ON AN ALGEBRA OBTAINED FROM BCI

Kiyoshi ISEKI

Dedicated to the 60th anniversary of Professor Masami Ito

Received December 21, 2001

Abstract. In any BCI, there exist two important subsets. One of them is BCK part. Roughly speaking, another is the set consisting of incomparable elements containing 0. It makes the basic BCI. This is a system which is obtained by an Abelian group. We concern with the later problem in this paper.

In my recent paper [2],[3], we proved that any partially ordered set X with the smallest element 0 has at least one BCK-structure. More general question, can we introduce a BCI structure on a partially ordered set? A set without any order is considered as a pure set. In [4], we proved that any finite pure set admitss a BCI-structure. On the other hand, for many useful informations on BCK-algebra, see [5].

If there exists at least one BCI structure on a pure set X, it satisfies the following identities and (3):
(1) $(x * y) * (x * z) = z * y$,
(2) $x * (x * y) = y$,
(3) $x * y = 0$ implies $x = y$.

An example with a BCI-structure is any additively written Abelian group G. We definee $x * y$ as $x - y$ for $x, y \in G$. Then G has the BCI structure just mentioned.

This system becomes to be a basic part to study BCI, so this system is refered as a **basic BCI**. We then mention some elementary results of a basic BCI.

From (1) and (2), we obtain

$$(x * y) * z = (x * y) * (x * (x * z)) = (x * z) * y.$$

Then we have the following

Proposition 1. *The* ***permutation rule***

$$(x * y) * z = (x * z) * y$$

holds in a basic BCI.

Proposition 2. *For every x,*

$$x * 0 = x.$$

Proof. Put y = 0 in (2). Then $x * (x * 0) = 0$. By (3),

$$x * 0 = x$$

is obtained.

2000 *Mathematics Subject Classification.* 03G25, 06F35, 06D99.
Key words and phrases. BCI, Abelian groups.

Proposition 3. *In a system satisfying the conditions* (1), (2) *and* $a * 0 = a$ *for some* a,

$$x * 0 = x$$

holds for all x.

Proof. (1) implies

$$(a * 0) * (a * x) = x * 0.$$

Since $a * 0 = a$, we have $a * (a * x) = x * 0$. By (2), $x = x * 0$ is obtained.

Corollary 1. *In a system satifying* (1), (2) *and* $0 * 0 = 0$, $x * 0 = x$ *holds for every* x.

But a basic BCI is also characterized as follows:

Proposition 4. *A system with a binary opeartion* $*$ *and a constant* 0 *is a basic BCI if and only if*

(2) $x * (x * y) = y$,
(4) $(x * y) * z = (x * z) * y$,
(3) $x * y = 0$ *implies* $x = y$.

Proof. As already seen, (2),(4) and (3) hold in a basic BCI. Let a system satisfies (2), (4). Then

$$(x * y) * (x * z) = (x * (x * z)) * y = z * y,$$

which is just (1).

Proposition 5. *If a system with a binary opeation* $*$ *and a constant* 0 *satisfies the following conditions*:

(2) $x * (x * y) = y$,
(4) $(x * y) * z = (x * z) * x$,
(5) $x * 0 = x$,

then it is a basic BCI.

Proof. Let $x * y = 0$. By (2), we have

$$x * (x * y) = x * 0 = y.$$

The condition (5) implies $x = y$. Hence it follows from Proposition 3 that this proposition holds.

Remark. From Propsition 4, we know that a basic BCI is defined by identities. Therefore, a basic BCI is an algebra.

To obtain some important results, we consider $0 * x$. Then we hve the following

(6) $0 * (0 * x) = x$.
(2) implies $(0 * (0 * x) = x$.
(7) $0 * x = 0 * y$ implies $x = y$.
$0 * x = 0 * y \to 0 * (0 * x) = 0 * (0 * y)$. By (6), $x = y$.
(8) $x * (0 * y) = y * (0 * x)$.

To prove this, we use (6) and the permutation rule:Its basic idea is due to T.Lei and C.Xi [5].

$$x * (0 * y) = (0 * (0 * x)) * (0 * y)$$

$$= (0 * (0 * y)) * (0 * x) = y * (0 * x).$$

If we define $x + y$ as $x * (0 * y)$, then (8) means $x + y = y + x$. Moreover,$x + (0 * x) = x * (0 * (0 * x)) = x * x = 0$. If we define $-x$ by $0 * x$, then $x + (-x) = 0$. In this system, a binary operation + and unary operation - are defined and + is commutative, and $-x$ acts as the inverse of x. It is obvious $x + (-y) = x * (0 * (0 * y)) = x * y$.

Finally we prove that the operation + is associative. In the proof, we use the commutativity of +.

$$x + (y + z) = x + (y * (0 * z)) = x * (0 * (y * (0 * z)))$$
$$= (0 * (0 * x)) * (0 * (y * (0 * z))) = (y * (0 * z)) * (0 * x)$$
$$= (y * (0 * x)) * (0 * z) = (y + x) * (0 * z)$$
$$= (y + x) + z = (x + y) + z.$$

Theorem. *In a basic BCI, if a binary operation $x + y$, an unary opearion $-x$ are introduced by $x * (0 * y), 0 * x$ respectively, then it is an Abelian group wth respect to +. $-x$ is the inverse of x.*

References

[1] M.Abe and K.Iseki, *A survey on BCK and BCI algebras*, Congresso de Logica Aplicada Techologia, LAPTEC 2000, 431-443.

[2] K.Iseki, *Some Fundamental Theorems on BCK*, Words, Semigroups and Transductions, Festschrift in Hornor of Gabriel Thierrin, World Sci. (2001), 231-238.

[3] K.Iseki, *Basic structure theorems on BCK*, to be published in LPTEC 2001.

[4] K.Iseki, *A way to BCI*, to be published in LAPTEC.

[5] T.Lei and C.Xi, *p-radical in BCI-algebras*, Math. Japonica, 30(1985), 511-517.

[6] J.Meng and Y.B.Jun, *BCK Algebras*, Kyung Moon Ss. Co. 1994.

14-6, Sakuragaoka Kitamachi, Takatsuki, Osaka 569-0817, Japan
e-mail: kiseki@mb.neweb.ne.jp