Scientiae Mathematicae Japonicae Online, **Vol. 5**, (2001), 545–548

LÖWNER-HEINZ THEOREM AND OPERATOR MEANS

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Received October 9, 2001; revised October 17, 2001

Abstract. Based on the Kubo-Ando theory of operator means, we give a proof of the well-known Löwner-Heinz theorem which asserts that for bounded linear operators A and B, if $A \geq B \geq 0$ then $A^p \geq B^p$ for $0 \leq p \leq 1$. A key fact for the proof of the theorem is its special case for $p = 1/2$: if $A \geq B \geq 0$ then $A^{1/2} \geq B^{1/2}$, which says that the geometric mean $X^{1/2}$ of the identity operator 1 and a positive operator X is monotone. We give a short proof of this fact, using the arithmetic-harmonic mean defined by J. I. Fujii.

1. Throughout this note, a capital letter means a (bounded linear) operator on a Hilbert space H. An operator A is said to be positive, denoted by $A \geq 0$, if $(Ax, x) \geq 0$ for all $x \in H$. Then it induces the order $A \geq B$ for selfadjoint operators A and B. The following result called Löwner-Heinz theorem [8], [12] is well-known:

Theorem A. *Let A and B be positive operators. Then*

$$A \geq B \quad \textit{implies} \quad A^p \geq B^p \quad \textit{for} \quad 0 \leq p \leq 1. \tag{1}$$

This theorem says that the function $t \mapsto t^p$ with $0 \leq p \leq 1$ is operator monotone on $[0, \infty)$. Recently Furuta [7] presented an exquisite extension of the inequality (1), called Furuta inequality, which enjoys the great worth of the theorem.

The proof of the theorem was initiated by Löwner [12] in the complete description of operator monotone functions, later a clear expression of the theorem was given by Heinz [8], and a completely operator theoretic proof was given by Kato [10]. A lot of authors since then gave proofs of the theorem ([1], [3], [9], [13], etc.). Among them it is noted that Pedersen [13] gave a proof, using fundamental properties of the spectral radius of an operator, and that Ando [1] obtained the theorem from operator monotonicity of the geometric mean defined on positive operators.

A proof of the theorem is to take full advantage of the following reduced inequality for $p = 1/2$, that is,

Theorem B. *Let A and B be positive operators. Then*

$$A \geq B \quad \textit{implies} \quad A^{1/2} \geq B^{1/2}. \tag{2}$$

Several authors ([2], [6], [9], [13], [14], etc.) have already indicated, explicitly or implicitly, that Theorem B is equivalent to Theorem A.

In this note, we first give an elementary proof of Theorem B, using a technique due to J. I. Fujii [4], [5], and next show a proof of Theorem A based on the Kubo-Ando theory of operator means [11].

2000 *Mathematics Subject Classification.* 47A63, 47A30.

Key words and phrases. positive operator, operator monotone function, Löwner-Heinz theorem, operator mean.

2. Following [11], we recall the basic three operator means: For $A,\ B \geq 0$,

$$(A) \quad \textit{arithmetic mean} \qquad A\nabla B = \frac{1}{2}(A+B),$$

$$(H) \quad \textit{harmonic mean} \qquad A \,!\, B = \left\{\frac{1}{2}(A^{-1}+B^{-1})\right\}^{-1} (= 2A(A+B)^{-1}B),$$

and

$$(G) \quad \textit{geometric mean} \qquad A\#B = A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}.$$

In the above definitions (H) and (G), both A and B (or at least one of them) must be assumed to be invertible. Without any assumption they are well-defined as the (strong operator) limits of $(A+\varepsilon 1) \,!\, (B+\varepsilon 1)$ and $(A+\varepsilon 1)\#(B+\varepsilon 1)$ as $\varepsilon \downarrow 0$ respectively. (1 is the identity operator.) For simplicity of discussions, from now on we assume that all positive operators are *invertible*.

Among the three operator means $(A), (H)$ and (G), the following fundamental inequalities hold (cf. [11]):

$$A\nabla B \geq A\#B \geq A \,!\, B. \tag{3}$$

(It is very easy to obtain these inequalities, in particular, if A and B commute.) For the monotone property of those means, we can easily see that if $A \geq B$ and $C \geq D$ then

$$A\nabla C \geq B\nabla D \quad \text{and} \quad A \,!\, C \geq B \,!\, D. \tag{4}$$

However, it need some device to obtain

$$A\#C \geq B\#D. \tag{5}$$

Since $A\#1 = A^{1/2}$, we see that (2) of Theorem B is nothing but a special case of (5). Now at the present, we prove the special case, employing the arithmetic-harmonic mean technique presented by J. I. Fujii [4], [5]:

Proof of Theorem B. First let $X_0 = 1,\ Y_0 = A$, and following [4], [5], define

$$X_n = X_{n-1}\nabla Y_{n-1} \quad \text{and} \quad Y_n = X_{n-1} \,!\, Y_{n-1} \quad (n = 1, 2, \dots).$$

Then we see that X_n and Y_n commute for all n, and that

$$X_nY_n = X_{n-1}Y_{n-1} = \cdots = X_0Y_0 = A.$$

Since $X_n \geq Y_n$ for $n \geq 1$ by (3) we can see that

$$X_1 \geq \cdots \geq X_n \geq Y_n \geq \cdots \geq Y_1.$$

Furthermore, $2(X_n - X_{n+1}) = X_n - Y_n \geq 0$, so that we obtain $A^{1/2} = 1\#A$ as the common limit of $\{X_n\}$ and $\{Y_n\}$, which is nothing but the arithmetic-harmonic mean of $X_0 = 1$ and $Y_0 = A$. Next in the same manner as before, putting $Z_0 = 1, W_0 = B$,

$$Z_n = Z_{n-1}\nabla W_{n-1} \quad \text{and} \quad W_n = Z_{n-1} \,!\, W_{n-1} \quad (n = 1, 2, \dots),$$

we can similarly obtain $B^{1/2}$ as the common limit of $\{Z_n\}$ and $\{W_n\}$. It is also not difficult to see that $X_n \geq Z_n$ and $Y_n \geq W_n$ for $n = 1, 2, \ldots$. Taking the limits, we then obtain $A^{1/2} \geq B^{1/2}$. □

From the definition (G) and Theorem B we now obtain the following fact, a little weaker than (5). (Afterwords we shall mention of (5) again.)

$$A \geq B \quad \text{and} \quad C \geq 0 \quad \text{imply} \quad C\#A \geq C\#B. \tag{6}$$

In fact, since $DAD \geq DBD$ for any $D \geq 0$, we have, applying Theorem B,

$$C\#A = C^{1/2}(C^{-1/2}AC^{-1/2})^{1/2}C^{1/2} \geq C^{1/2}(C^{-1/2}BC^{-1/2})^{1/2}C^{1/2} = C\#B.$$

At this stage we give

Proof of Theorem A. (cf. [6, Lemma 1]) By norm continuity of $p \mapsto A^p$, we may show that (1) holds for every p such that $p = m/2^k$, $k = 1, 2, \ldots$ and $m = 1, 2, \ldots, 2^k$ for each k. We take the mathematical induction with respect to k. For the first step, (1) clearly holds for $k = 1$. For the next step, assuming that (1) holds for $k = n$, we may show it for $k = n+1$. We then consider the two cases (i) $1 \leq m \leq 2^n$ and (ii) $2^n + 1 \leq m \leq 2^{n+1}$. For (i), since $A^{m/2^n} \geq B^{m/2^n}$ by assumption, we have $A^{m/2^{n+1}} = 1\#A^{m/2^n} \geq 1\#B^{m/2^n} = B^{m/2^{n+1}}$. For (ii),

$$\begin{aligned}
A^{m/2^{n+1}} &= B^{1/2}\{(B^{-1/2}A^{m/2^{n+1}}B^{-1/2})^2\}^{1/2}B^{1/2} \\
&= B\#(A^{m/2^{n+1}}B^{-1}A^{m/2^{n+1}}) \\
&\geq B\#(A^{m/2^{n+1}}A^{-1}A^{m/2^{n+1}}) \quad (\text{by } B^{-1} \geq A^{-1} \text{ and } (6)) \\
&= B\#A^{m/2^n - 1} \\
&\geq B\#B^{m/2^n - 1} \quad (\text{by } A^{m/2^n - 1} \geq B^{m/2^n - 1} \text{ and } (6)) \\
&= B^{m/2^{n+1}}.
\end{aligned}$$

□

Remark 1. By uniqueness of the square root of a positive operator we can see that $X = A\#B$ if and only if $XA^{-1}X = B$ or $(A^{-1/2}XA^{-1/2})^2 = A^{-1/2}BA^{-1/2}$ for $X \geq 0$. Since $XA^{-1}X = B$ is equivalent to $XB^{-1}X = A$, we have

$$A\#B = B\#A. \tag{7}$$

With this identity (7) and the inequality (6) we now obtain the desired (5) as follows : if $A \geq B$ and $C \geq D$, then

$$A\#C \geq A\#D = D\#A \geq D\#B = B\#D.$$

In [1], Ando defined $A\#B$ by

$$A\#B = \max\left\{X \geq 0; \begin{bmatrix} A & X \\ X & B \end{bmatrix} \geq 0\right\},$$

which is equivalent to the definition (G). From this definition he obtained (5) immediately.

Remark 2. Using (5), we can give another proof of Theorem A (cf. [1, Corollary I. 2. 2]): Let $\triangle = \{p \in [0,1]; A^p \geq B^p\}$. Then by norm continuity of $p \mapsto A^p$, $\triangle$ is closed, so that we may only show that $\triangle$ is convex. Let $q, r \in \triangle$, that is, $A^q \geq B^q$ and $A^r \geq B^r$. Then

$$A^{(q+r)/2} = A^q \# A^r \geq B^q \# B^r = B^{(q+r)/2},$$

which implies $(q+r)/2 \in \triangle$, that is, convexity of $\triangle$.

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