ON THE RATES OF POINTWISE APPROXIMATION OF CONJUGATE FUNCTIONS

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Abstract. There are estimated the rates of approximation of function $\widetilde{f}$, conjugated to the Lebesgue integrable function f , by the Cesàro and Abel-Poisson means of conjugate Fourier series. As a measure of such approximation the characteristics constructed on the base of definition of the points of differentiation of the indefinite Lebesgue integral are used.

1. Introduction

Let L be the class of all real-valued functions $2\pi-$periodic and Lebesgue-integrable on the interval $[-\pi,\pi]$. Let $S[f]$ be the trigonometric Fourier series of $f\in L$,and let $\widetilde{S}[f]$ be the conjugate trigonometric Fourier series

$$\sum_{k=1}^{\infty}\left[a_k\left(f\right)\sin kx-b_k\left(f\right)\cos kx\right], \tag{1}$$

of f, and let $\widetilde{f}$ be the conjugate function of f, i.e.([1],p.519 or [7],p.51,I)

$$\widetilde{f}(x)=-\frac{1}{\pi}\int_0^{\pi}\frac{f\left(x+t\right)-f\left(x-t\right)}{2\tan(t/2)}dt=-\frac{1}{\pi}\lim_{\epsilon\to0^+}\widetilde{f}(x;\epsilon), \tag{2}$$

where

$$\widetilde{f}(x;\epsilon):=\int_{\epsilon}^{\pi}\frac{f\left(x+t\right)-f\left(x-t\right)}{2\tan(t/2)}dt.$$

It is known ([7],p.254,I) that if $\widetilde{f}\in L$, then $\widetilde{S}[f]=S[\widetilde{f}]$.

(i) Let $\widetilde{\sigma}_n^{\alpha}\left(f;x\right)=\widetilde{\sigma}_n^{\alpha}\left(x\right)$ be the (C,α) means of $\widetilde{S}[f]$,where $(\alpha\geq0)$ (see [7] ,p.95,I), thus

$$l\widetilde{\sigma}_n^{\alpha}\left(x\right)=-\frac{1}{\pi}\int_0^{\pi}[f\left(x+t\right)-f\left(x-t\right)]\widetilde{K}_n^{\alpha}(t)dt, \tag{3}$$

where

$$\widetilde{K}_n^{\alpha}\left(t\right)=\sum_{\nu=0}^{n}A_{n-\nu}^{\alpha-1}\widetilde{D}_{\nu}\left(t\right)/A_n^{\alpha}\qquad\qquad (n=0,1,2,...)$$

is the conjugate (C,α) kernel and because

$$\widetilde{D}_{\nu}\left(t\right)=\sum_{\mu=1}^{\nu}\sin\mu t=\frac{\cos\frac{t}{2}-\cos\left(\nu+\frac{1}{2}\right)t}{2\sin\frac{t}{2}}$$

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so we can rewritten

$$\widetilde{K}_n^{\alpha}(t) = \frac{1}{2}ctg\frac{1}{2}t - \frac{1}{A_n^{\alpha}}\sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1}\frac{\cos\left(\nu+\frac{1}{2}\right)t}{2\sin\frac{t}{2}} =: \frac{1}{2}ctg\frac{1}{2}t - \widetilde{H}_n^{\alpha}(t). \tag{4}$$

(ii) Let $\widetilde{H}_r(f,x;h) = \widetilde{H}_r(x;h)$ be the generalized Abel-Poisson means of $\widetilde{S}[f]$, where h is a suitable real-valued, bounded function defined on $[r_0;1]$, $(r_0>0)$, (see [5], p.105). Then

$$\widetilde{H}_r(x;h) = \sum_{k=1}^{\infty}(1+(1-r)kh(r))r^k\left[a_k(f)\sin kx - b_k(f)\cos kx\right],$$

or in the integral form

$$\widetilde{H}_r(x;h) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x-t)\widetilde{J}_r(t;h)\,dt, \tag{5}$$

where the conjugate kernel $\widetilde{J}_r(t;h) = \widetilde{J}_r(t)$ is given by the formula

$$\widetilde{J}_r(t) = \sum_{k=1}^{\infty}(1+(1-r)kh(r))r^k\sin kt \qquad (0<r<1).$$

Given any function $f\in L$ we write

$$w_x(f;\delta) = w_x(\delta) = \sup_{0<u\le\delta}\left|\frac{1}{\delta}\int_0^u \psi_x(t)dt\right|, \tag{6}$$

where $\psi_x(t) =: f(x+t) - f(x-t)$ .We observe that the function $\delta w_x(\delta)$ is non - decreasing in δ and $w_x(\delta)\to 0$, $(\delta\to 0+)$ almost everywhere in x. The symbols C_j $(j=1,2,...)$ will denote the suitable positive constants depending sametimes on parameters α or β. Assuming further that

$$\Omega_x(u) := \int_0^u \psi_x(t)\,dt,$$

it is easy to show that

$$|\Omega_x(u)| \le u w_x(u). \tag{7}$$

Further we use the following relations:

1) Let

$$u(\beta,n,t) = \sum_{v=0}^{n} A_v^{\beta}e^{-ivt},$$

then summation by parts gives

$$u(\beta,n,t) = \left\{-A_n^{\beta}e^{-i(n+1)t} + u(\beta-1,n,t)\right\}(1-e^{-it})^{-1}. \tag{8}$$

2) If $0\le\alpha\le m+1$, $m\in N$ and $n=1,2...$ then

$$\left|\widetilde{K}_n^{\alpha}(t)\right| \le C_1 n, \qquad (0\le t\le\pi), \tag{9}$$

$$\left|\frac{d^m}{dt^m}\widetilde{K}_n^{\alpha}(t)\right| \leq C_2 n^{m+1}, \qquad (0 \leq t \leq \pi), \tag{10}$$

$$\left|\frac{d^m}{dt^m}\widetilde{H}_n^{\alpha}(t)\right| \leq C_3 \frac{1}{n^{\alpha-m}t^{\alpha+1}}, \qquad (\frac{1}{n} \leq t \leq \pi), \tag{11}$$

(cf.[7],p.64,II).

2. Main results

Privalov proved that the conjugate-Fejér sums $\widetilde{\sigma}_n^1(f;x)$ of f at x will satisfy the relation

$$\lim_{n\to\infty}\left(\widetilde{\sigma}_n^1(f;x) - \widetilde{f}(x;\frac{1}{n})\right) = 0$$

provided that x is a $\widetilde{B}$ point of f, i.e.

$$\int_0^h |f(x+t) - f(x-t)|\, dt = o(h), \qquad (\mathrm{h} \to 0).$$

Now, we will extend this result to the case (C,α) means with $\alpha > 1$ and present it in approximation version

THEOREM 1. *If* $f \in L$ *and* $\alpha > 1$, *then*

$$\left|\widetilde{\sigma}_n^{\alpha}(f,x) - \widetilde{f}(x;\frac{1}{n})\right| \leq \frac{C_4}{n^{\alpha-1}} \int_{1/n}^{\pi} \frac{w_x(f,t)}{t^{\alpha}}dt \tag{12}$$

for all real x *and every positive integer* n.

From Theorem 1 we can immediately derive

COROLLARY 1. *If* $f \in L$, $\alpha > 1$ *and* $w_x(t) = O(t^{\gamma})$, *for* $\gamma > 0$, *then*

$$\left|\widetilde{\sigma}_n^{\alpha}(f,x) - \widetilde{f}(x;\frac{1}{n})\right| \leq \begin{cases} C_5 n^{1-\alpha} & if \quad \gamma > \alpha - 1, \\ C_6 n^{-\gamma}\ln n & if \quad \gamma = \alpha - 1, \\ C_7 n^{-\gamma} & if \quad \gamma < \alpha - 1. \end{cases} \tag{13}$$

PROOF of THEOREM 1. It is easy to see, that

$$\left|\widetilde{\sigma}_n^{\alpha}(x) - \widetilde{f}(x;\frac{1}{n})\right| \leq \frac{1}{\pi}\left|\int_0^{1/n}\psi_x(t)\,\widetilde{K}_n^{\alpha}(t)\,dt\right| + \frac{1}{\pi}\left|\int_{1/n}^{\pi}\psi_x(t)\,\widetilde{H}_n^{\alpha}(t)\,dt\right| =: \frac{1}{\pi}|I_n| + \frac{1}{\pi}|Y_n|.$$

By partial integration we obtain

$$|I_n| \leq \left|\Omega_x\left(\frac{1}{n}\right)\widetilde{K}_n^{\alpha}\left(\frac{1}{n}\right)\right| + \left|\int_0^{1/n}\Omega_x(t)\,\frac{d}{dt}\widetilde{K}_n^{\alpha}(t)\,dt\right|,$$

$$|Y_n| \leq \left|\Omega_x\left(\frac{1}{n}\right)\widetilde{H}_n^{\alpha}\left(\frac{1}{n}\right)\right| + \left|\int_{1/n}^{\pi}\Omega_x(t)\,\frac{d}{dt}\widetilde{H}_n^{\alpha}(t)\,dt\right|.$$

1) Case 1< $\alpha<2$

$$\widetilde{H}_n^{\alpha}(t)=\frac{1}{A_n^{\alpha}}\sum_{\nu=0}^{n}A_{n-\nu}^{\alpha-1}\frac{\cos\left(\nu+\frac{1}{2}\right)t}{2\sin\frac{t}{2}}=Re\{\frac{1}{A_n^{\alpha}}\frac{e^{i(n+\frac{1}{2})t}}{2\sin\frac{t}{2}}\sum_{\nu=0}^{n}A_{\nu}^{\alpha-1}e^{-i\nu t}\}.$$

For the sum in the curly brackets we use the relation (8)

$$\widetilde{H}_n^{\alpha}(t)=Re\{\frac{-ie^{i(n+1)t}}{A_n^{\alpha}(2\sin\frac{t}{2})^2}\sum_{\nu=0}^{n}A_{\nu}^{\alpha-2}e^{-i\nu t}\},$$

so

$$\left|\widetilde{H}_n^{\alpha}(t)\right|\leq\frac{1}{A_n^{\alpha}(2\sin\frac{t}{2})^2}\left|Re\{-ie^{i(n+1)t}\sum_{\nu=0}^{n}A_{\nu}^{\alpha-2}e^{-i\nu t}\}\right|=$$

$$=\frac{1}{A_n^{\alpha}(2\sin\frac{t}{2})^2}\left|Re\{-ie^{i(n+1)t}\left(\sum_{\nu=0}^{\infty}A_{\nu}^{\alpha-2}e^{-i\nu t}-\sum_{\nu=n+1}^{\infty}A_{\nu}^{\alpha-2}e^{-i\nu t}\right)\}\right|.$$

The Cesàro numbers $A_{\nu}^{\alpha-2}$are positive,decreasing in ν (-1<$\alpha-2<0$) and the sequence of the sums $\sum\limits_{\nu=n+1}^{\infty}e^{-i\nu t}$ is bounded. Then from the theorem of Abel, we have

$$\left|\sum_{\nu=n+1}^{\infty}A_{\nu}^{\alpha-2}e^{-i\nu t}\right|\leq 2A_{n+1}^{\alpha-1}\left|1-e^{-it}\right|^{-1}.$$

Thus, for $\frac{1}{n}\leq t\leq\pi$

$$\left|\widetilde{H}_n^{\alpha}(t)\right|\leq\frac{1}{A_n^{\alpha}(2\sin\frac{t}{2})^2}\left|1-e^{-it}\right|^{-(\alpha-1)}+\frac{2A_{n+1}^{\alpha-2}}{A_n^{\alpha}(2\sin\frac{t}{2})^2}\left|1-e^{-it}\right|^{-1}\leq$$

$$\leq\frac{C_8}{n^{\alpha}t^{\alpha+1}}+\frac{C_9}{n^2t^3}\leq\frac{C_{10}}{n^{\alpha}t^{\alpha+1}}. \tag{14}$$

Furhermore, using the estimates (9), (10),(11) and (14) we obtain

$$\left|\widetilde{\sigma}_n^{\alpha}(x)-\widetilde{f}(x;\frac{1}{n})\right|\leq\frac{1}{\pi}\left(\frac{1}{n}w_x\left(\frac{1}{n}\right)n+C_{11}n^2\int_0^{1/n}\left|\Omega_x(t)\right|dt\right)+$$

$$+\frac{1}{\pi}\left(\frac{1}{n}w_x\left(\frac{1}{n}\right)n+\frac{C_{10}}{n^{\alpha-1}}\int_{1/n}^{\pi}\left|\Omega_x(t)\right|\frac{1}{t^{\alpha+1}}dt\right)\leq\frac{C_4}{n^{\alpha-1}}\int_{1/n}^{\pi}\frac{w_x(t)}{t^{\alpha}}dt.$$

2) Case $\alpha\geq 2$

$$\widetilde{H}_n^{\alpha}(t)=Re\{\frac{-ie^{i(n+1)t}}{A_n^{\alpha}(2\sin\frac{t}{2})^2}\sum_{\nu=0}^{n}A_{\nu}^{\alpha-2}e^{-i\nu t}\}.$$

Similarly, for the sum in the curly brackets we use the relation (8)

$$\widetilde{H}_n^{\alpha}(t)=Re\{\frac{-1}{A_n^{\alpha}(2\sin\frac{t}{2})^3}\left(\sum_{\nu=0}^{n}A_{\nu}^{\alpha-3}e^{i(n+\frac{3}{2}-\nu)t}-A_n^{\alpha-2}e^{i\frac{t}{2}}\right)$$

and

$$\left|\widetilde{H}_n^{\alpha}(t)\right| \leq \frac{1}{A_n^{\alpha}\left|(2\sin\frac{t}{2})^3\right|}\sum_{\nu=0}^{n}A_{\nu}^{\alpha-3} + \frac{A_n^{\alpha-2}}{A_n^{\alpha}\left|(2\sin\frac{t}{2})^3\right|} \leq$$

$$\leq \frac{C_{11}}{n^{\alpha}t^3}\sum_{\nu=0}^{n}A_{\nu}^{\alpha-3} + \frac{C_{12}}{n^2t^3} \leq \frac{C_{13}}{n^2t^3}.$$

Thus the proof of the first theorem is completed.

In the sequel M_j $(j = 1, 2, ...)$, constructed to the $\widetilde{H}_r$, will denote the suitable absolute positive constants or constans depending on γ.

THEOREM 2. *If* $f \in L$ *and* $r \in \left[\frac{1}{2}, 1\right)$, *then*

$$\left|\widetilde{H}_r(f, x; h) - \widetilde{f}(x; \pi(1-r))\right| \leq$$

$$\leq 130\pi^3[1 + |1 - h(r)|](1-r)^2 \int_{\pi(1-r)}^{\pi} \frac{w_x(t)}{t^3}dt. \tag{15}$$

We notice that, if $h(r) \equiv 0$, then H_r is the ordinary Abel - Poisson mean and the inequality (15) leads us to the inequality (see[3])

$$\left|\widetilde{H}_r(f, x; h) - \widetilde{f}(x; \pi(1-r))\right| \leq M_1 (1-r)^2 \int_{\pi(1-r)}^{\pi} \frac{w_x(t)}{t^3}dt.$$

From Theorem 2 we can also derive

COROLLARY 2. *If* $f \in L$, $r \in \left[\frac{1}{2}, 1\right)$ *and* $w_x(t) = O(t^{\gamma})$, *where* $\gamma > 0$, *then*

$$\left|\widetilde{H}_r(x; h) - \widetilde{f}(x; \pi(1-r))\right| \leq \begin{cases} M_2 (1-r)^{\gamma}[1 + |1 - h(r)|] & if \quad 0 < \gamma < 2; \\ M_3 (1-r)^2 [1 + |1 - h(r)|]\,|\ln(1-r)| & if \quad \gamma = 2. \end{cases}$$

PROOF of THEOREM 2. It is easy to see that

$$\left|\widetilde{H}_r(x; h) - \widetilde{f}(x; \pi(1-r))\right| \leq \left|\widetilde{H}_r(x; h) - \widetilde{B}_r(x)\right| + \left|\widetilde{B}_r(x) + \frac{1}{\pi}\int_{\pi(1-r)}^{\pi} \frac{f(x+t) - f(x-t)}{2\tan(t/2)}dt\right|,$$

where $\widetilde{B}_r$ is defined as

$$\widetilde{B}_r(x) = \widetilde{B}_r(f, x) = \sum_{k=1}^{\infty}\left\{1 + \frac{k}{2}\left(1 - r^2\right)\right\} r^k(a_k \sin kx - b_k \cos kx) =$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t) Q_r(x-t)\,dt,$$

and

$$Q_r(t) = \frac{r(3 + r^2 - 4r\cos t)\sin t}{2(1 + r^2 - 2r\cos t)^3}$$

(see [4], p.159). The method of this proof goes in the similar way as in ([5] and [4]). Thus, we can show that

$$\left|\widetilde{H}_r(x;h)-\widetilde{B}_r(x)\right|\leq\frac{2}{\pi}\left|h\left(r\right)-\frac{1+r}{2}\right|(1-r)^2\left|\int\limits_0^{\pi}\psi_x(t)R_r\left(t\right)dt\right|,$$

where

$$(1-r^2)rR_r\left(t\right):=\sum_{k=1}^{\infty}kr^k\sin kx=\frac{(1-r^2)r\sin t}{(1+r^2-2r\cos t)^2}.$$

By partial integration we obtain

$$\left|\int\limits_0^{\pi}\psi_x(t)R_r\left(t\right)dt\right|=\left|R_r\left(\pi\right)\Omega_x\left(\pi\right)-\int\limits_0^{\pi}\Omega_x\left(t\right)\frac{d}{dt}R_r\left(t\right)dt\right|=$$

$$=\left|\left(\int\limits_0^{\pi(1-r)}+\int\limits_{\pi(1-r)}^{\pi}\right)\Omega_x\left(t\right)\frac{d}{dt}R_r\left(t\right)dt\right|\leq\left|I_1\right|+\left|I_2\right|.$$

Since

$$\frac{d}{dt}R_r\left(t\right)=\frac{\cos t[(1-r)^2+4r\sin^2\frac{t}{2}]-4r\sin^2 t}{[(1-r)^2+4r\sin^2\frac{t}{2}]^3}$$

and

$$\left|\frac{d}{dt}R_r\left(t\right)\right|\leq\begin{cases}\frac{1}{(1-r)^4}+\frac{5t^2}{(1-r)^6} & if\quad 0\leq t\leq\pi(1-r),\\ 3\pi^4t^{-4} & if\quad \pi(1-r)\leq t\leq\pi,\end{cases}$$

then

$$\left|I_1\right|\leq\frac{1}{(1-r)^4}\int\limits_0^{\pi(1-r)}\left|\Omega_x\left(t\right)\right|dt+\frac{5}{(1-r)^6}\int\limits_0^{\pi(1-r)}\left|\Omega_x\left(t\right)\right|t^2dt\leq$$

$$\leq\frac{1}{(1-r)^4}\int\limits_0^{\pi(1-r)}tw_x\left(t\right)dt+\frac{5}{(1-r)^6}\int\limits_0^{\pi(1-r)}t^3w_x\left(t\right)dt,$$

and by the monotonicity of the function $\delta w_x\left(\delta\right)$ we have

$$\left|I_1\right|\leq\frac{18\pi^2}{(1-r)^2}w_x\left(\pi(1-r)\right).$$

The second component $\left|I_2\right|$ we can immediatly estimate as follows

$$\left|I_2\right|\leq3\pi^4\int\limits_{\pi(1-r)}^{\pi}\frac{\left|\Omega_x\left(t\right)\right|}{t^4}dt\leq3\pi^4\int\limits_{\pi(1-r)}^{\pi}\frac{w_x\left(t\right)}{t^3}dt.$$

We observe that

$$(1-r)^2\int\limits_{\pi(1-r)}^{\pi}\frac{w_x\left(t\right)}{t^3}dt\geq\frac{7}{24\pi^2}w_x\left(\pi(1-r)\right).$$

Thus

$$\left|\widetilde{H}_r(x;h)-\widetilde{B}_r(x)\right|\leq 130\pi^3\left|h\left(r\right)-\frac{1+r}{2}\right|(1-r)^2\int\limits_{\pi(1-r)}^{\pi}\frac{w_x\left(t\right)}{t^3}dt.$$

Similarly, a simple calculation shows that

$$\left|\widetilde{B}_r(x)+\frac{1}{\pi}\int\limits_{\pi(1-r)}^{\pi}\frac{f\left(x+t\right)-f\left(x-t\right)}{2\tan(t/2)}dt\right|=\frac{1}{\pi}\left|\int\limits_{0}^{\pi(1-r)}\psi_x(t)Q_r\left(t\right)dt-\int\limits_{\pi(1-r)}^{\pi}\psi_x(t)P_r\left(t\right)dt\right|,$$

where

$$P_r\left(t\right):=\frac{1}{2}ctg\frac{t}{2}-Q_r\left(t\right).$$

From ([4],p.167,168) we know that

$$\left|\widetilde{B}_r(x)+\frac{1}{\pi}\int\limits_{\pi(1-r)}^{\pi}\frac{f\left(x+t\right)-f\left(x-t\right)}{2\tan(t/2)}dt\right|\leq\frac{1}{2\pi}\left|\Omega_x\left(\pi(1-r)\right)ctg\frac{\pi(1-r)}{2}\right|+$$

$$+\frac{1}{\pi}\left|\int\limits_{0}^{\pi(1-r)}\Omega_x\left(t\right)\frac{d}{dt}Q_r\left(t\right)dt\right|+\frac{1}{\pi}\left|\int\limits_{\pi(1-r)}^{\pi}\Omega_x\left(t\right)\frac{d}{dt}P_r\left(t\right)dt\right|.$$

The first component in the above inequality we can easly estimate

$$\frac{1}{2\pi}\left|\Omega_x\left(\pi(1-r)\right)ctg\frac{\pi(1-r)}{2}\right|\leq\frac{1}{2}w_x\left(\pi(1-r)\right).$$

Since

$$\left|\frac{d}{dt}Q_r\left(t\right)\right|\leq\frac{3}{(1-r)^2}+\frac{11t^2}{(1-r)^4}+\frac{2t^4}{(1-r)^6}\qquad\text{for } 0\leq t\leq\pi(1-r),$$

$$\left|\frac{d}{dt}P_r\left(t\right)\right|\leq\frac{11\pi^4(1-r)^2}{8t^4}+\frac{5\pi^6(1-r)^4}{16t^6}\qquad\text{for } \pi(1-r)\leq t\leq\pi,$$

then

$$\left|\int\limits_{0}^{\pi(1-r)}\Omega_x\left(t\right)\frac{d}{dt}Q_r\left(t\right)dt\right|\leq 79\pi w_x\left(\pi(1-r)\right),$$

$$\left|\int\limits_{\pi(1-r)}^{\pi}\Omega_x\left(t\right)\frac{d}{dt}P_r\left(t\right)dt\right|\leq\frac{5\pi^5}{16}(1-r)^4\int\limits_{\pi(1-r)}^{\pi}\frac{w_x\left(t\right)}{t^5}dt+\frac{11\pi^3}{8}(1-r)^2\int\limits_{\pi(1-r)}^{\pi}\frac{w_x\left(t\right)}{t^3}dt.$$

Taking together the above estimates and observing that

$$(1-r)^2\int\limits_{\pi(1-r)}^{\pi}\frac{w_x\left(t\right)}{t^3}dt\geq\frac{7}{24\pi^2}w_x\left(\pi(1-r)\right),$$

$$(1-r)^4 \int_{\pi(1-r)}^{\pi} \frac{w_x(t)}{t^5} dt \le \frac{1}{\pi^2}(1-r)^2 \int_{\pi(1-r)}^{\pi} \frac{w_x(t)}{t^3} dt,$$

we obtain

$$\left| \widetilde{B}_r(x) + \frac{1}{\pi} \int_{\pi(1-r)}^{\pi} \frac{f(x+t) - f(x-t)}{2\tan(t/2)} dt \right| \le 274\pi^3 (1-r)^2 \int_{\pi(1-r)}^{\pi} \frac{w_x(t)}{t^3} dt$$

and thus the proof is completed.

REMARK. From our estimations we can obtain the analogical inequalities for the norm approximation. Using the classical modulus of continuity

$$\omega(f,\delta)_X = \sup_{0<h\le\delta} \|f(\circ + h) - f(\circ)\|_X$$

with respect to the suitable norms in the spaces $X = X^p = C$, where $p = \infty$ or $X = X^p = L^p$, where $1 \le p < \infty$, we obtain the estimations

$$\left\| \widetilde{\sigma}_n^{\alpha}(f,\circ) - \widetilde{f}(\circ, \frac{1}{n}) \right\|_X \le \frac{C_{14}}{n^{\alpha-1}} \int_{1/n}^{\pi} \frac{\omega(f,t)_X}{t^{\alpha}} dt,$$

$$\left\| \widetilde{H}_r(f,\circ;h) - \widetilde{f}(\circ, \pi(1-r)) \right\|_X \le M_4 \left[|1-h(r)| + 1 \right] (1-r)^2 \int_{\pi(1-r)}^{\pi} \frac{\omega(f,t)_X}{t^3} dt,$$

for $f \in X$, every positive integer n, $\alpha > 1$, and all $r \in \left[\frac{1}{2}, 1\right)$.

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