re $p(z) > 0$ and $p(0) = 1$. The logarithmic differentiation of (2.4) yields

$$1+z\frac{s''(z)}{s'(z)}-z\frac{s'(z)}{s(z)}=\frac{z(1-\alpha)\cos\lambda p'(z)}{(1-\alpha)\cos\lambda p(z)+\alpha\cos\lambda+i\sin\lambda}. \tag{2.5}$$

By Lemma 2.2 and putting $\dfrac{1}{p(z)} = U + iV$, we have

$$\begin{aligned}
&\left|1+z\frac{s''(z)}{s'(z)}-z\frac{s'(z)}{s(z)}\right| \\
&=\left|\frac{\frac{zp'(z)}{p(z)}}{1+\frac{\alpha}{1-\alpha}\frac{1}{p(z)}+i\frac{1}{1-\alpha}\tan\lambda\frac{1}{p(z)}}\right| \\
&=(1-\alpha)\left|\frac{\frac{zp'(z)}{p(z)}}{(1-\alpha)+\alpha\frac{1}{p(z)}+i\tan\lambda\frac{1}{p(z)}}\right| \\
&\le\frac{(1-\alpha)\frac{2r}{1-r^2}}{\left|(1-\alpha)+\alpha\frac{1}{p(z)}+i\tan\lambda\frac{1}{p(z)}\right|} \\
&\le\frac{(1-\alpha)\frac{2r}{1-r^2}}{U|\tan\lambda|}.
\end{aligned} \tag{2.6}$$

Using Lemma 2.3 and (2.6), we have

$$\left|1+z\frac{s''(z)}{s'(z)}-z\frac{s'(z)}{s(z)}\right|\le\frac{2(1-\alpha)r}{(1-r)^2|\tan\lambda|}. \tag{2.7}$$

And by Lemma 2.3 and (2.7), we get

$$\begin{aligned}
&\left|1+z\frac{s''(z)}{s'(z)}-\frac{1+\{2(1-\alpha)\cos\lambda e^{i\lambda}-1\}r^2}{1-r^2}\right| \\
&\le\frac{2(1-\alpha)r\{1+r+(1-r)|\sin\lambda|\}\cos\lambda}{(1-r)^2(1+r)|\sin\lambda|}.
\end{aligned} \tag{2.8}$$

(ii) For $\lambda = 0$, from (2.4) we get

$$\frac{zs'(z)}{s(z)}-\alpha=(1-\alpha)p(z). \tag{2.9}$$

Using Lemma 2.2 and (2.9), by similar method as $\lambda \neq 0$,

$$\left|1+z\frac{s''(z)}{s'(z)}-z\frac{s'(z)}{s(z)}\right|\le\frac{2r(1-\alpha)}{\{(1-\alpha)(1+r)+\alpha(1-r)\}(1-r)}. \tag{2.10}$$

¿From Lemma 2.1 ($\lambda = 0$), we get

$$\left|1+z\frac{s''(z)}{s'(z)}-\frac{1+(1-2\alpha)r^2}{1-r^2}\right| \tag{2.11}$$

$$\leq \frac{4r(1-\alpha)\{1+(1-\alpha)r\}}{(1-r^2)\{(1-\alpha)(1+r)+\alpha(1-r)\}}.$$

□

Theorem 2.1. *If $s(z) \in \mathcal{S}^*_\lambda(\alpha)$ $(\lambda \neq 0)$, then $s(z)$ is convex in $|z| < R(\lambda,\alpha)$, where $R(\lambda,\alpha)$ is the smallest positive root of the equation*

$$\begin{aligned} T(r) = & r^3|\sin\lambda|\{2(1-\alpha)\cos^2\lambda - 1\} - r^2[2(1-\alpha)\{\cos^2\lambda|\sin\lambda| \\ & - (1-|\sin\lambda|)\cos\lambda\} - |\sin\lambda|] + r\{|\sin\lambda| \\ & + 2(1-\alpha)(1+|\sin\lambda|)\cos\lambda\} - |\sin\lambda|. \end{aligned} \tag{2,12}$$

The result is sharp.

Proof. ¿From Lemma 2.4, we obtain

$$\begin{aligned} \operatorname{Re}\left(1 + z\frac{s''(z)}{s'(z)}\right) & \\ \geq & -\frac{r^3|\sin\lambda|\{2(1-\alpha)\cos^2\lambda - 1\}}{(1-r)^2(1+r)|\sin\lambda|} \\ & + \frac{r^2[2(1-\alpha)\{\cos^2\lambda|\sin\lambda| - (1-|\sin\lambda|)\cos\lambda\} - |\sin\lambda|]}{(1-r)^2(1+r)|\sin\lambda|} \\ & - \frac{r\{|\sin\lambda| - 2(1-\alpha)(1+|\sin\lambda|)\cos\lambda\} - |\sin\lambda|}{(1-r)^2(1+r)|\sin\lambda|}. \end{aligned} \tag{2.13}$$

Since $T(0) < 0$ and $T(1) > 1$, there exists a real root of $T(r) = 0$ in $(0,1)$. Let $R(\lambda,\alpha)$ be the smallest positive root of $T(r) = 0$ in $(0,1)$. Then $s(z)$ is convex in $|z| < R(\lambda,\alpha)$. Sharpness is attained for the function,

$$s(z) = \frac{z}{(1-z)^{2(1-\alpha)\cos\lambda\exp(-i\lambda)}}. \tag{2.14}$$

□

Remark 1. In the case $\lambda = 0$, from Lemma 4(ii), we get

$$\begin{aligned} \operatorname{Re}\left(1 + z\frac{s''(z)}{s'(z)}\right) & \\ \geq & \frac{1+(1-2\alpha)r^2}{1-r^2} - \frac{4r(1-\alpha)\{1+r-\alpha r\}}{\{(1+r)(1-\alpha)+\alpha(1-r)\}(1-r^2)} \\ = & \frac{(1-2\alpha)^2r^3 - (4\alpha^2-6\alpha+3)r^2 + (2\alpha-3)r + 1}{(1-r^2)\{(1-\alpha)(1+r)+\alpha(1-r)\}}. \end{aligned} \tag{2.15}$$

We have $s(z)$ is convex in $|z| < R(\alpha)$, where $R(\alpha)$ is the smallest positive root of the equation

$$T(r) = (1-2\alpha)^2r^3 - (4\alpha^2-6\alpha+3)r^2 + (2\alpha-3)r + 1. \tag{2.16}$$

Remark 2. If $\alpha = 0$ in (2.16), we get $r = 2 - \sqrt{3}$. This result is obtained by Libera [2].

Theorem 2.2. *If $s(z) \in \mathcal{S}^*_\lambda(\alpha)$ $(\lambda \neq 0)$, then $s(z)$ is β-convex in $|z| < R(\lambda, \alpha, \beta)$, where $R(\lambda, \alpha, \beta)$ is the smallest positive root of the equation*

$$\begin{aligned} T(r) = & r^3|\sin\lambda|\{2(1-\alpha)\cos^2\lambda - 1\} - r^2[2(1-\alpha)\{\cos^2\lambda|\sin\lambda| \\ & - (\beta - |\sin\lambda|)\cos\lambda\} - |\sin\lambda|] \\ & + r\{2(1-\alpha)\cos\lambda(\beta + |\sin\lambda|) + |\sin\lambda|\} - |\sin\lambda|. \end{aligned} \tag{2.17}$$

The result is sharp.

Proof. ¿From the inequatlity (2.1), we have

$$\operatorname{Re} z\frac{s'(z)}{s(z)} \geq \frac{\{2(1-\alpha)\cos^2\lambda - 1\}r^2 - 2(1-\alpha)r\cdot\cos\lambda + 1}{1-r^2}. \tag{2.18}$$

For $0 \leq \beta \leq 1$, if we multiply both sides of (2.18) by $(1-\beta)$ and of (2.13) by β, then

$$\begin{aligned} & \operatorname{Re} J(\beta, s(z)) \\ \geq & \frac{-r^3|\sin\lambda|\{2(1-\alpha)\cos^2\lambda - 1\} + r^2[2(1-\alpha)\{\cos^2\lambda|\sin\lambda|}{(1-r)^2(1+r)|\sin\lambda|} \\ & \frac{-(\beta - |\sin\lambda|)\cos\lambda\} - |\sin\lambda|] - r\{2(1-\alpha)\cos\lambda(\beta + |\sin\lambda|)}{(1-r)^2(1+r)|\sin\lambda|} \\ & \frac{+|\sin\lambda|\} + |\sin\lambda|}{(1-r)^2(1+r)|\sin\lambda|}. \end{aligned} \tag{2.19}$$

Since $T(0) < 0$ and $T(1) > 0$, there exist a real root of $T(r) = 0$ in $(0, 1)$. Let $R(\lambda, \alpha, \beta)$ be the smallest positive root $T(r) = 0$ in $(0, 1)$. Then $s(z)$ is β-convex in $|z| < R(\lambda, \alpha, \beta)$. We obtain the sharpness for the extremal function given by (2.14).

□

Corollary 2.1. *If $\beta = 1$, then we obtain the radius of convexity for the class of λ-spirallike of order α functions which is given in* Theorem 2.1.

Corollary 2.2. *If $\beta = 0$, then*

$$r = \frac{(1-\alpha)\cos\lambda - \sqrt{1 - (1-\alpha^2)\cos^2\lambda}}{2(1-\alpha)\cos^2\lambda - 1}.$$

Remark 3. If $\alpha = 0$ in Corollary 2.2, then

$$r = \frac{1}{|\sin\lambda| + \cos\lambda}.$$

This is the radius of starlikness for λ-spirallike functions, which was obtained by Robertson [10] and Libera [2].

3. The radius of β-convexity for p-fold λ-spirallike functions

Theorem 3.1. *If $f(z) \in S^*_{\lambda p}(\alpha)$ $(\lambda \neq 0)$, then $f(z)$ is β-convex in $|z| < R(\lambda, \alpha, \beta, p)$, where $R(\lambda, \alpha, \beta, p)$ is the smallest positive root of the equation*

$$\begin{aligned} T(r) =& r^{3p}|\sin\lambda|\{2(1-\alpha)\cos^2\lambda - 1\} - r^{2p}[2(1-\alpha)\{\cos^2\lambda|\sin\lambda| \\ &- (\beta p - |\sin\lambda|)\cos\lambda\} - |\sin\lambda|] \\ &+ r^p\{2(1-\alpha)\cos\lambda(\beta p + |\sin\lambda|) + |\sin\lambda|\} - |\sin\lambda|. \end{aligned} \tag{3.1}$$

Proof. ¿From the relation (1.8) we obtain

$$1 + z^p\frac{s''(z^p)}{s'(z^p)} = \frac{1}{p}\left(1 + z\frac{f'(z)}{f(z)}\right) + \left(1 - \frac{1}{p}\right)z\frac{f'(z)}{f(z)}.$$

¿From a simple caculation of (1.8), (2.13) and (2.16), we obtain

$$\begin{aligned} &\operatorname{Re}\left\{J\left(\frac{1}{p}, f(z)\right)\right\} \\ &\geq -\frac{r^{3p}|\sin\lambda|\{2(1-\alpha)\cos^2\lambda - 1\}}{(1-r^p)^2(1+r^p)|\sin\lambda|} \\ &\quad + \frac{r^{2p}[2(1-\alpha)\{\cos^2\lambda|\sin\lambda| - (1 - |\sin\lambda|)\cos\lambda\} - |\sin\lambda|]}{(1-r^p)^2(1+r^p)|\sin\lambda|} \\ &\quad - \frac{r^p\{|\sin\lambda| + 2(1-\alpha)(1 + |\sin\lambda|)\cos\lambda\} - |\sin\lambda|}{(1-r^p)^2(1+r^p)|\sin\lambda|}, \end{aligned} \tag{3.2}$$

$$\operatorname{Re}\left\{z\frac{f'(z)}{f(z)}\right\} \geq \frac{\{2(1-\alpha)\cos^2\lambda - 1\}r^{2p} - 2(1-\alpha)r^p\cos\lambda + 1}{1 - r^{2p}}. \tag{3.3}$$

If we multiply both sides of (3.2) by γ and (3.3) by $1-\gamma$, and add the corresponding members, we obtain

$$\begin{aligned} &\operatorname{Re}\left\{J\left(\frac{\gamma}{p}, f(z)\right)\right\} \\ &\geq -\frac{r^{3p}|\sin\lambda|\{2(1-\alpha)\cos^2\lambda - 1\}}{(1-r^p)^2(1+r^p)|\sin\lambda|} \\ &\quad + \frac{r^{2p}[2(1-\alpha)\{\cos^2\lambda|\sin\lambda| - (\gamma - |\sin\lambda|)\cos\lambda\} - |\sin\lambda|]}{(1-r^p)^2(1+r^p)|\sin\lambda|} \\ &\quad - \frac{r^p\{2(1-\alpha)\cos\lambda(\gamma + |\sin\lambda|) + |\sin\lambda|\} - |\sin\lambda|}{(1-r^p)^2(1+r^p)|\sin\lambda|} \end{aligned} \tag{3.4}$$

where $0 \leq \gamma \leq 1$. If we take $\dfrac{\gamma}{p} = \beta$ the inequality (3.2) can be written in the form

$$\begin{aligned} &\operatorname{Re}\{J(\beta, f(z))\} \\ &\geq -\frac{r^{3p}|\sin\lambda|\{2(1-\alpha)\cos^2\lambda - 1\}}{(1-r^p)^2(1+r^p)|\sin\lambda|} \\ &\quad + \frac{r^{2p}[2(1-\alpha)\{\cos^2\lambda|\sin\lambda| - (\beta p - |\sin\lambda|)\cos\lambda\} - |\sin\lambda|]}{(1-r^p)^2(1+r^p)|\sin\lambda|} \\ &\quad - \frac{r^p\{2(1-\alpha)\cos\lambda(\beta p + |\sin\lambda|) + |\sin\lambda|\} - |\sin\lambda|}{(1-r^p)^2(1+r^p)|\sin\lambda|} \end{aligned} \tag{3.5}$$

where $0 \leq \beta \leq 1$. Since $T(0) < 0$ and $T(1) > 0$, there exist a real root of $T(r) = 0$ in $(0,1)$. Let $R(\lambda, \alpha, \beta, p)$ be the smallest positive root $T(r) = 0$ in $(0,1)$. Then $f(z)$ is β-convex in $|z| < R(\lambda, \alpha, \beta, p)$. We obtain the sharpness because the extremal function is $f(z) = z/(1-z^p)^{2(1-\alpha)\cos\lambda\exp(-i\lambda)/p}$. This shows that the theorem is true.

□

Corollary 3.1. *If $p = 1$, then we obtain the radius of β-convexity for the class of λ-spirallike of order α functions which is given in* Theorem 2.1.

Corollary 3.2. *If $\alpha = 0$, then we obtain the radius of β-convexity for the class of λ-spirallike functions.*

Corollary 3.3. *For $\beta = 0$ we obtain*

$$r = \sqrt[p]{\frac{(1-\alpha)\cos\lambda - \sqrt{1-(1-\alpha)\cos^2\lambda}}{2(1-\alpha)\cos^2\lambda - 1}}.$$

This is the radius of starlikeness for p-fold λ-spirallike functions. If we take $p = 1$, $\alpha = 0$ and $\beta = 0$, we obtain $r = (|\sin\lambda| + \cos\lambda)^{-1}$, which was obtained by Roberston [8] and Libera [2].

Corollary 3.4. *In the case $\lambda = 0$, we obtain the radius of β-convexity for the class of p-fold starlike of order α functions. If we take $\alpha = 0$ we obtain the radius of β-convexity for the class of p-fold starlike functions.*

For $p = 1$, $\beta = 0$, $\lambda = 0$ and $\alpha = 0$, we obtain $r = 2 - \sqrt{3}$, which was obtained by Libera [2].

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