Scientiae Mathematicae Japonicae Online, **Vol. 7**, (2002), 1–7

MONOIDS WITH SUBGROUPS OF FINITE INDEX AND THE BRAID INVERSE MONOID

Isamu Inata and Yuji Kobayashi

Dedicated to Professor Masami Ito on his 60th birthday

Received December 10, 2001

Abstract. We investigate finite presentability of monoids with finitely presented subgroups of finite index. If such a monoid satisfies some additional conditions, we can find a finite presentation of it. As an application of the result, we exhibit a finite presentation of the braid inverse monoid. The braid inverse monoid naturally contains the braid group as a subgroup of finite index.

1 Introduction.

The following result is well-known (see for example [3]).

Result 1.1 *Let H be a subgroup of a group G of finite index. If H is finitely presented, then G is also finitely presented.*

In the case of monoids, a similar result holds for very special submonoids. A submonoid N of a monoid M such that $M \setminus N$ is a finite set is called a *large submonoid* of M. The following result can be found in [4].

Result 1.2 *Let N be a large submonoid of a monoid M. If N is finitely presented, then M is also finitely presented.*

We are interested in generalizing the above results. For a submonoid N of a monoid M, N is said to have *finite right* (*resp. left*) *index* in M, if there is a finite subset C of M such that $M = \bigcup_{x \in C} Nx$ (resp. $M = \bigcup_{x \in C} xN$). Of course, subgroups of a group of finite index as well as large submonoids of a monoid have finite right and left index.

In this paper we investigate finite presentability of monoids with a finitely presented submonoid of finite index in the above sense. In Section 2 we first exhibit a counter example and see that we cannot simply generalize Results 1.1 and 1.2 to monoids with submonoids (more strongly subgroups) of finite right (or left) index. Next we give a result (Theorem 2.2) which generalizes Result 1.1 by adding some conditions. In Section 3 we consider the monoid of partial braids which contains the braid group as a subgroup of finite index and give an explicit finite presentation of it by confirming that it satisfies the conditions given in Theorem 2.2.

2 Monoids with a submonoid of finite index.

We consider the following problem. Let N be a submonoid of a monoid M of finite right (or left) index. If N is finitely presented, then is M also finitely presented? First we give a negative answer to this problem by exhibiting an example.

2000 *Mathematics Subject Classification.* Primary 20M05, Secondary 20F36.
Key words and phrases. finite presentation, finite index, braid group, inverse monoid.

Let H be a finitely generated group and F be a finitely generated free group with an epimorphism $\phi : F \to H$. We may assume that $F \cap H = \emptyset$. Set $M = F \cup H$ and define a multiplication $\cdot$ on M as follows. Let $x, y \in M$. If $x, y \in F$ or $x, y \in H$, then $x \cdot y$ is just the product xy in F or H. If $x \in F$ and $y \in H$ (resp. $x \in H$ and $y \in F$), then $x \cdot y$ is the product $\phi(x)y$ (resp. $x\phi(y)$) in H. Let e be the identity element of F. It is easy to see that $(M, \cdot)$ is a monoid with the identity element e.

Lemma 2.1 *In the above situation, we have the following.*
(1) *$M = F \cup F \cdot \phi(e) = F \cup \phi(e) \cdot F$, that is, F has finite right and left index.*
(2) *M is finitely presented if and only if H is finitely presented.*

Proof. (1) For any $x \in H$, there is $a \in F$ such that $x = \phi(a)$. Therefore, $x = x\phi(e) = a \cdot \phi(e) \in F \cdot \phi(e)$. Similarly $x \in \phi(e) \cdot F$. Thus, we have $M = F \cup H = F \cup F \cdot \phi(e) = F \cup \phi(e) \cdot F$.

(2) ($\Rightarrow$) Assume that M has a finite monoid presentation (A, R), and let $f : A^* \to M$ be the natural surjection where A^* is the free monoid generated by A. Set $B = A \cap f^{-1}(F)$ and $C = A \cap f^{-1}(H)$. Define a homomorphism $\psi : A^* \to H$ by

$$\psi(a) = \begin{cases} \phi \circ f(a) & \text{if } a \in B \\ f(a) & \text{if } a \in C. \end{cases}$$

Choose $z \in A^*$ such that $f(z) = \phi(e)$. Let $R' = \{(az, a) \mid a \in B\}$ and set $S = R' \cup R$. We claim that (A, S) is a monoid presentation of H under the homomorphism ψ.

Let $x = x_1 x_2 \cdots x_k \in A^*$. Since $F \cap H = \emptyset$, $F \cdot F \subset F$ and $F \cdot H = H \cdot F \subset H$ in M, the following condition is satisfied.

($\dagger$) $f(x) \in H$ if and only if some of x_i's is in C, and if $f(x) \in H$, then $f(x) = \psi(x)$.

The above condition shows that ψ is surjective, that is, A generates H. Next we shall show that, for any $u, v \in A^*$, $\psi(u) = \psi(v)$ in H if and only if $u =_S v$, where $=_S$ is the congruence on A^* generated by S. First for any $a \in B$, $\psi(az) = \psi(a)\psi(z) = \phi(f(a))\phi(e) = \phi(f(a)) = \psi(a)$. Next for $(u, v) \in R$, we have $f(u) = f(v)$ in M. Here, if $f(u), f(v) \in H$, then by condition ($\dagger$), $\psi(u) = f(u) = f(v) = \psi(v)$. On the other hand, if $f(u), f(v) \in F$, then $\psi(u) = \phi(f(u)) = \phi(f(v)) = \psi(v)$. Thus, $u =_S v$ implies $\psi(u) = \psi(v)$ in H.

To show the converse, let $u, v \in A^*$ such that $\psi(u) = \psi(v)$ in H. If $f(u), f(v) \in H$, then, by condition ($\dagger$), $\psi(u) = f(u)$ and $\psi(v) = f(v)$. Hence, $f(u) = f(v)$ in M and so $u =_R v$, a *fortiori* $u =_S v$. On the other hand, if $f(u), f(v) \in F$, then $f(uz), f(vz) \in H$ and $\psi(uz) = \psi(vz)$ in H. So, by the above discussion, we see $uz =_S vz$. Further, since $u =_{R'} uz$ and $v =_{R'} vz$, we have $u =_S v$.

Hence, we have proved that H is presented by the finite monoid presentation (A, S).

($\Leftarrow$) Assume that H has a finite monoid presentation (A, R). Let $g : A^* \to H$ be the natural surjection and B be a finite monoid-generating set of F. Set $C = A \cup B$. We extend g to a homomorphism from C^* to M by $g(b) = b$ for all $b \in B$. Let $R_1 = \{(bb^{-1}, \epsilon), (b^{-1}b, \epsilon) \mid b \in B\}$, where ϵ is the empty word. For each $b \in B$, choose $x \in A^*$ such that $\phi(b) = g(x)$ and let $R_2 = \{(ab, ax), (ba, xa) \mid a \in A \text{ and } b \in B\}$. Set $S = R \cup R_1 \cup R_2$. We claim that (C, S) is a monoid presentation of M under the homomorphism g. Since $M = F \cup H$, C generates M. It remains to show that, for $u, v \in C^*$, $g(u) = g(v)$ if and only if $u =_S v$. It is easy to see that $g(x) = g(y)$ in M for each $(x, y) \in S$. Hence for each $u, v \in C^*$, $u =_S v$ implies $g(u) = g(v)$ in M.

To show the converse, let $u = u_1 u_2 \cdots u_k, v = v_1 v_2 \cdots v_\ell \in C^*$ such that $g(u) = g(v)$ in M. As we see in condition ($\dagger$), $g(u), g(v) \in H$ if and only if u_i and v_j are in A for some i and

j. If $g(u), g(v) \in H$, then $u =_{R_2} w$, $v =_{R_2} w'$ and $g(w) = g(w')$ in H for some $w, w' \in A^*$. So, $u =_{R_2} w =_R w' =_{R_2} v$ and we have $u =_S v$. On the other hand, if $g(u), g(v) \in F$, then $u_i, v_j \in B \cup \{\epsilon\}$ for all i, j. Hence, $u =_{R_1} v$ and $u =_S v$. Thus, M is presented by the finite monoid presentation (C, S). ■

The above lemma tells us that even if a monoid contains a finitely presented submonoid (more strongly subgroup) of finite right and left index, it is not necessarily finitely presented. In fact, take H to be finitely generated but not finitely presented, then M is not finitely presented though it contains the finitely generated free subgroup F, which is finitely presented and of finite index. So we cannot simply generalize Results 1.1 and 1.2, and we need to consider additional conditions. The following result generalizes Result 1.1 in some sense.

Theorem 2.2 *Let H be a subgroup of a monoid M of finite right index and C be a finite subset of M such that $M = \bigcup_{x \in C} Hx$. If H is finitely presented and, for every $x \in C$, the subgroup $H(x) = \{g \in H \mid gx = x \text{ in } M\}$ of H is finitely generated, then M is finitely presented.*

Proof. Let (B, S) be a finite monoid presentation of H. Set $B = \{b_1, b_2, \ldots, b_m\}$ and $C = \{x_1, x_2, \ldots, x_n\}$. It is easy to verify that the set $A = B \cup C$ generates M. Because both B and C are finite, M is finitely generated.

Removing redundant elements from C, we may assume that $x_i \notin Hx_j$ if $i \neq j$. First, for each i, j with $1 \leq i \leq j \leq n$, there is a unique k with $1 \leq k \leq n$ such that $x_i x_j \in Hx_k$. Choose $u \in B^*$ such that $x_i x_j = u x_k$ in M and define a set R_1 of relations with respect to the generating set A of M by

$$R_1 = \{(x_i x_j, u x_k) \mid 1 \leq i \leq j \leq n\}.$$

Next, for each i, j with $1 \leq i \leq n$ and $1 \leq j \leq m$ there is a unique k with $1 \leq k \leq n$ such that $x_i b_j \in Hx_k$. Choose $v \in B^*$ such that $x_i b_j = v x_k$ in M and define a set R_2 of relations with respect the generating set A of M by

$$R_2 = \{(x_i b_j, v x_k) \mid 1 \leq i \leq n \text{ and } 1 \leq j \leq m\}.$$

Finally, for each i with $1 \leq i \leq n$, let $D_i \subset B^*$ be a finite generating set of $H(x_i)$ and define a set R_3 of relations with respect the generating set A of M by

$$R_3 = \bigcup_{i=1}^{n} \{(d x_i, x_i) \mid d \in D_i\}.$$

Set $R = S \cup R_1 \cup R_2 \cup R_3$. We claim that (A, R) is a finite monoid presentation of M. Since S, all R_i and all D_j are finite, R is finite, and it is clear that all the relations in R hold in M. So the only thing we must prove is that, for any $u, v \in A^*$, if $u = v$ in M, then it is a consequence of the relations in R. Assume that $u = v$ in M. By using relations in $R_1 \cup R_2$, there exist $w, w' \in B^*$ and i with $1 \leq i \leq n$ such that $u =_{R_1 \cup R_2} w x_i$ and $v =_{R_1 \cup R_2} w' x_i$. Since H is a group, we have $w^{-1} w' x_i =_S x_i$. So $w^{-1} w' \in H(x_i)$ and there exist $d_1, d_2, \ldots, d_\ell \in D_i$ such that $w^{-1} w' =_S d_1 d_2 \cdots d_\ell$. Hence,

$$v =_{R_1 \cup R_2} w' x_i =_S w w^{-1} w' x_i =_S w d_1 d_2 \cdots d_\ell x_i =_{R_3} w x_i =_{R_1 \cup R_2} u,$$

and we obtained $u =_R v$. This completes the proof of the theorem. ■

In the next section, we give a monoid (called the braid inverse monoid) which contains the braid group and give a finite monoid presentation of it by confirming that it satisfies the conditions in Theorem 2.2.

3 Braid groups and Braid inverse monoids. The braid group B_n is a group defined by the following finite monoid presentation (see [1] or [2])

$$\begin{aligned}\text{generators}:&\ \sigma_1^{\pm 1}, \sigma_2^{\pm 1}, \dots, \sigma_{n-1}^{\pm 1}, \text{ and}\\ \text{relations}:&\ \text{(G0) } \sigma_i\sigma_i^{-1} = 1 \text{ and } \sigma_i^{-1}\sigma_i = 1 \text{ for } 1 \le i \le n-1,\\ &\ \text{(G1) } \sigma_i\sigma_j = \sigma_j\sigma_i \text{ for } 1 \le i, j \le n-1 \text{ such that } i \le j-2 \text{ and}\\ &\ \text{(G2) } \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \text{ for } 1 \le i \le n-2.\end{aligned}$$

The braid group has the following geometrical interpretation. A *braid* on n strings is defined as a system of n strings in $\mathbb{R}^2 \times [0,1] \subset \mathbb{R}^3$. It consists of disjoint intertwining n strings which join n fixed points in the upper plane $\mathbb{R}^2 \times \{0\}$ and n fixed points in the lower plane $\mathbb{R}^2 \times \{1\}$, and intersecting each intermediate plane $\mathbb{R}^2 \times \{t\}$ in exactly n points. A string attached to the upper plane at the i-th position is called the *i-th string*.

By $B(n)$, we denote the set of isotopy classes of braids on n strings. We usually identify a braid and its isotopy class. So, an element in $B(n)$ is actually an isotopy class of braids, but it is called simply a braid. $B(n)$ has a group structure as follows. The product of two braids β_1 and β_2, denoted by juxtaposition $\beta_1\beta_2$, is defined as follows. First attach β_2 under β_1 identifying the upper plane of β_2 and the lower plane of β_1, and then remove the center plane. The *trivial braid* is the braid in which all strings go straight from the upper plane to the lower plane. And the *inverse* of a braid is defined as the mirror image of it with respect to the vertical direction.

For each i with $1 \le i \le n-1$, let $\widetilde{\sigma}_i$ be the braid in which the i-th string overcrosses the $(i+1)$-th string once and all other strings go straight from the upper plane to the lower plane.

The following result can be found in [1] or [2], and we identify $B(n)$ with B_n.

Result 3.1 *The groups B_n and $B(n)$ are isomorphic under the mapping $\sigma_i \mapsto \widetilde{\sigma}_i$.*

In the above discussion, we obtain a finitely presented group B_n. Now we construct a monoid containing B_n as a subgroup and satisfies the conditions in Theorem 2.2.

A *partial braid* on n strings is defined as a subsystem of a braid on n strings, that is, it consists of disjoint intertwining m strings ($0 \le m \le n$) which join m points of the n fixed points in the upper plane $\mathbb{R}^2 \times \{0\}$ and m points of the n fixed points in the lower plane $\mathbb{R}^2 \times \{1\}$, and intersecting each intermediate plane $\mathbb{R}^2 \times \{t\}$ in exactly m points. Accordingly, a partial braid on n strings can be obtained from a braid on n strings by removing some (possibly all or no) strings. For example, in Fig.1, the right-hand side is a partial braid that is obtained from the braid at the left-hand side by removing the fourth string. By BI_n, we denote the set of isotopy classes of partial braids.

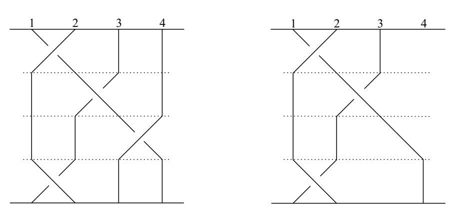

Fig.1 (a braid and a partial braid on 4 strings)

We define the product of two partial braids β_1 and β_2, denoted by juxtaposition $\beta_1\beta_2$, as follows. First attach β_2 under β_1 identifying the upper plane of β_2 and the lower plane of β_1. Then remove every string in β_1 (resp. β_2) that has no corresponding string in β_2 (resp. β_1). Lastly remove the center plane. For example, in Fig.2, we remove the second string in β_1, because it has no corresponding string in β_2. We also remove the fourth string in β_2 for the same reason.

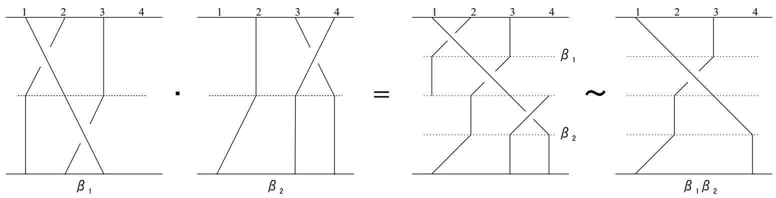

Fig.2 (the product of two partial braids β_1 and β_2 on 4 strings)

Then BI_n forms a monoid with this operation and contains B_n as a subgroup. In the following, we shall show that B_n and BI_n satisfy the conditions in Theorem 2.2.

For each i with $1 \leq i \leq n$, let γ_i be the partial braid that is obtained from the trivial braid by removing the i-th string (see Fig.3).

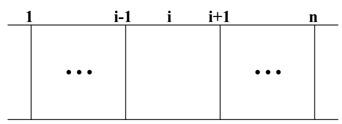

Fig.3

It is easy to verify that the following two types (I1-2) of relations hold in BI_n.

$$\begin{array}{ll} \text{(I1)} & \gamma_i^2 = \gamma_i \quad \text{for } 1 \leq i \leq n, \\ \text{(I2)} & \gamma_i\gamma_j = \gamma_j\gamma_i \quad \text{for } 1 \leq i < j \leq n. \end{array}$$

By E_n, we denote the submonoid of BI_n generated by the set $\{\gamma_i \mid i = 1, 2, \ldots, n\}$.

Lemma 3.2 *E_n is a finite set and every partial braid can be expressed as $\beta\gamma$ with $\beta \in B_n$ and $\gamma \in E_n$, that is, $BI_n = \bigcup_{\gamma \in E_n} B_n\gamma$ and so B_n is finite right index in BI_n. Moreover, for any $\beta, \beta' \in B_n$ and $\gamma, \gamma' \in E_n$, if $\beta\gamma = \beta'\gamma'$ in BI_n, then $\gamma = \gamma'$.*

Proof. By relations in (I1-2), every element in E_n can be expressed in the form $\gamma_1^{\epsilon_1}\gamma_2^{\epsilon_2}\cdots\gamma_n^{\epsilon_n}$ where $\epsilon_i \in \{0, 1\}$ for all i, and so E_n is finite. Any partial braid can be obtained from a braid by removing some strings and it is realized by applying an element of E_n to the braid. Thus every partial braid expressed as $\beta\gamma$ with $\beta \in B_n$ and $\gamma \in E_n$. Finally, if $\beta\gamma = \beta'\gamma'$ in BI_n, then the same strings must be removed in the partial braids $\beta\gamma$ and $\beta'\gamma'$, and so $\gamma = \gamma'$. ■

A string in a braid is called *pure* if it is attached to the upper and lower plane at the same position and a braid is called *pure* if all the strings in it are pure. By PB_n we denote the set of pure braids on n strings. It is clear that PB_n is a subgroup of B_n.

Result 3.3 (see [1] or [2]) *The group PB_n is generated by the set*

$$\{a_{sr} = \sigma_{r-1}\sigma_{r-2}\cdots\sigma_{s+1}\sigma_s^2\sigma_{s+1}^{-1}\cdots\sigma_{r-2}^{-1}\sigma_{r-1}^{-1} \mid 1 \leq s < r \leq n\} \quad \text{(see Fig.4)}.$$

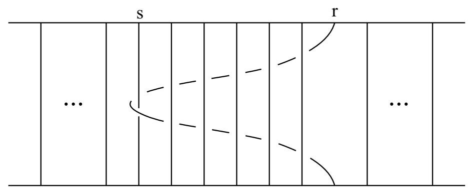

Fig.4 (the pure braid a_{sr})

Lemma 3.4 *For any s, r and i with $1 \leq s < r \leq n$ and $1 \leq i \leq n$, $a_{sr}^{\pm 1}\gamma_i = \gamma_i a_{sr}^{\pm 1}$ in BI_n. Further $a_{sr}^{\pm 1}\gamma_i = \gamma_i$ in BI_n if and only if $s = i$ or $r = i$.*

Proof. Clearly, removing the s-th or r-th string from $a_{sr}^{\pm 1}$ yields the partial braid γ_s or γ_r (see Fig.4). ■

Lemma 3.5 *For each $\gamma \in E_n$, the subgroup $PB_n(\gamma) = \{\beta \in PB_n \mid \beta\gamma = \gamma \text{ in } BI_n\}$ of PB_n is finitely generated.*

Proof. Let $\gamma = \gamma_{k_1}\gamma_{k_2}\cdots\gamma_{k_\ell} \in E_n$ with $1 \leq k_1 < k_2 < \cdots < k_\ell \leq n$ and $\beta = a_{s_1r_1}^{\epsilon_1}a_{s_2r_2}^{\epsilon_2}\cdots a_{s_hr_h}^{\epsilon_h} \in PB_n(\gamma)$, where $\epsilon_j \in \{-1, 1\}$ for all j. By Lemma 3.4, $a_{s_jr_j}\gamma = \gamma a_{s_jr_j}$ for each j with $1 \leq j \leq h$, and $a_{s_jr_j}\gamma = \gamma$ if $s_j = k_i$ or $r_j = k_i$ for some i with $1 \leq i \leq \ell$. Let β' be the pure braid obtained from β by deleting all $a_{s_jr_j}^{\epsilon_j}$ such that $s_j = k_i$ or $r_j = k_i$ for some i with $1 \leq i \leq \ell$. Then, we have $\beta'\gamma = \beta\gamma = \gamma$. By the construction of β', the k_i-th strings in β' with $1 \leq i \leq \ell$ go straight from the upper plane to the lower plane and do not influence the other strings. Moreover, because $\beta'\gamma = \gamma$, the k-th strings in β' with $k \notin \{k_1, k_2, \ldots, k_\ell\}$ do not essentially intertwine any other strings. So β' must be isotopic to the trivial braid in B_n. It follows that β can be isotopically deformed to the braid in which only the k_i-th strings with $1 \leq i \leq \ell$ move and the other strings go straight from the upper plane to the lower plane. Hence, β is written as a product of elements of $\bigcup_{i=1}^{\ell}\{a_{sk_i}^{\pm 1}, a_{k_ir}^{\pm 1} \mid 1 \leq s < k_i < r\}$. It follows that the subgroup $PB_n(\gamma)$ of PB_n is finitely generated. ■

Let S_n be the symmetric group on the set $I = \{1, 2, \ldots, n\}$ and $\tau : B_n \to S_n$ be the natural mapping, that is, it sends $\sigma_i^{\pm 1}$ to the transposition $(i, i+1)$ for all i with $1 \leq i \leq n-1$. For $p \in S_n$, let $I(p) = \{i \in I \mid p(i) \neq i\}$. For each $p \in S_n$, there is a braid β_p such that $\tau(\beta_p) = p$ and only the i-th string for $i \in I(p)$ moves and the other strings go straight from the upper plane to the lower plane in β_p. Choose one such braid β_p for each $p \in S_n$, and set $P = \{\beta_p \mid p \in S_n\}$. Clearly P is finite.

Lemma 3.6 *Let $\gamma \in E_n$ and $\beta \in B_n$. If $\beta\gamma = \gamma$ in BI_n, then there exist $p \in S_n$ such that $\beta\beta_p \in PB_n(\gamma)$.*

Proof. Let $\gamma = \gamma_{k_1}\gamma_{k_2}\cdots\gamma_{k_\ell}$ with $1 \leq k_1 < k_2 < \cdots < k_\ell \leq n$ and $p = \tau(\beta^{-1}) \in S_n$. Then, we see $I(p) \subseteq \{k_1, k_2, \ldots, k_\ell\}$, $\beta_p\gamma = \gamma$ and $\beta\beta_p \in PB_n$. Hence, we have $\beta\beta_p\gamma = \beta\gamma = \gamma$. ■

Corollary 3.7 *For each $\gamma \in B_n$, the subgroup $B_n(\gamma) = \{\beta \in B_n \mid \beta\gamma = \gamma \text{ in } BI_n\}$ of B_n is finitely generated.*

Proof. By Lemma 3.5, $PB_n(\gamma)$ is generated by some finite set $X = \{\beta_1, \beta_2, \ldots, \beta_m\}$. Let $Y = P \cap B_n(\gamma)$. We claim that that $X \cup Y$ generates $B_n(\gamma)$. Let $\beta \in B_n(\gamma)$. By Lemma 3.6, there exists $p \in S_n$ such that $\beta_p \in Y$ and $\beta\beta_p \in PB_n(\gamma)$. So, $\beta\beta_p = \beta_{k_1}^{\epsilon_1}\beta_{k_2}^{\epsilon_2}\ldots\beta_{k_\ell}^{\epsilon_\ell}$ in B_n for some $\beta_{k_1}, \beta_{k_2}, \ldots, \beta_{k_\ell} \in X$ and $\epsilon_j \in \{-1, 1\}$. Thus, $\beta = \beta_{k_1}^{\epsilon_1}\beta_{k_2}^{\epsilon_2}\ldots\beta_{k_\ell}^{\epsilon_\ell}\beta_p^{-1}$ in B_n. ■

By Lemma 3.2 and Corollary 3.7, B_n is a subgroup of BI_n satisfying the conditions in Theorem 2.2. Thus, we can obtain a finite monoid presentation of BI_n along the proof of the theorem. Omitting the detail calculation here, we present a monoid presentation of BI_n in a simplified form.

Theorem 3.8 *The monoid BI_n is defined by the generators $\sigma_1^{\pm 1}$, $\sigma_2^{\pm 1}$, ..., $\sigma_{n-1}^{\pm 1}$ and γ_1, γ_2, ..., γ_n, and the relations*

(G0) $\sigma_i \sigma_i^{-1} = 1$, $\sigma_i^{-1} \sigma_i = 1$ *for* $1 \leq i \leq n-1$,

(G1) $\sigma_i \sigma_j = \sigma_j \sigma_i$ *for* $1 \leq i, j \leq n-1$ *such that* $i \leq j-2$,

(G2) $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$ *for* $1 \leq i \leq n-2$,

(I1) $\gamma_i^2 = \gamma_i$ *for* $1 \leq i \leq n$,

(I2) $\gamma_i \gamma_j = \gamma_j \gamma_i$ *for* $1 \leq i < j \leq n$,

(I3) $\gamma_{i+1} \sigma_i = \sigma_i \gamma_i$ *for* $1 \leq i \leq n-1$,

(I4) $\gamma_i \sigma_i = \sigma_i \gamma_{i+1}$ *for* $1 \leq i \leq n-1$,

(I5) $\gamma_j \sigma_i = \sigma_i \gamma_j$ *for* $1 \leq i \leq n-1$ *and* $1 \leq j \leq n$ *such that* $j \neq i, i+1$,

(I6) $\sigma_i^2 \gamma_i = \gamma_i$ *for* $1 \leq i \leq n-1$ *and*

(I7) $\sigma_i \gamma_i \gamma_{i+1} = \gamma_i \gamma_{i+1}$ *for* $1 \leq i \leq n-1$.

Remark 3.9 *The relations in Theorem 3.8 are related to the sets R_1, R_2 and R_3 of relations which are used in the proof of Theorem 2.2. In fact, relations* (I1-2) *come from R_1, relations* (I3-5) *from R_2, and relations* (I6-7) *from R_3.*

Remark 3.10 *The monoid BI_n forms an inverse monoid with the semillatice E_n of idempotents and is called the braid inverse monoid on n strings. Actually, for any partial braid, the unique inverse is the mirror image of it in the vertical direction in the same way as ordinary braids.*

References

[1] J. Birman, *Braids, links, and mapping class groups*, Ann. of Math. Stud. **82**, Princeton Univ. Press, 1975.

[2] V.L. Hansen, *Braids and Coverings: selected topics*, London Math. Soc. Student Texts 18, Cambridge Univ. Press, 1989.

[3] D.L. Johnson, *Presentations of Groups*, London Math. Soc. Student Texts 15, Cambridge Univ. Press, 1990.

[4] N. Ruškuc, *On large subsemigroups and finiteness conditions of semigroups*, Proc. London Math. Soc. (3) **76** (1998) 383-405.

Department of Information Science, Toho University, Funabashi 274-8510, Japan