CONTINUATION SEMANTICS AND CPS-TRANSLATION OF $\lambda\mu$-CALCULUS

To the honor of Professor Masami Ito on his 60th birthday

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Abstract. We investigate relations between denotational semantics of $\lambda\mu$-calculus and syntactic interpretation by the so-called CPS-translation. It is shown that continuation denotational semantics of $\lambda\mu$-calculus has a simulation relation to direct denotational semantics following the CPS-translation.

1 Introduction Parigot [Pari92, Pari97] introduced the $\lambda\mu$-calculus from the viewpoint of classical logic, and established an extension of the Curry-Howard isomorphism [How80, Grif90, Murt91]. From the motivation of universal computation, we study denotational semantics of type-free $\lambda\mu$-calculus [Fuji02]. Given domains $U \times U \cong U \cong [U \to U]$ such as in Lambek-Scott [LS86], first we introduce continuation denotational semantics of the $\lambda\mu$-calculus. Next we define a syntactic translation, called a CPS-translation in the similitude of Plotkin [Plot75], from $\lambda\mu$-calculus to λ-calculus, and then give direct denotational semantics *à la* Scott [Scot72] of type-free λ-calculus. Finally we show that a simulation relation holds between the continuation denotational semantics and the CPS-translation followed by the direct denotational semantics.

2 $\lambda\mu$-calculus We give the definition of *type free* $\lambda\mu$-calculus [BHF99, BHF01]. The syntax of the $\lambda\mu$-terms is defined from variables denoted by x, λ-abstraction, application, μ-abstraction over names denoted by α, or named term in the form of $[\alpha]M$.

$$\Lambda\mu \ni M ::= x \mid \lambda x.M \mid MM \mid \mu\alpha.N$$
$$N ::= [\alpha]M$$

We write $\Lambda\mu$ to denote the set of $\lambda\mu$-terms. The set of reduction rules consists of the following rules:

Definition 1 ($\lambda\mu$-calculus)

(β) $(\lambda x.M_1)M_2 \to M_1[x := M_2]$

(η) $\lambda x.Mx \to M$ *if* $x \notin FV(M)$

(μ) $(\mu\alpha.N_1)M_2 \to \mu\alpha.N_1[\alpha \Leftarrow M_2]$

(μ_β) $[\alpha](\mu\beta.N) \to N[\beta := \alpha]$

(μ_η) $\mu\alpha.[\alpha]M \to M$ *if* $\alpha \notin FN(M)$

$FV(M)$ stands for the set of free variables in M, and $FN(M)$ for the set of free names in M. The $\lambda\mu$-term $M_1[\alpha \Leftarrow M_2]$ denotes a term obtained by replacing each subterm of the form $[\alpha]M$ in M_1 with $[\alpha](MM_2)$. This operation is inductively defined as follows:

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1. $x[\alpha \Leftarrow M] = x$

2. $(\lambda x.M_1)[\alpha \Leftarrow M] = \lambda x.M_1[\alpha \Leftarrow M]$

3. $(M_1 M_2)[\alpha \Leftarrow M] = (M_1[\alpha \Leftarrow M])(M_2[\alpha \Leftarrow M])$

4. $(\mu\beta.N)[\alpha \Leftarrow M] = \mu\beta.N[\alpha \Leftarrow M]$

5. $([\beta]M_1)[\alpha \Leftarrow M] = \begin{cases} [\beta]((M_1[\alpha \Leftarrow M])M), & \text{for } \alpha \equiv \beta \\ [\beta](M_1[\alpha \Leftarrow M]), & \text{otherwise} \end{cases}$

The binary relation $=_{\lambda\mu}$ over $\Lambda\mu$ denotes the symmetric, reflexive and transitive closure of the one step reduction relation, i.e., the equivalence relation induced from the reduction rules.

The λ-calculus together with surjective pairing is defined in the following:
$\Lambda^{\langle\rangle} \ni M ::= x \mid \lambda x.M \mid MM \mid \pi_1(M) \mid \pi_2(M) \mid \langle M, M\rangle$
We write $\Lambda^{\langle\rangle}$ for the set of λ-terms. The reudction rules of $\Lambda^{\langle\rangle}$ are defined as follows:

Definition 2 (λ-calculus with surjective pairing)

(β) $(\lambda x.M_1)M_2 \to M_1[x := M_2]$

(η) $\lambda x.Mx \to M$ *if* $x \notin FV(M)$

(π_i) $\pi_i\langle M_1, M_2\rangle \to M_i$ *for* $i = 1, 2$

(sp) $\langle \pi_1 M, \pi_2 M\rangle \to M$

We employ the notation $=_\lambda$ to indicate the symmetric, reflexive and transitive closure of the one step reduction of $\Lambda^{\langle\rangle}$.

3 Denotational semantics Along the line of denotational semantics such as in Stoy [Stoy77], a semantic function will interpret $\lambda\mu$-terms as elements in domain D of a cpo:
(1) there exists a least element $\bot \in D$;
(2) for every directed $X \subseteq D$ the supremum $\sqcup X \in D$ exits.
We say that a map $f : D \to D'$ is continuous if $f(\sqcup X) = \sqcup f(X)$ for any directed $X \subseteq D$.

Given cpo's $(D_1, \sqsubseteq_1)$ and $(D_2, \sqsubseteq_2)$, we define a cpo $[D_1 \to D_2] \stackrel{\text{def}}{=} \{f : D_1 \to D_2 \mid f \text{ continuous}\}$. Clearly $[D_1 \to D_2]$ is a poset under the partial order $f \sqsubseteq g$ iff $\forall x \in D_1.f(x) \sqsubseteq_2 g(x)$. For readability we sometimes write $D_2^{D_1}$ instead of $[D_1 \to D_2]$.

3.1 Direct denotational semantics Due to Scott [Scot72], domains for interpreting λ-terms can be constructed by the inverse limit of an inverse system of cpo's, so that one obtains recursive domains D such that $D \cong [D \to D]$. In order to simplify our discussion we assume that recursively defined domains are already given together with isomorphisms, as follows [LS86]:
$D \times D \cong D \cong [D \to D]$
with $\sigma : [D \times D \to D]$ and $\psi : [[D \to D] \to D]$
Let f be a function. Then $f(x\!:\!d)$ is an updated function as follows:
$f(x\!:\!d) : y \mapsto \begin{cases} d & \text{for } y = x \\ f(y) & \text{for } y \neq x \end{cases}$
We write ρ for an environment of semantics such that
$\rho : \{x_0, x_1, x_2, \ldots,\}_{\texttt{Var}} \cup \{\alpha_0, \alpha_1, \alpha_2, \ldots,\}_{\texttt{Name}} \to D$.
Then the following semantic function $\mathcal{D}[\![-]\!]$ defines *direct denotational semantics* of $\Lambda^{\langle\rangle}$:
$\mathcal{D}[\![-]\!] : \Lambda^{\langle\rangle} \times \texttt{Env} \to D$

Definition 3 (Direct denotational semantics of $\Lambda^{\langle\rangle}$)

1. $\mathcal{D}[\![x]\!]_\rho = \rho(x)$

2. $\mathcal{D}[\![\lambda x.M]\!]_\rho = \psi(\lambda d \in D.\mathcal{D}[\![M]\!]_{\rho(x:d)})$

3. $\mathcal{D}[\![M_1 M_2]\!]_\rho = \psi^{-1}\ \mathcal{D}[\![M_1]\!]_\rho\ \mathcal{D}[\![M_2]\!]_\rho$

4. $\mathcal{D}[\![\langle M_1, M_2\rangle]\!]_\rho = \sigma(\langle \mathcal{D}[\![M_1]\!]_\rho, \mathcal{D}[\![M_2]\!]_\rho\rangle)$

5. $\mathcal{D}[\![\pi_i(M)]\!]_\rho = (\lambda\langle d_1, d_2\rangle \in D \times D.d_i)(\sigma^{-1}(\mathcal{D}[\![M]\!]))$

Proposition 1 *For any $M_1, M_2 \in \Lambda^{\langle\rangle}$, if we have $M_1 =_\lambda M_2$ then $\mathcal{D}[\![M_1]\!]_\rho = \mathcal{D}[\![M_2]\!]_\rho$.*

Proof. See [Scot72, Bare84].

3.2 Continuation denotational semantics A continuation semantics provides a denotation which is a function sending the rest of the computation, called a continuation, to the final result. Let U be a continuation semantics domain, i.e., domains for our denotations. Then we should have $U = [K \to R]$, where K is a domain for continuations and R is for final results. Following the discussion in [Fuji01], we consider continuations K such as infinite lists $K \cong U \times K$. For continuation denotational semantics, we have to construct recursive domains such that $U = [K \to R]$ and $K \cong U \times K$. Due to [SR98], for non-empty R the recursive domain equation $K \cong R^K \times K$ can be solved by an inverse limit, so that one finally obtains $R^K \cong R_\infty$ with $R_\infty \cong [R_\infty \to R_\infty]$ of Scott domain [Scot72]. For continuation denotational semantics of $\Lambda\mu$, it is enought to assume again the recursive domains and the isomorphisms [LS86]:

$U \times U \cong U \cong [U \to U]$

with $\sigma : [U \times U \to U]$ and $\psi : [[U \to U] \to U]$

By the isomorphisms, we define continuous functions Ψ and Ψ^{-1}:

$$\begin{cases} \Psi \stackrel{\text{def}}{=} \lambda d' \in U^{U\times U}.\psi(d' \circ \sigma^{-1}) : [[U \times U \to U] \to U] \\ \Psi^{-1} \stackrel{\text{def}}{=} \lambda d \in U.\psi^{-1}(d) \circ \sigma : [U \to [U \times U \to U]] \end{cases}$$

Then functional compositions of them constitute identity functions by the definitions:

$\Psi \circ \Psi^{-1} = id_{U\to U}$ and $\Psi^{-1} \circ \Psi = id_{[U\times U \to U]\to[U\times U\to U]}$.

We write e to denote an environment for continuation semantics, such that

$e : \{x_0, x_1, x_2, \dots,\}_{\texttt{Var}} \cup \{\alpha_0, \alpha_1, \alpha_2, \dots,\}_{\texttt{Name}} \to [U \times U \to U]$.

Then *continuation denotational semantics* is defined by the semantic function $\mathcal{C}[\![-]\!]$, see also [HS97, SR98, Seli01]:

$\mathcal{C}[\![-]\!] : \Lambda\mu \times \texttt{Env} \to [U \times U \to U]$

Definition 4 (Continuation denotational semantics of $\Lambda\mu$)

1. $\mathcal{C}[\![x]\!]_e = e(x)$

2. $\mathcal{C}[\![\lambda x.M]\!]_e = \texttt{lam}\ (\lambda d \in U^{U\times U}.\mathcal{C}[\![M]\!]_{e(x:d)})$

3. $\mathcal{C}[\![M_1 M_2]\!]_e = \texttt{app}\ \mathcal{C}[\![M_1]\!]_e\ \mathcal{C}[\![M_2]\!]_e$

4. $\mathcal{C}[\![\mu\alpha.N]\!]_e = \texttt{Lam}(\lambda d \in U^{U\times U}.\mathcal{C}[\![N]\!]_{e(\alpha:d)})$

5. $\mathcal{C}[\![[\beta]M]\!]_e = \texttt{App}\ \mathcal{C}[\![M]\!]_e\ (e(\beta))$

where

(i) $\texttt{lam} = \lambda f.\lambda\langle d_1, d_2\rangle.f\ (\Psi^{-1}(d_1))\ (\sigma^{-1}(d_2)) : [[U^{U\times U} \to U^{U\times U}] \to U^{U\times U}]$
for $f \in [U^{U\times U} \to U^{U\times U}]$ *and* $d_1, d_2 \in U$.

(ii) $\texttt{app} = \lambda f.\lambda g.\lambda k.f\ \langle \Psi(g), \sigma(k)\rangle : [U^{U\times U} \to [U^{U\times U} \to U^{U\times U}]]$
for $f, g \in [U \times U \to U]$ *and* $k \in U \times U$.

(iii) $\texttt{Lam} = \lambda f.\lambda k.\Psi(f(\Psi^{-1}(\sigma k))) : [[U^{U\times U} \to U^{U\times U}] \to U^{U\times U}]$
for $f \in [U^{U\times U} \to U^{U\times U}]$ *and* $k \in U \times U$.

(iv) $\texttt{App} = \lambda f.\lambda g.\Psi^{-1}(f(\sigma^{-1}(\Psi g))) : [U^{U\times U} \to [U^{U\times U} \to U^{U\times U}]]$
for $f, g : [U \times U \to U]$.

Lemma 1 *All of the following functions are identity functions:*
$\texttt{lam} \circ \texttt{app} = id : [[U \times U \to U] \to [U \times U \to U]]$
$\texttt{app} \circ \texttt{lam} = id : [[U^{U\times U} \to U^{U\times U}] \to [U^{U\times U} \to U^{U\times U}]]$
$\texttt{Lam} \circ \texttt{App} = id : [[U \times U \to U] \to [U \times U \to U]]$
$\texttt{App} \circ \texttt{Lam} = id : [[U^{U\times U} \to U^{U\times U}] \to [U^{U\times U} \to U^{U\times U}]]$

Proof. By the definitions of `lam`, `app`, `Lam`, and `App`. □

Lemma 2 **(i)** $\mathcal{C}[\![M_1[x := M_2]]\!]_e = \mathcal{C}[\![M_1]\!]_{e(x:\mathcal{C}[\![M_2]\!]_e)}$

(ii) $\mathcal{C}[\![M_1[\alpha \Leftarrow M_2]]\!]_e = \mathcal{C}[\![M_1]\!]_{e(\alpha:K)}$ *where* $K = (\Psi^{-1} \circ \sigma)\langle \Psi(\mathcal{C}[\![M_2]\!]_e), \Psi(e(\alpha))\rangle$

Proof. By induction on the structure of M_1. We show only the case M_1 of $[\alpha]M$ for (ii).
$\mathcal{C}[\![([\alpha]M)[\alpha \Leftarrow M_2]]\!]_e$
$= \mathcal{C}[\![[\alpha]((M[\alpha \Leftarrow M_2])M_2)]\!]_e$
$= \texttt{App}\ (\texttt{app}\ \mathcal{C}[\![M[\alpha \Leftarrow M_2]]\!]_e\ \mathcal{C}[\![M_2]\!]_e)\ (e(\alpha))$
$= \texttt{App}\ (\lambda k.\mathcal{C}[\![M[\alpha \Leftarrow M_2]]\!]_e\ \langle \Psi(\mathcal{C}[\![M_2]\!]_e), \sigma(k)\rangle)\ (e(\alpha))$
$= \Psi^{-1}((\lambda k.\mathcal{C}[\![M[\alpha \Leftarrow M_2]]\!]_e\ \langle \Psi(\mathcal{C}[\![M_2]\!]_e), \sigma(k)\rangle)\ (\sigma^{-1}(\Psi(e(\alpha)))))$
$= \Psi^{-1}(\mathcal{C}[\![M[\alpha \Leftarrow M_2]]\!]_e\ \langle \Psi(\mathcal{C}[\![M_2]\!]_e), \Psi(e(\alpha))\rangle)$
$= \Psi^{-1}(\mathcal{C}[\![M]\!]_{e(\alpha:K)}\ \langle \Psi(\mathcal{C}[\![M_2]\!]_e), \Psi(e(\alpha))\rangle)$
where $K = (\Psi^{-1} \circ \sigma)\langle \Psi(\mathcal{C}[\![M_2]\!]_e), \Psi(e(\alpha))\rangle$ by the induction hypothesis
$= \Psi^{-1}(\mathcal{C}[\![M]\!]_{e(\alpha:K)}\ ((\sigma^{-1} \circ \Psi)(e(\alpha:K)(\alpha))))$
$= \texttt{App}\ \mathcal{C}[\![M]\!]_{e(\alpha:K)}\ (e(\alpha:K)(\alpha))$
$= \mathcal{C}[\![[\alpha]M]\!]_{e(\alpha:K)}$ □

Proposition 2 *For any* $M_1, M_2 \in \Lambda\mu$, *if we have* $M_1 =_{\lambda\mu} M_2$ *then* $\mathcal{C}[\![M_1]\!]_e = \mathcal{C}[\![M_2]\!]_e$.

Proof. By induction on the derivation of $=_{\lambda\mu}$ together with the lemma above. We show some of the base cases.
Case of (β):
$\mathcal{C}[\![(\lambda x.M_1)M_2]\!]_e$
$= \texttt{app}\ (\texttt{lam}(\lambda d.\mathcal{C}[\![M_1]\!]_{e(x:d)}))\ \mathcal{C}[\![M_2]\!]_e$
$= (\lambda d.\mathcal{C}[\![M_1]\!]_{e(x:d)})\mathcal{C}[\![M_2]\!]_e$ by Lamma 1
$= \mathcal{C}[\![M_1]\!]_{e(x:\mathcal{C}[\![M_2]\!]_e)}$
$= \mathcal{C}[\![M_1[x := M_2]]\!]_e$ by Lemma 2
Case of (μ):
$\mathcal{C}[\![(\mu\alpha.N)M]\!]_e$
$= \texttt{app}\ (\texttt{Lam}(\lambda d.\mathcal{C}[\![N]\!]_{e(\alpha:d)}))\ \mathcal{C}[\![M]\!]_e$
$= \texttt{app}\ (\lambda k.\Psi((\lambda d.\mathcal{C}[\![N]\!]_{e(\alpha:d)})(\Psi^{-1}(\sigma(k)))))\ \mathcal{C}[\![M]\!]_e$

$$\begin{aligned}
&= \texttt{app}\ (\lambda k.\Psi(\mathcal{C}[\![N]\!]_{e(\alpha:\Psi^{-1}(\sigma(k)))}))\ \mathcal{C}[\![M]\!]_e \\
&= \lambda\alpha.(\lambda k.\Psi(\mathcal{C}[\![N]\!]_{e(\alpha:\Psi^{-1}(\sigma(k)))}))\ \langle\Psi(\mathcal{C}[\![M]\!]_e), \sigma(\alpha)\rangle \\
&= \lambda\alpha.\Psi(\mathcal{C}[\![N]\!]_{e(\alpha:K)})\ \text{ where } K = (\Psi^{-1}\circ\sigma)\langle\Psi(\mathcal{C}[\![M]\!]_e), \sigma(\alpha)\rangle \\
&= \lambda\alpha.\Psi(\mathcal{C}[\![N]\!]_{e(\alpha:\Psi^{-1}(\sigma(\alpha)))(\alpha:L)}) \\
&\quad \text{where } L = (\Psi^{-1}\circ\sigma)\langle\Psi(\mathcal{C}[\![M]\!]_{e(\alpha:\Psi^{-1}(\sigma(\alpha)))}), \Psi(\Psi^{-1}(\sigma(\alpha)))\rangle \\
&\qquad\qquad = (\Psi^{-1}\circ\sigma)\langle\Psi(\mathcal{C}[\![M]\!]_e), \sigma(\alpha)\rangle \quad \text{since } \alpha \notin FN(M) \\
&\qquad\qquad = K \\
&= \lambda\alpha.\Psi(\mathcal{C}[\![N[\alpha \Leftarrow M]]\!]_{e(\alpha:\Psi^{-1}(\sigma(\alpha)))})\ \text{ by Lemma 2} \\
&= \lambda k.\Psi((\lambda d.\mathcal{C}[\![N[\alpha \Leftarrow M]]\!]_{e(\alpha:d)})(\Psi^{-1}(\sigma(k)))) \\
&= \texttt{Lam}(\lambda d.\mathcal{C}[\![N[\alpha \Leftarrow M]]\!]_{e(\alpha:d)}) \\
&= \mathcal{C}[\![\mu\alpha.N[\alpha \Leftarrow M]]\!]_e
\end{aligned}$$ □

4 CPS-translation As an extension of Plotkin [Plot75], see also [Fuji01] for the essential distinction, we next define a syntactic translation called a *CPS-translation* from $\Lambda\mu$ to $\Lambda^{\langle\rangle}$:

Definition 5 (CPS-translation from $\Lambda\mu$ to $\Lambda^{\langle\rangle}$)

1. $\underline{x} = \lambda k.xk$

2. $\underline{\lambda x.M} = \lambda k.\underline{M}(\pi_2 k)[x := \pi_1 k]$

3. $\underline{M_1 M_2} = \lambda k.\underline{M_1}\langle\underline{M_2}, k\rangle$

4. $\underline{\mu\alpha.N} = \lambda\alpha.\underline{N}$

5. $\underline{[\alpha]M} = \underline{M}\alpha$

Lemma 3 (i) $\underline{M_1}[x := \underline{M_2}] =_\lambda \underline{M_1[x := M_2]}$

(ii) $\underline{M_1[\alpha \Leftarrow M_2]} =_\lambda \underline{M_1}[\alpha := \langle\underline{M_2}, \alpha\rangle]$

Proof. By induction on the structure of M_1. We show the case M_1 of $[\alpha]M$ for (ii).

$$\begin{aligned}
&\underline{([\alpha]M)[\alpha \Leftarrow M_2] = [\alpha]((M[\alpha \Leftarrow M_2])M_2)} \\
&= (\lambda k.\underline{M[\alpha \Leftarrow M_2]}\langle\underline{M_2}, k\rangle)\alpha \\
&=_\beta \underline{M[\alpha \Leftarrow M_2]}\langle\underline{M_2}, \alpha\rangle \\
&=_\lambda \underline{M}[\alpha := \langle\underline{M_2}, \alpha\rangle]\langle\underline{M_2}, \alpha\rangle \quad \text{by the induction hypothesis} \\
&= (\underline{M}\alpha)[\alpha := \langle\underline{M_2}, \alpha\rangle] \\
&= \underline{[\alpha]M}[\alpha := \langle\underline{M_2}, \alpha\rangle]
\end{aligned}$$ □

Proposition 3 *For any $M_1, M_2 \in \Lambda\mu$, if we have $M_1 =_{\lambda\mu} M_2$ then $\underline{M_1} =_\lambda \underline{M_2}$.*

Proof. By induction on the derivation of $=_{\lambda\mu}$ together with the lemma above. We show some of the base cases.

Case of (β):

$$\begin{aligned}
&\underline{(\lambda x.M_1)M_2} = \lambda k.\underline{\lambda x.M_1}\langle\underline{M_2}, k\rangle \\
&= \lambda k.(\lambda k.\underline{M_1}(\pi_2 k)[x := \pi_1 k])\langle\underline{M_2}, k\rangle \\
&=_\beta \lambda k.\underline{M_1}(\pi_2\langle\underline{M_2}, k\rangle)[x := \pi_1\langle\underline{M_2}, k\rangle] \\
&=_\pi \lambda k.\underline{M_1}k[x := \underline{M_2}] \\
&=_\beta \underline{M_1}[x := \underline{M_2}] \\
&=_\lambda \underline{M_1[x := M_2]} \quad \text{by Lemma 3 (i)}
\end{aligned}$$

Case of (μ):

$$\begin{aligned}
&\underline{(\mu\alpha.N_1)M_2} = \lambda k.\underline{\mu\alpha.N_1}\langle\underline{M_2}, k\rangle \\
&= \lambda k.(\lambda\alpha.\underline{N_1})\langle\underline{M_2}, k\rangle \\
&=_\beta \lambda k.\underline{N_1}[\alpha := \langle\underline{M_2}, k\rangle] = \lambda\alpha.\underline{N_1}[\alpha := \langle\underline{M_2}, \alpha\rangle] \\
&=_\lambda \lambda\alpha.\underline{N_1[\alpha \Leftarrow M_2]} \quad \text{by Lemma 3 (ii)} \\
&= \underline{\mu\alpha.N_1[\alpha \Leftarrow M_2]}
\end{aligned}$$ □

5 Direct with CPS semantics Let $\rho : \mathtt{Var} \cup \mathtt{Name} \to D$ be an environment. Then we define the following semantic function $\mathcal{D}^C[\![-]\!] : \Lambda\mu \times \mathtt{Env} \to D$, called direct with CPS semantics here.

Definition 6 (Direct with CPS semantics)

1. $\mathcal{D}^C[\![x]\!]_\rho = \rho(x)$

2. $\mathcal{D}^C[\![\lambda x.M]\!]_\rho = \mathtt{cur}(\lambda d \in D.\mathcal{D}^C[\![M]\!]_{\rho(x:d)})$

3. $\mathcal{D}^C[\![M_1 M_2]\!]_\rho = \mathtt{ev}\ \mathcal{D}^C[\![M_1]\!]_\rho\ \mathcal{D}^C[\![M_2]\!]_\rho$

4. $\mathcal{D}^C[\![\mu\alpha.N]\!]_\rho = \psi(\lambda d \in D.\mathcal{D}^C[\![N]\!]_{\rho(\alpha:d)})$

5. $\mathcal{D}^C[\![[\beta]M]\!]_\rho = \psi^{-1}\ \mathcal{D}^C[\![M]\!]_\rho\ (\rho(\beta))$

where

(i) $\mathtt{cur} = \lambda f \in D^D.\psi(\lambda d \in D.\psi^{-1}\ (f(\mathtt{p}_1(\sigma^{-1}d)))\ (\mathtt{p}_2(\sigma^{-1}d))) : [[D \to D] \to D]$

(ii) $\mathtt{ev} = \lambda f \in D.\lambda g \in D.\psi(\lambda d \in D.\psi^{-1}\ f\ (\sigma\langle g, d\rangle)) : [D \to [D \to D]]$

(iii) $\mathtt{p_i} = \lambda\langle d_1, d_2\rangle \in D \times D.d_i : [D \times D \to D] \quad (i = 1, 2)$

Lemma 4 *Following functions are identity functions:*
$\mathtt{cur} \circ \mathtt{ev} = id : [D \to D]$
$\mathtt{ev} \circ \mathtt{cur} = id : [[D \to D] \to [D \to D]]$

Proof. By the definitions of cur and ev. □

Proposition 4 $\forall M \in \Lambda\mu.\ \ \mathcal{D}[\![\underline{M}]\!]_\rho = \mathcal{D}^C[\![M]\!]_\rho$

Proof. By straightforward induction on the structure of M. We show some of the cases here.

Case M of $\lambda x.M_1$:

$\mathcal{D}[\![\underline{\lambda x.M_1}]\!]_\rho$
$= \mathcal{D}[\![\lambda k.\underline{M_1}(\pi_2 k)[x := \pi_1 k]]\!]_\rho$
$= \psi(\lambda d.(\psi^{-1}\ \mathcal{D}[\![\underline{M_1}[x := \pi_1 k]]\!]_{\rho(k:d)}\ \mathcal{D}[\![\pi_2 k]\!]_{\rho(k:d)}))$
$= \psi(\lambda d.(\psi^{-1}\ \mathcal{D}[\![\underline{M_1}]\!]_{\rho(k:d)(x:[\![\pi_1 k]\!]_{\rho(k:d)})}\ \mathcal{D}[\![\pi_2 k]\!]_{\rho(k:d)}))$
$= \psi(\lambda d.(\psi^{-1}\ \mathcal{D}[\![\underline{M_1}]\!]_{\rho(k:d)(x:\mathtt{p}_1(\sigma^{-1}(d)))}\ (\mathtt{p}_2(\sigma^{-1}(d)))))$
$= \psi(\lambda d.(\psi^{-1}\ \mathcal{D}[\![\underline{M_1}]\!]_{\rho(x:\mathtt{p}_1(\sigma^{-1}(d)))}\ (\mathtt{p}_2(\sigma^{-1}(d)))))$ since $k \notin FV(\underline{M_1})$
$= \mathtt{cur}(\lambda d.\mathcal{D}[\![\underline{M_1}]\!]_{\rho(x:d)})$
$= \mathtt{cur}(\lambda d.\mathcal{D}^C[\![M_1]\!]_{\rho(x:d)})$ by the induction hypothesis
$= \mathcal{D}^C[\![\lambda x.M_1]\!]_\rho$

Case M of $\mu\alpha.N$:

$\mathcal{D}[\![\underline{\mu\alpha.N}]\!]_\rho = \mathcal{D}[\![\lambda\alpha.\underline{N}]\!]_\rho$
$= \psi(\lambda d.\mathcal{D}[\![\underline{N}]\!]_{\rho(\alpha:d)})$
$= \psi(\lambda d.\mathcal{D}^C[\![N]\!]_{\rho(\alpha:d)})$ by the induction hypothesis
$= \mathcal{D}^C[\![\mu\alpha.N]\!]_\rho$ □

In order to investigate relations between continuation denotational semantics and direct denotational semantics with CPS-translation, from Proposition 4 we study relations between $\mathcal{C}[\![-]\!]$ and $\mathcal{D}^C[\![-]\!]$ in the next section.

6 Relations between continuation denotational semantics and direct with CPS semantics Let D' be $[U \times U \to U]$. We write $\bot_D$ for the least element of the cpo D. Along the line of Reynolds [Reyn74], we define a *simulation relation* $\mathcal{S}$ over cpo's $D \times D'$ coinductively as follows:

Definition 7 (Simulation relation $\mathcal{S}$)
$(d_1\ \mathcal{S}\ d_2)$
if and only if
$[(d_1 = \bot_D) \wedge (d_2 = \bot_{D'})]$
$\vee$
$\exists f_1, f_2.\ \{[(d_1 = \mathtt{cur}(f_1)) \wedge (d_2 = \mathtt{lam}(f_2)) \wedge (\forall d \in D, d' \in D'.(d\ \mathcal{S}\ d') \Longrightarrow (f_1 d\ \mathcal{S}\ f_2 d'))]$
$\qquad \vee\ [(d_1 = \psi(f_1)) \wedge (d_2 = \mathtt{Lam}(f_2)) \wedge (\forall d \in D, d' \in D'.(d\ \mathcal{S}\ d') \Longrightarrow (f_1 d\ \mathcal{S}\ f_2 d'))]\}$
$\vee$
$\exists a_1, a_2, a_3, a_4.\ \{[(d_1 = \mathtt{ev}\ a_1\ a_3) \wedge (d_2 = \mathtt{app}\ a_2\ a_4) \wedge (a_1\ \mathcal{S}\ a_2) \wedge (a_3\ \mathcal{S}\ a_4)]$
$\vee\ [(d_1 = \psi^{-1}\ a_1\ a_3) \wedge (d_2 = \mathtt{App}\ a_2\ a_4) \wedge (a_1\ \mathcal{S}\ a_2) \wedge (a_3\ \mathcal{S}\ a_4)]\}$

It is obtained that the two semantic definitions $\mathcal{C}[\![-]\!]$ and $\mathcal{D}[\![\underline{-}]\!]$ give denotations between which the simulation relation holds if so does each environment.

Theorem 1 *If* $(\rho(x)\ \mathcal{S}\ e(x))$ *for any* $x \in \mathtt{Var} \cup \mathtt{Name}$, *then we have* $(\mathcal{D}[\![\underline{M}]\!]_\rho\ \mathcal{S}\ \mathcal{C}[\![M]\!]_e)$.

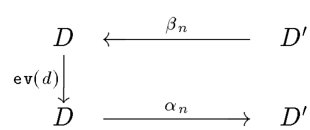
Proof. From Proposition 4, we will prove $(\mathcal{D}^C[\![M]\!]_\rho\ \mathcal{S}\ \mathcal{C}[\![M]\!]_e)$ by induction on the structure of M. We show only the case M of $\lambda x.M_1$. From the definitions of $\mathcal{D}^C[\![-]\!]$ and $\mathcal{C}[\![-]\!]$, we have $\mathcal{D}^C[\![\lambda x.M_1]\!]_\rho = \mathtt{cur}(f_1)$ and $\mathcal{C}[\![\lambda x.M_1]\!]_e = \mathtt{lam}(f_2)$ where $f_1 = \lambda d \in D.\mathcal{D}^C[\![M_1]\!]_{\rho(x:d)}$ and $f_2 = \lambda d \in U^{U \times U}.\mathcal{C}[\![M_1]\!]_{e(x:d)}$. It is enough to prove that $(f_1 d_1\ \mathcal{S}\ f_2 d_2)$ for any d_1 and d_2 such that $(d_1\ \mathcal{S}\ d_2)$. Assume that $(d_1\ \mathcal{S}\ d_2)$. Then $(\rho(x:d_1)(y)\ \mathcal{S}\ e(x:d_2)(y))$ for any $y \in \mathtt{Var} \cup \mathtt{Name}$. Hence the induction hypothesis gives $(\mathcal{D}^C[\![M_1]\!]_{\rho(x:d_1)}\ \mathcal{S}\ \mathcal{C}[\![M_1]\!]_{e(x:d_2)})$, that is, $(f_1 d_1\ \mathcal{S}\ f_2 d_2)$. □

Let $\alpha : [D \to D']$ and $\beta : [D' \to D]$ be the least upper bounds, respectively defined simultaneously in the following:

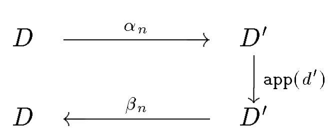
$$\alpha = \bigsqcup_{n=0}^{\infty} \alpha_n \qquad \beta = \bigsqcup_{n=0}^{\infty} \beta_n$$

where

$$\begin{cases} \alpha_0(d) &= \bot_{D'} \\ \alpha_{n+1}(d) &= \mathtt{lam}(\alpha_n \circ \mathtt{ev}(d) \circ \beta_n) \end{cases}$$



$$\begin{cases} \beta_0(d') &= \bot_D \\ \beta_{n+1}(d') &= \mathtt{cur}(\beta_n \circ \mathtt{app}(d') \circ \alpha_n) \end{cases}$$



Moreover, let I_D and $J_{D'}$ be the least upper bounds, respectively defined as follows:

$$I_D = \bigsqcup_{n=0}^{\infty} I_n \qquad J_{D'} = \bigsqcup_{n=0}^{\infty} J_n$$

where

$$\begin{cases} I_0(d) & = & \bot_D \\ I_{n+1}(d) & = & \mathtt{cur}(I_n \circ \mathtt{ev}(d) \circ I_n) \end{cases} \qquad \begin{cases} J_0(d') & = & \bot_{D'} \\ J_{n+1}(d') & = & \mathtt{lam}(J_n \circ \mathtt{app}(d') \circ J_n) \end{cases}$$

Then one can show the following lemma:

Lemma 5 **(1)** $\forall d \in D.(I_n(d)\ \mathcal{S}\ \alpha_n(d))$

(2) $\forall d \in D, \forall d' \in D'.(d\ \mathcal{S}\ d') \Longrightarrow I_n(d) = \beta_n(d')$

Proof. By simultaneous induction on n.
Base cases:
From the definition we have that $(\bot_D\ \mathcal{S}\ \bot_{D'})$ and $I_0(d) = \bot_D = \beta_0(d')$.
Step case for (1):
We have that $\begin{cases} I_{n+1}(d) = \mathtt{cur}(f_1) & \text{where } f_1 = I_n \circ \mathtt{ev}(d) \circ I_n; \\ \alpha_{n+1}(d) = \mathtt{lam}(f_2) & \text{where } f_2 = \alpha_n \circ \mathtt{ev}(d) \circ \beta_n. \end{cases}$
We will show that $(f_1 d_1\ \mathcal{S}\ f_2 d_2)$ for any d_1, d_2 such that $(d_1\ \mathcal{S}\ d_2)$. Assume that $(d_1\ \mathcal{S}\ d_2)$. Then $f_2 d_2 = \alpha_n(\mathtt{ev}(d)(\beta_n(d_2))) = \alpha_n(\mathtt{ev}(d)(I_n(d_1)))$ by the induction hypothesis of (2). Hence from the induction hypothesis of (1), we have $(I_n(\mathtt{ev}(d)(I_n(d_1)))\ \mathcal{S}\ \alpha_n(\mathtt{ev}(d)(I_n(d_1))))$, that is, $(f_1 d_1\ \mathcal{S}\ f_2 d_2)$.
Step case for (2):
We have that $\begin{cases} I_{n+1}(d) = \mathtt{cur}(f_1) & \text{where } f_1 = I_n \circ \mathtt{ev}(d) \circ I_n; \\ \beta_{n+1}(d') = \mathtt{cur}(f_2) & \text{where } f_2 = \beta_n \circ \mathtt{app}(d') \circ \alpha_n. \end{cases}$
It is enought to show that $f_1 a = f_2 a$ for any $a \in D$. The induction hypothesis of (1) gives that $(I_n a\ \mathcal{S}\ \alpha_n a)$ for any $a \in D$. Then we have that $((\mathtt{ev}\ d\ (I_n a))\ \mathcal{S}\ (\mathtt{app}\ d'\ (\alpha_n a)))$ from the assumption of $(d\ \mathcal{S}\ d')$. Now the induction hypothesis of (2) proves that $I_n(\mathtt{ev}\ d\ (I_n a)) = \beta_n(\mathtt{app}\ d'\ (\alpha_n a))$, and hence we have $f_1 = f_2$, which gives $I_{n+1}(d) = \beta_{n+1}(d')$. □

For any n we have the following facts:

Fact 1 **(i)** $I_n \circ I_n = I_n = \beta_n \circ \alpha_n$ *and* $J_n \circ J_n = J_n = \alpha_n \circ \beta_n$

(ii) $I_n \sqsubseteq I_{n+1}$, $\alpha_n \sqsubseteq \alpha_{n+1}$, $\beta_n \sqsubseteq \beta_{n+1}$, *and* $J_n \sqsubseteq J_{n+1}$

Let R be a relation between D and D'. Following Reynolds [Reyn74], R is called *directed complete* (or *admissible*) if and only if $R(x, y)$ whenever $x \overset{\text{def}}{=} \sqcup\{x_n \mid n \geq 0\}$ and $y \overset{\text{def}}{=} \sqcup\{y_n \mid n \geq 0\}$ for two ω-chains $x_0 \sqsubseteq x_1 \sqsubseteq x_2 \sqsubseteq \cdots$ and $y_0 \sqsubseteq y_1 \sqsubseteq y_2 \sqsubseteq \cdots$ such that $R(x_n, y_n)$ for any n.

Proposition 5 *Assume that $\mathcal{S}$ is directed complete.*

(1) $\forall d \in D.(I_D(d)\ \mathcal{S}\ \alpha(d))$

(2) $\forall d \in D, \forall d' \in D'.(d\ \mathcal{S}\ d') \Longrightarrow I_D(d) = \beta(d')$

Proof. From Lemma 5. □

Proposition 6 *Assume that $\mathcal{S}$ is directed complete.*

(1) $\forall d' \in D'.(\beta(d')\ \mathcal{S}\ J_{D'}(d'))$

(2) $\forall d \in D, \forall d' \in D'.(d\ \mathcal{S}\ d') \Longrightarrow \alpha(d) = J_{D'}(d')$

Proof. Following the similar pattern to the proof of Proposition 5. □

Now it can be shown that the functions α and β make the continuation denotational semantics related to the direct denotational semantics following the CPS-translation:

$$\begin{array}{ccccccc} M & \in & \Lambda\mu & \xrightarrow{CPS} & \Lambda^{\langle\rangle} & \ni & \underline{M} \\ {\scriptstyle continuation}\downarrow & & & & & & \downarrow{\scriptstyle direct} \\ \mathcal{C}[\![M]\!] & \in & D' & \xleftarrow{\alpha}\!\!\xrightarrow{\beta} & D & \ni & \mathcal{D}[\![\underline{M}]\!] \end{array}$$

The CPS-translation followed by the direct denotational semantics is essentially the same as the continuation denotational semantics. The continuation denotational semantics establishes an interpretation which involves the effect of the CPS-translation at the syntactic level, i.e., $\mathcal{C}[\![-]\!]$ is a semantical counterpart of the syntactic interpretation by the CPS-translation.

Theorem 2 *$I_D(\mathcal{D}[\![\underline{M}]\!]_{I_D\circ\rho}) = \beta(\mathcal{C}[\![M]\!]_{\alpha\circ\rho})$ and $\alpha(\mathcal{D}[\![\underline{M}]\!]_{\beta\circ e}) = J_{D'}(\mathcal{C}[\![M]\!]_{J_{D'}\circ e})$, provided that $\mathcal{S}$ is directed complete.*

Proof. $\forall x \in \texttt{Var} \cup \texttt{Name}.\ (I_D(\rho(x))\ \mathcal{S}\ \alpha(\rho(x)))$ by Proposition 5
$\Longrightarrow (\mathcal{D}[\![\underline{M}]\!]_{I_D\circ\rho}\ \mathcal{S}\ \mathcal{C}[\![M]\!]_{\alpha\circ\rho})$ by Theorem 1
$\Longrightarrow I_D(\mathcal{D}[\![\underline{M}]\!]_{I_D\circ\rho}) = \beta(\mathcal{C}[\![M]\!]_{\alpha\circ\rho})$ by Proposition 5
The another statement can be verified similarly by Theorem 1 and Proposition 6. □

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