NOTE ON THE NUMBER OF SEMISTAR-OPERATIONS, V

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Dedicated to Professor Masami Ito on his 60th birthday

ABSTRACT. Let D be an n-dimensional integral domain with $n > 1$ which is not quasi-local, and let $\Sigma'(D)$ be the set of semistar-operations on D. We show that $|\Sigma'(D)| \geq n+5$. Also we answer to a problem posed in [MSi], and answer to a conjecture posed in [MSu].

This is a continuation of our [M2]. Let D be an integral domain, and let $\mathrm{F}(D)$ be the set of non-zero fractional ideals of D. A mapping $I \longmapsto I^*$ of $\mathrm{F}(D)$ into itself is called a $star-operation$ on D if it satisfies the following conditions:

(1) $(a)^* = (a)$ for each non-zero element a of K, where K is the quotient field of D.

(2) $(aI)^* = aI^*$ for each non-zero element a of K and for each element $I \in \mathrm{F}(D)$.

(3) $I \subset I^*$ for each element $I \in \mathrm{F}(D)$.

(4) $I \subset J$ implies $I^* \subset J^*$ for all elements I and J in $\mathrm{F}(D)$.

(5) $(I^*)^* = I^*$ for each element $I \in \mathrm{F}(D)$.

Let $\mathrm{F}'(D)$ be the set of non-zero D-submodules of K. A mapping $I \longmapsto I^*$ of $\mathrm{F}'(D)$ into itself is called a $semistar-operation$ on D if it satisfies the following conditions:

(1) $(aI)^* = aI^*$ for each non-zero element a of K and for each element $I \in \mathrm{F}'(D)$.

(2) $I \subset I^*$ for each element $I \in \mathrm{F}'(D)$.

(3) $I \subset J$ implies $I^* \subset J^*$ for all elements I and J in $\mathrm{F}'(D)$.

(4) $(I^*)^* = I^*$ for each element $I \in \mathrm{F}'(D)$.

The set of star-operations (resp. semistar-operations) on D is denoted by $\Sigma(D)$ (resp. $\Sigma'(D)$).

In [M1] we showed the followings,

Theorem 1. Let D be an n-dimensional domain with $n > 2$ which is not quasi-local. Then $|\Sigma'(D)| \geq n+4$.

Theorem 2. Let D be an n-dimensional domain with $n > 2$ which is not quasi-local. Then $|\Sigma'(D)| = n+4$ if and only if the following conditions hold:

(1) D is a Prüfer domain with exactly two maximal ideals M and N.

(2) There exist prime ideals $P_1, P_2, \cdots, P_{n-1}$ of D such that $M \cap N \supsetneqq P_{n-1} \supsetneqq \cdots \supsetneqq P_1 \supsetneqq (0)$.

(3) $P_iD_{P_i}$ is a principal ideal of D_{P_i} for each i, MD_M is a principal ideal of D_M, and ND_N is a principal ideal of D_N.

(4) D has exactly two star-operations.

For each positive integer $n > 1$, in [M1], we gave an example of domains which satisfies the conditions (1),(2) and (3) in Theorem 2.

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In [MSi], we posed the following problem: Let D be an integrally closed domain with dimension 4. If $5 \leq |\Sigma'(D)| \leq 9$, is D a valuation domain?

In [MSu], we conjectured the following: Let D be an integrally closed domain with dimension n. If $n+1 \leq |\Sigma'(D)| \leq 2n+1$, then D is a valuation domain.

In this note we show the followings,

Theorem 3. Let D be an n-dimensional domain with $n > 1$ which is not quasi-local. Then $|\Sigma'(D)| \geq n+5$.

Proposition 1. Let D be an integrally closed domain with dimension 4. If $5 \leq |\Sigma'(D)| \leq 9$, then D is a valuation domain.

Proposition 2. For each positive integer n with $n \geq 5$, there exists an integrally closed domain D with dimension n such that $n+1 \leq |\Sigma'(D)| \leq 2n+1$ and which is not a valuation domain.

The identity mapping $\mathrm{d}=\mathrm{d}_D$ on $\mathrm{F}(D)$ is a star-operation, and is called the d-*operation* on D. The mapping $I \longmapsto I^v = I^{v_D} = (I^{-1})^{-1}$ of $\mathrm{F}(D)$ is a star-operation, and is called the v-*operation* on D. The identity mapping $\mathrm{d}' = \mathrm{d}'_D$ on $\mathrm{F}'(D)$ is a semistar-operation on D, and is called the d$'$-*operation* on D. We set $I^{v'} = I^v$ for each element $I \in \mathrm{F}(D)$, and set $I^{v'} = K$ for each element $I \in \mathrm{F}'(D) - \mathrm{F}(D)$, where K is the quotient field of D. Then $\mathrm{v}' = \mathrm{v}'_D$ is a semistar-operation on D, and is called the v$'$-*operation* on D. If we set $I^e = K$ for each $I \in \mathrm{F}'(D)$, then $\mathrm{e} = \mathrm{e}_D$ is a semistar-operation on D, and is called the e-*operation* on D. Let $*$ be a star-operation on D, and let $*'$ be a semistar-operation on D. If the restriction of $*'$ to $\mathrm{F}(D)$ coincides with $*$, then $*'$ is called an extension of $*$ to a semistar-operation. Let R be a domain, let D be a subdomain of R, and let $*$ be a semistar-operation on D. If we set $I^{\alpha(*)} = I^*$ for each $I \in \mathrm{F}'(R)$, then $\alpha(*) = \alpha_{R/D}(*)$ is a semistar-operation on R, and is called the *ascent* of $*$ to R. Let $*$ be a semistar-operation on R. If we set $I^{\delta(*)} = (IR)^*$ for each $I \in \mathrm{F}'(D)$, then $\delta(*) = \delta_{R/D}(*)$ is a semistar-operation on D, and is called the *descent* of $*$ to D. In this note, D denotes a domain, K denotes the quotient field of D, n denotes a positive integer, and the descent $\delta_{R/D}(d'_R)$ of the d$'$-operation d'_R on R is also denoted by $*_R$, where R is an overring of D.

Lemma 1 ([H, Lemma 5.2] and [AA, Proposition 12]). Let V be a non-trivial valuation domain on a field, and let M be its maximal ideal. If M is principal, then $|\Sigma(V)| = 1$, and if M is not principal, then $|\Sigma(V)| = 2$.

Lemma 2 ([H, Theorem 5.1]). Let D be an integrally closed domain. Then each non-zero ideal of D is divisorial if and only if D is a Prüfer domain, the maximal ideals of D are finitely generated, each ideal of D has only finitely many minimal primes, and, each non-zero prime ideal P of D is not contained in two different maximal ideals of D.

Lemma 3 ([MSu, Corollary 6]). Let D be an integrally closed quasi-local domain with dimension n. Then D is a valuation domain if and only if $n+1 \leq |\Sigma'(D)| \leq 2n+1$.

Lemma 4. Let D be an n-dimensional Prüfer domain with $n > 2$ and with exactly two maximal ideals M and N. Assume that there exist prime ideals $P_1, P_2, \cdots, P_{n-1}$ of D such that $M \cap N \supsetneqq P_{n-1} \supsetneqq \cdots \supsetneqq P_2 \supsetneqq P_1 \supsetneqq (0)$, and that there exist elements $\pi_1, \pi_2, \cdots, \pi_{n-1}, p$ and q of D such that $P_i D_{P_i} = \pi_i D_{P_i}$ for each i, $M = (p)$ and $N = (q)$. Then

(1) Each non-zero element x of K can be expressed as $\pi_1^{a_1}\pi_2^{a_2}\cdots\pi_{n-1}^{a_{n-1}}p^bq^c$ up to a unit of D with integers a_i, b, c. This expression is unique for the element x.

(2) Each finitely generated ideal of D is principal.

(3) Define the fractional ideals $A_{n-1} = \cup_1^\infty(1/\pi_{n-1}^m), \cdots, A_2 = \cup_1^\infty(1/\pi_2^m)$, $A = \cup_1^\infty(1/p^m)$, $B = \cup_1^\infty(1/q^m)$ and $C = \cup_1^\infty(1/(pq)^m)$ of D. Then each non-finitely generated fractional ideal I of D is of the form xA_{n-1} or $\cdots$ or xA_2 or xA or xB or xC with $x \in K$.

(4) We have $P_1 = P_1^v = \pi_1 A_2, P_2 = P_2^v = \pi_2 A_3, \cdots, P_{n-1} = P_{n-1}^v = \pi_{n-1}C = \pi_{n-1}A^v = \pi_{n-1}B^v, C = C^v$, $A_2 = A_2^v, \cdots\cdots, A_{n-1} = A_{n-1}^v$, and $A \neq A^v, B \neq B^v$.

(5) A is not of the form xB, and B is not of the form xA, where $x \in K$.

(6) For a non-zero fractionl ideal I of D, set $I^{*_1} = I$ if I is of the form xA, and set $I^{*_1} = I^v$ otherwise, where $x \in K$. Then $*_1$ is a star-operation on D. Set $I^{*_2} = I$ if I is of the form xB, and set $I^{*_2} = I^v$ otherwise. Then $*_2$ is a star-operation on D.

(7) In (6), $*_1$ differs from $*_2$, and each of $*_1, *_2$ differs from each of d and v.

Proof. (1) and (2) are straightforward.

(3) We may assume that there exist principal fractional ideals $I_n = (x_n)$ of D such that $I_1 \subsetneqq I_2 \subsetneqq I_3 \subsetneqq \cdots$ and $I = \cup_1^\infty I_n$. We may assume that each x_i is of the form $\pi_1^{a_{i,1}}\pi_2^{a_{i,2}}\cdots\pi_{n-1}^{a_{i,n-1}}p^{b_i}q^{c_i}$ with integers $a_{i,j}, b_i, c_i$. Next, we may assume that $x_i = \pi_2^{a_{i,2}}\cdots\pi_{n-1}^{a_{i,n-1}}p^{b_i}q^{c_i}$ with integers $a_{i,j}, b_i, c_i$. If inf $(a_{i,2}) = -\infty$, then $I = A_2$. If inf $(a_{i,2}) > -\infty$, then we may assume that $x_i = \pi_3^{a_{i,3}}\cdots\pi_{n-1}^{a_{i,n-1}}p^{b_i}q^{c_i}$. If inf $(a_{i,3}) = -\infty$, then $I = A_3$. If inf $(a_{i,3}) > -\infty$, we may assume that $x_i = \pi_4^{a_{i,4}}\cdots\pi_{n-1}^{a_{i,n-1}}p^{b_i}q^{c_i}$. $\cdots$. If inf $(a_{i,n-1}) = -\infty$, then $I = A_{n-1}$. If inf $(a_{i,n-1}) > -\infty$, then we may assume that $x_i = p^{b_i}q^{c_i}$. If inf $(b_i) > -\infty$, we may assume that $x_i = q^{c_i}$. Then $I = B$. If inf $(c_i) > -\infty$, we may assume that $x_i = p^{b_i}$. Then $I = A$. If inf $(b_i) =$ inf $(c_i) = -\infty$, then $I = C$.

(4) We have $P_1 = \cap_1^\infty(\pi_2^m)$, and hence $P_1 = P_1^v$. $P_2 = \cap_1^\infty(\pi_3^m)$, and hence $P_2 = P_2^v$. $\cdots$. $P_{n-2} = \cap_1^\infty(\pi_{n-1}^m)$, and hence $P_{n-2} = P_{n-2}^v$. Next, $P_{n-1} = \cap_1^\infty(pq)^m$, and hence $P_{n-1} = P_{n-1}^v$. Next, $\pi_{n-1}C = \cup_1^\infty(\pi_{n-1}/(pq)^m) = P_{n-1}$, and hence $C = C^v$. Next, $\pi_{n-2}A_{n-1} = \cup_1^\infty(\pi_{n-2}/\pi_{n-1}^m) = P_{n-2}$, and hence $A_{n-1} = A_{n-1}^v$. $\cdots$. $\pi_1 A_2 = \cup_1^\infty(\pi_1/\pi_2^m) = P_1$, and hence $A_2 = A_2^v$. Assume that $\pi_{n-1}A \subset (\alpha)$ for an element $\alpha \in K$. It follows that $P_{n-1} \subset (\alpha)$. Hence $\pi_{n-1}A^v = P_{n-1}$. Similarly, $\pi_{n-1}B^v = P_{n-1}$. Clearly, $A \neq A^v$ and $B \neq B^v$.

(5) This is straightforward.

(6) Let I and J be non-zero fractional ideals of D such that $I \subset J$. We must show that $I^{*_1} \subset J^{*_1}$. We may assume that I is not of the form xA, and that J is of the form xA. Next, we may assume that $I = B$ and $J = xA$ for an element $x \in K$. x is expressed as $\pi_1^{a_1}\cdots\pi_{n-1}^{a_{n-1}}p^bq^c$ up to a unit of D with integers a_i, b, c. Then we see that either $a_1 < 0$ or $a_1 = 0 > a_2$ or $a_1 = a_2 = 0 < a_3$ or $\cdots$ or $a_1 = a_2 = \cdots = a_{n-2} = 0 > a_{n-1}$. Hence $I^{*_1} = P_{n-1}/\pi_{n-1} \subset \pi_1^{a_1}\cdots\pi_{n-1}^{a_{n-1}}p^bq^cA = J$. Similarly, $*_2$ is a star-operation on D.

(7) This follows from (4) and (5).

Remark 1. Let D be a 1-dimensional Prüfer domain with exactly two maximal ideals M and N, and assume that M and N are principal ideals of D. Then we have $\mid \Sigma'(D) \mid= 5$, and $\Sigma'(D) = \{\mathrm{e}, *_V, *_W, \mathrm{d}', \mathrm{v}'\}$.

Proof. There exist elements p, q of D such that $M = (p)$ and $N = (q)$. Define $A = \cup_1^\infty(1/p^m)$ and $B = \cup_1^\infty(1/q^m)$. Each non-finitely generated D-submodule of K is K or p^aB or q^bA. Let $*$ be a semistar-operation on D such that $D^* = D$, $* \neq \mathrm{d}'$, and $* \neq \mathrm{v}'$. If $A^* = A$ and $B^* = B$, then $* = \mathrm{d}'$. If $A^* = B^* = K$, then $* = \mathrm{v}'$. Thus we may assume that $A \subsetneqq A^* \subsetneqq K$. Since $K = \cup_1^\infty(1/(pq)^m)$, there exists m such that $A^* \ni 1/(pq)^m$ and

$A^* \not\ni 1/(pq)^{m+1}$. Then $A^* = (1/q^m)A$. It follows that $(A^*)^* \neq A^*$; a contradiction.

Proposition 3. Let D be an n-dimensional domain with $n > 1$ which satisfies the following conditions:

(1) D is a Prüfer domain with exactly two maximal ideals M and N.

(2) There exist prime ideals $P_1, \cdots, P_{n-1}$ of D such that $M \cap N \supsetneqq P_{n-1} \supsetneqq \cdots \supsetneqq P_1 \supsetneqq (0)$.

(3) $M = (p)$ and $N = (q)$ are principal ideals of D.

(4) Define the fractrional ideals $A = \cup_1^\infty (1/p^m)$ and $B = \cup_1^\infty (1/q^m)$. Set, for each non-zero fractional ideal I of D, $I^{*_1} = I$ if I is of the form xA, and $I^{*_1} = I^v$ otherwise, where $x \in K$. Set $I^{*_2} = I$ if I is of the form xB, and $I^{*_2} = I^v$ otherwise.

Then we have that $*_1$ and $*_2$ are star-operations on D, that each of $*_1$ and $*_2$ differs from each of d and v, and that $*_1 \neq *_2$. It follows that $\mid \Sigma'(D) \mid \geq n + 6$.

Proof. At first, each non-zero element x of K can be expressed as $\pi_1^{a_1} \cdots \pi_{n-1}^{a_{n-1}} p^b q^c$ up to a unit of D, where $\pi_i \in P_i - P_{i-1}$ and $a_i, b, c \in \mathbf{Z}$ for each i.

Next, each finitely generated ideal of D is principal.

Let $* = *_1$, and let I, J be non-zero fractional ideals of D such that $I \subset J$. We must show that $I^* \subset J^*$. We may assume that I is not finitely generated, that I is not of the form xA, and that $J = A$. It suffices to show that $I \subset D$. Thus suoppose that $I \not\subset D$, and take $x \in I - D$. We may assume that x is of the form $\pi_1^{a_1} \cdots \pi_{n-1}^{a_{n-1}} p^b q^c$. Suppose that all of the a_i are not 0, and let $a_1 = \cdots = a_{k-1} = 0 \neq a_k$. If $a_k < 0$, then $x \notin A$; a contradiction. If $a_k > 0$, then $x \in D$; a contradiction. Thus we may assume that x is of the form $p^b q^c$. If $y \in I$, then $(y, x) = (y')$ with $y' \notin D$. Hence y' is of the form $p^{b'} q^{c'}$. It follows that $I = \cup_1^\infty (x_i)$, $(x_1) \subsetneqq (x_2) \subsetneqq \cdots$, $x_i \notin D$ for each i, and x_i is of the form $p^{b_i} q^{c_i}$ for each i. Since $x_i \in A$, we have $c_i \geq 0$ for each i. Thus we may assume that $c = c_i$ is a constant, and $I = q^c (\cup_1^\infty (p^{b_i})$. Then we have $I = q^c A$; a contradiction.

Next, assume that $A \subset (x)$ with $x \in K$. Express $x = \pi_1^{a_1} \cdots \pi_{n-1}^{a_{n-1}} p^b q^c$. Then $a_1 = \cdots = a_{k-1} = 0 > a_k$ for some $k < n$. It follows that $A \neq A^v$. Similarly, $B \neq B^v$. Clearly, B (resp. A) is not of the form xA (resp. xB). The proof is complete.

Proposition 4. Let D be an n-dimensional domain with $n > 1$ which satisfies the following conditions:

(1) D is a Prüfer domain with exactly two maximal ideals M and N.

(2) There exist prime ideals $P_1, \cdots, P_{n-1}$ of D such that $M \cap N \supsetneqq P_{n-1} \supsetneqq \cdots \supsetneqq P_1 \supsetneqq (0)$.

(3) MD_M is a principal ideal of D_M, $P_i D_{P_i}$ is a principal ideal of D_{P_i} for each i, and ND_N is not a principal ideal of D_N.

For a non-zero fractional ideal I of D, set $I^* = I$ if I is of the form xN, and set $I^* = I^v$ otherwise, where $x \in K$. Then $*$ is a star-operation on D. $*$ differs from each of d and v.

Proof. At first, $M = (p)$ is a principal ideal of D, and each non-zero element x of K can be expressed as $\pi_1^{a_1} \cdots \pi_{n-1}^{a_{n-1}} p^b q^c$ up to a unit of D with $q \in N - M$ and with integers a_i, b, c.

Next, each finitely generated ideal of D is principal. Define the fractional ideals $A_{n-1} = \cup_1^\infty (1/\pi_{n-1}^m), \cdots, A_2 = \cup_1^\infty (1/\pi_2^m)$, $A = \cup_1^\infty (1/p^m)$. Then we have $P_1 = P_1^v = \pi_1 A_2, \cdots$, $P_{n-2} = P_{n-2}^v = \pi_{n-2} A_{n-1}, P_{n-1} = P_{n-1}^v = \pi_{n-1} A^v, A_2 = A_2^v, \cdots, A_{n-1} = A_{n-1}^v$, and $A \neq A^v$.

Next, $N^v = D$, and A is not of the form xN. Let I and J be non-zero fractional ideals of D such that $I \subset J$. We must show that $I^* \subset J^*$. We may assume that I is not of the form xN, and that $J = N$. We may assume that there exist elements $x_1, x_2, \cdots$ of D such that $(x_1) \subsetneqq (x_2) \subsetneqq \cdots$ and that $I = \cup_1^\infty (x_i)$. We may assume that x_i is of the form $\pi_1^{a_{i,1}} \cdots \pi_{n-1}^{a_{i,n-1}} p^{b_i} q^{c_i}$ with integers $a_{i,j}, b_i, c_i$. We may assume that $a_{i,1} \geq 0$ is a constant. If $a_{i,1} > 0$, then $I \subset P_1$, and hence $I^v \subset N$. Thus we may assume that each x_i is of the form $\pi_2^{a_{i,2}} \cdots \pi_{n-1}^{a_{i,n-1}} p^{b_i} q^{c_i}$. We may assume that $a_{i,2} \geq 0$ is a constant. If $a_{i,2} > 0$, then $I \subset P_2$, and hence $I^v \subset N$. $\cdots$. Thus we may assume that each x_i is of the form $\pi_{n-1}^{a_{i,n-1}} p^{b_i} q^{c_i}$. We may assume that $a_{i,n-1} \geq 0$ is a constant. If $a_{i,n-1} > 0$, then $I \subset P_{n-1}$, and hence $I^v \subset N$. Thus we may assume that each x_i is of the form $p^{b_i} q^{c_i}$. We may assume that $b = b_i \geq 0$ is a constant. Then we have $I = p^b I_0$ and $I_0 = \cup_1^\infty (q_i)$. If inf ${}_i w(q_i) = 0$, then $I_0 = N$, and hence $I = p^b N$; a contradiction. If inf ${}_i w(q_i) > 0$, there exists $q \in N - M$ such that $w(q) <$inf ${}_i w(q_i)$. Then $I_0 \subset (q)$, and hence $I \subset (p^b q)$. Then $I^v \subset (p^b q) \subset N$. The proof is complete.

Remark 2. If $n = 1$ in Proposition 4, the star-operation $*$ coincides with d.

Proof. Let I be a non-zero ideal of D. We must show that $I^* = I$. We may assume that I is not finitely generated, and that I is not of the form xN. There exist elements x_i of D such that $I = \cup_1^\infty (x_i)$ and $(x_1) \subsetneqq (x_2) \subsetneqq \cdots$. Express $x_i = p^{a_i} q^{b_i}$ with $a_i, b_i \geq 0$. We may assume that $a = a_i$ is a constant, and that $q_i = q^{b_i} \in N - M$. If $\inf_i w(q_i) = 0$, then $I = p^a N$; a contradiction. Thus $\inf_i w(q_i) > 0$. Let $\{q \in N - M \mid w(q) \leq \inf_i w(q_i)\} = \{q_\lambda \mid \Lambda\}$. Then $I = \cap_\lambda (p^a q_\lambda)$ or $I = p^a qN$.

Proof of Proposition 1: Suppose the contrary. Then, by Lemma 3, we may assume that D is not quasi-local. Easily we may assume that D is a Prüfer domain with exactly two maximal ideals M and N, that there exist prime ideals P_1, P_2, P_3 such that $M \cap N \supsetneqq P_3 \supsetneqq P_2 \supsetneqq P_1 \supsetneqq (0)$. If MD_M is a principal ideal of D_M and ND_N is a principal ideal of D_N, then there arise star-operations $*_1$ and $*_2$ in Proposition 3. If MD_M is principal and ND_N is not principal, then we may assume that $P_i D_{P_i}$ is principal for each i. Then we have the semistar-operation $\delta(\mathrm{v}'_W)$ and a star-operation $*$ in Proposition 4. If neither MD_M nor ND_N is principal, then we have $\delta(\mathrm{v}'_V)$ and $\delta(\mathrm{v}'_W)$; a contradiction.

Proof of Theorem 3: Suppose the contrary. First, assume that $n \geq 3$. Then $\mid \Sigma'(D) \mid = n + 4$ by Theorem 1. Then D satisfies the conditions (1),(2),(3) and (4) in Theorem 2 by Theorem 2.

Since MD_M is a principal ideal of D_M, we may take an element p of D such that $M = (p)$. Also we may take q such that $N = (q)$. Then the above condition (4) contradicts to the Lemma 4 (7).

Next, assume that $n = 2$. We may assume that D is a Prüfer domain with exactly two maximal ideals M and N, and that there exists a prime ideal P of D such that $M \supsetneqq P \supsetneqq (0)$. Set $V = D_M, W = D_N$ and $U = D_P$.

The case that $P \subsetneqq M \cap N$: If MD_M is a principal ideal of D_M, and ND_N is a principal ideal of D_N, we have $\mid \Sigma'(D) \mid \geq 8$ by Proposition 3. If ND_N is not principal, then there arizes a semistar-operation $\delta(V'_W)$.

The case that $P \not\subset M \cap N$ and height$(N) = 2$: There exists a prime ideal Q of D such that $N \supsetneqq Q \supsetneqq (0)$. Set $U \cap D_Q = R$. Then there arizes a semistar-operation $*_R$ on D.

The case height$(N) = 1$: Set $R = U \cap W$. If PD_P is principal, and ND_N is principal, then $\mid \Sigma'(D) \mid = 5$ by Remark 1. Hence $\mid \Sigma'(D) \mid \geq 7$. If PU is not principal, then we have a semistar-operation $\delta(\mathrm{v}'_U)$. If NW is not principal, then we have $\delta(\mathrm{v}'_V)$. The proof is complete.

Theorem 4. Let D be an n-dimensional domain with $n > 1$ which satisfies the conditions (1),(2) and (3) in Theorem 2. Then we have $\mid \Sigma'(D) \mid = n+6$, and $\Sigma'(D) = \{\mathrm{e}, *_{U_1}, \cdots, *_{U_{n-1}}, *_V, *_W, \mathrm{d}', \mathrm{v}', *_1', *_2'\}$, where $*_1'$ (resp. $*_2'$) is the canonical extension of $*_1$ (resp. $*_2$) in Lemma 4 to a semistar-operation on D.

Proof. Suppose the contrary. We confer the proof of Theorem 3. Then there exists a star-operation $*$ on D which differs from each of d,v, $*_1, *_2$. If $A^* = A$ and $B^* = B$, then $* =$d. If $A^* = A^v$ and $B^* = B^v$, then $* =$v. Thus we may assume that $A \subsetneqq A^* \subsetneqq A^v$. Since $A^v = \cup_1^\infty (1/(pq)^m)$, there exists m such that $A^* \ni 1/(pq)^m$ and $A^* \not\ni 1/(pq)^{m+1}$. Then $A^* = (1/q^m)A$. It follows that $(A^*)^* \neq A^*$; a contradiction.

Proof of Proposition 2. Let D be an integral domain with dimension n which satisfies conditions (1), (2) and (3) in Theorem 2. Then, by Theorem 4, we have $\mid \Sigma'(D) \mid = n+6$.

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