CHAOTIC ORDER AMONG MEANS OF POSITIVE OPERATORS

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Abstract. M. Fujii and R. Nakamoto discuss the monotonity of the operator function $F(r) = ((1-\mu)A^r + \mu B^r)^{\frac{1}{r}}$ $(r \in \mathbf{R})$ for given $A, B > 0$ and $\mu \in [0,1]$. They proved it under the usual operator order: $F(r) \le F(s)$ if $1 \le r \le s$ or $1 \le s \le 2r$. Furthermore, they proved it under the chaotic order: $F(r) \ll F(s)$ if $r < s$ and consequently $\mathbf{s} - \lim_{r\to 0} F(r) = A \diamondsuit_\mu B$, where $\diamondsuit_\mu$ is the chaotic geometric mean defined by $A \diamondsuit_\mu B := e^{(1-\mu)\log A + \mu \log B}$.

The aim of this paper is to generalize the above mentioned as follows:
Let $M_k^{[r]}(\mathbf{A}; w) := (\sum_{j=1}^k \omega_j \ A_j^r)^{1/r}$ $(r \in \mathbf{R}\backslash\{0\})$ be weighted power mean of positive operators A_j, $\mathsf{Sp}(A_j) \subseteq [m, M]$ $(j = 1, \ldots, k)$, $0 < m < M$ and $\omega_j \in \mathbf{R}_+$, $\sum_{j=1}^k \omega_j = 1$. Let $M_k^{[0]}(\mathbf{A}; w)$ be the corresponding chaotic geometric mean. If $r \le s$ then real constants α_1 and α_1 such that $\alpha_2 M_k^{[s]}(\mathbf{A}; w) \le M_k^{[r]}(\mathbf{A}; w) \le \alpha_1 M_k^{[s]}(\mathbf{A}; w)$, are determined, when $r \notin \langle -1, 1\rangle$, $r \ne 0$ or $s \notin \langle -1, 1\rangle$, $s \ne 0$. Furthermore, if $r \le s$ then real constant Δ such that $\Delta M_k^{[s]}(\mathbf{A}; w) \ll M_k^{[r]}(\mathbf{A}; w) \ll M_k^{[s]}(\mathbf{A}; w)$, is determined.

1 Introduction.

Let $\mathcal{B}(H)$ be the C^*-algebra of all bounded linear operators on a Hilbert space H, $\mathcal{B}_+(H)$ be the set of all positive operators of $\mathcal{B}(H)$ and $\mathsf{Sp}(A)$ be the spectrum of the operator A. We denote by $\ge$ the usual order among self-adjoint operator on H (i.e. $A \ge B$ if $A - B \in \mathcal{B}_+(H)$). We denote by $\gg$ the chaotic order among invertible operators of $\mathcal{B}_+(H)$ (i.e. for $A, B > 0$, $A \gg B$ if $\log A \ge \log B$).

M. Fujii and R. Nakamoto [2] discuss the monotonity of the operator function $F(r) = ((1-\mu)A^r + \mu B^r)^{\frac{1}{r}}$ $(r \in \mathbf{R})$ for given $A, B > 0$ and $\mu \in [0,1]$. They do it under the usual operator order:

Lemma A (M.Fujii-R.Nakamoto). *Let $A, B > 0$ and $\mu \in [0,1]$ be given. Then the operator function $F(r) = ((1-\mu)A^r + \mu B^r)^{\frac{1}{r}}$ $(r \in \mathbf{R})$ is monotone increasing on $[1, \infty\rangle$, i.e. $F(r) \le F(s)$ if $1 \le r \le s$. In addition $F(r) \le F(s)$ if $1 \le s \le 2r$, and $F(r)$ is not monotone increasing on $\langle 0, 1]$ in general.*

Next, they do it under the chaotic order:

Lemma B (M.Fujii-R.Nakamoto). *The operator function $F(r)$ is monotone increasing under the chaotic order, i.e. $F(r) \ll F(s)$ if $r < s$. In particular, $\mathbf{s}-\lim_{r\to 0} F(r) = A \diamondsuit_\mu B$, where $\diamondsuit_\mu$ is the chaotic geometric mean defined by $A \diamondsuit_\mu B := e^{(1-\mu)\log A + \mu \log B}$.*

We consider the following weighted power means of positive operators (see [6, 4, 1]). Let $A_j \in \mathcal{B}_+(H)$ with $\mathsf{Sp}(A_j) \subseteq [m, M]$, $0 < m < M$, $(j = 1, \ldots, k)$ and $\omega_j \in \mathbf{R}_+$ such that

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$\sum_{j=1}^{k} \omega_j = 1$. We define

$$M_k^{[r]}(\mathbf{A}; w) := \begin{cases} \left(\sum_{j=1}^{k} \omega_j \ A_j^r\right)^{1/r} & \text{if} \quad r \in \mathbf{R}\backslash\{0\}, \\ \exp\left(\sum_{j=1}^{k} \omega_j \ \log A_j\right) & \text{if} \quad r = 0. \end{cases} \tag{1}$$

The limit

$$\mathbf{s} - \lim_{r \to 0} M_k^{[r]}(\mathbf{A}; w) = M_k^{[0]}(\mathbf{A}; w)$$

exists (see [1] or Lemma 7) and $M_k^{[0]}(\mathbf{A}; w)$ reduces to the usual geometric mean in the case of commuting operators. To remind, we define usual geometric mean by $G(\mathbf{A}; w) := A_k^{1/2} \left(A_k^{-1/2} A_{k-1}^{1/2} \cdots \left(A_3^{-1/2} A_2^{1/2} \left(A_2^{-1/2} A_1 A_2^{-1/2}\right)^{u_1} A_2^{1/2} A_3^{-1/2}\right)^{u_2} \cdots A_{k-1}^{1/2} A_k^{-1/2}\right)^{u_{k-1}} A_k^{1/2}$ where $u_j = 1 - \omega_{j+1} / \sum_{l=1}^{j+1} \omega_l$ $(j = 1, \ldots, k-1)$.

The aim of this paper is to generalize the above results of Fujii-Nakamoto as follows: We shall determine real constants α_1 and α_1 such that

$$\alpha_2 M_k^{[s]}(\mathbf{A}; w) \le M_k^{[r]}(\mathbf{A}; w) \le \alpha_1 M_k^{[s]}(\mathbf{A}; w),$$

holds if $r \le s$, $r \notin \langle -1, 1 \rangle$, $r \neq 0$ or $s \notin \langle -1, 1 \rangle$, $s \neq 0$.

Furthermore, we shall determine real constant Δ such that

$$\Delta M_k^{[s]}(\mathbf{A}; w) \ll M_k^{[r]}(\mathbf{A}; w) \ll M_k^{[s]}(\mathbf{A}; w),$$

holds if $r \le s$.

2 The usual operator order among means.

In this section we discuss the usual operator order among power means (1) when $r \in \mathbf{R}\backslash\{0\}$.

Theorem 1. *Let $A_j \in \mathcal{B}_+(H)$ with $\mathsf{Sp}(A_j) \subseteq [m, M]$, $0 < m < M$, $(j = 1, \ldots, k)$ and $\omega_j \in \mathbf{R}_+$ such that $\sum_{j=1}^{k} \omega_j = 1$. If $r, s \in \mathbf{R}$, $r \le s$, then*

$$\alpha_2 M_k^{[s]}(\mathbf{A}; w) \le M_k^{[r]}(\mathbf{A}; w) \le \alpha_1 M_k^{[s]}(\mathbf{A}; w), \tag{2}$$

where

$$\alpha_2 = \ \Delta \quad \text{if} \quad \text{(vi)}, \qquad \alpha_1 = \begin{cases} 1 & \text{if} \quad \text{(i) or (ii) or (iii)}, \\ \Delta^{-1} & \text{if} \quad \text{(iv) or (v)}, \end{cases}$$

and

$$\Delta = \left\{ \frac{r(\kappa^s - \kappa^r)}{(s-r)(\kappa^r - 1)} \right\}^{-\frac{1}{s}} \left\{ \frac{s(\kappa^r - \kappa^s)}{(r-s)(\kappa^s - 1)} \right\}^{\frac{1}{r}}, \qquad \kappa = \frac{M}{m}.$$

Here we denote intervals from (i) to (vi) as on the Table 1 (see Figure 1).

Remark 2. *B. Mond and J. Pečarić [6, 4] proved the following inequalities*

$$M_k^{[r]}(\mathbf{A}; w) \le M_k^{[s]}(\mathbf{A}; w) \qquad \text{if} \quad \text{(i) or (ii) or (iii)},$$
$$M_k^{[s]}(\mathbf{A}; w) \le \Delta^{-1} M_k^{[r]}(\mathbf{A}; w) \qquad \text{if} \quad \text{(vi)}.$$

(i)	$s \geq r, \ \ s \notin \langle -1, 1\rangle, \ \ r \notin \langle -1, 1\rangle,$
(ii)	$s \geq 1 \geq r \geq 1/2,$
(iii)	$r \leq -1 \leq s \leq -1/2,$
(iv)	$s \geq 1, \ \ -1 < r < 1/2, \ \ r \neq 0,$
(v)	$r \leq -1, \ \ -1/2 < s < 1, \ \ s \neq 0,$
(vi)	$s > r, \ \ s \notin \langle -1, 1\rangle, \ \ r \neq 0$ or $r \notin \langle -1, 1\rangle, \ \ s \neq 0.$

Table 1: *Intervals from* (i) *to* (vi)

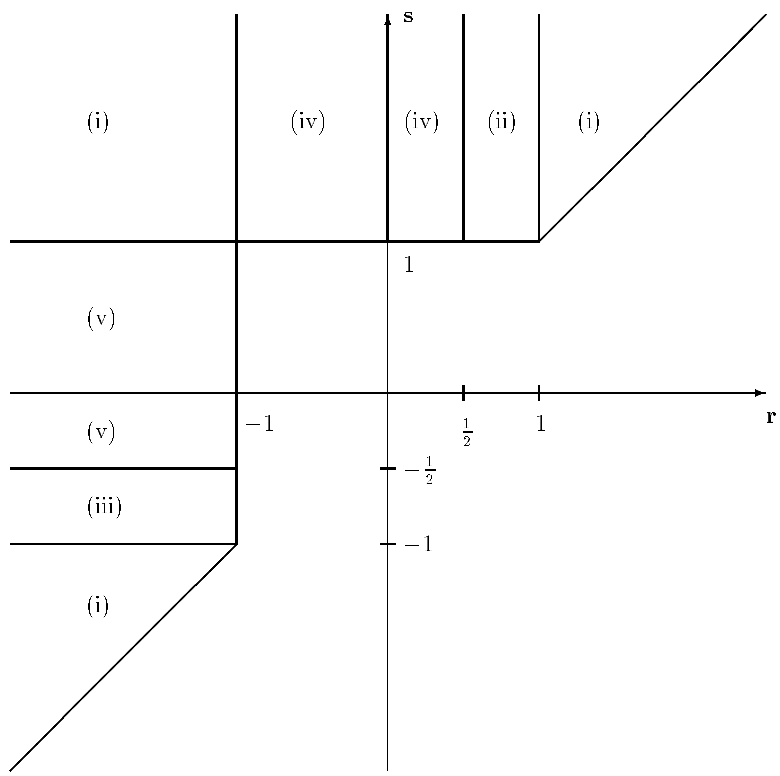

Figure 1

For the proof of Theorem 1 we need some results. If Jensen's inequality and Mond-Pečarić method applied, then the following two theorems hold:

Theorem J ([6, Theorem 1]). *Let* $\mathcal{J} \subseteq \mathbf{R}$ *be an interval. Let* $A_j \in \mathcal{B}_+(H)$ *with* $\mathsf{Sp}(A_j) \subseteq \mathcal{J}$ $(j = 1, \ldots, k)$ *and* $\omega_j \in \mathbf{R}_+$ *such that* $\sum_{j=1}^k \omega_j = 1$. *If* f *is a operator convex* *function on $\mathcal{J}$, then*

$$f\left(\sum_{j=1}^{k}\omega_j A_j\right) \le \sum_{j=1}^{k}\omega_j f(A_j). \tag{3}$$

Theorem MP ([5, Theorem 5]). *Let $A_j \in \mathcal{B}_+(H)$ with $\mathsf{Sp}(A_j) \subseteq [m, M]$, $0 < m < M$, $(j = 1, \dots, k)$ and $\omega_j \in \mathbf{R}_+$ such that $\sum_{j=1}^{k}\omega_j = 1$. Let f be a strictly convex twice differentiable function on $[m, M]$. Suppose in addition that either of the following conditions holds* (i) *$f > 0$ on $[m, M]$ or* (ii) *$f < 0$ on $[m, M]$. Then the following inequality*

$$\sum_{j=1}^{k}\omega_j f(A_j) \le \alpha\, f\left(\sum_{j=1}^{k}\omega_j A_j\right), \tag{4}$$

holds for some $\alpha > 1$ in case (i) *or $0 < \alpha < 1$ in case* (ii).

More precisely, a value of α for (4) may be determined as follows: Let $\mu_f = (f(M) - f(m))/(M - m)$. If $\mu_f = 0$, let $t = t_o$ be the unique solution of the equation $f'(t) = 0$ $(m < t_o < M)$; then $\alpha = f(m)/f(t_o)$ suffices for (4). If $\mu_f \neq 0$, let $t = t_o$ be the unique solution of the equation $\mu_f f(t) - f'(t)\,(f(m) + \mu_f(t - m)) = 0$; then $\alpha = \mu_f/f'(t_o)$ suffices for (4).

Corollary 3. *Let $A_j \in \mathcal{B}_+(H)$ with $\mathsf{Sp}(A_j) \subseteq [m, M]$, $0 < m < M$, $(j = 1, \dots, k)$ and $\omega_j \in \mathbf{R}_+$ such that $\sum_{j=1}^{k}\omega_j = 1$. If $p \in \mathbf{R}$, then*

$$\alpha_2\left(\sum_{j=1}^{k}\omega_j A_j\right)^p \le \sum_{j=1}^{k}\omega_j A_j^p \le \alpha_1\left(\sum_{j=1}^{k}\omega_j A_j\right)^p \tag{5}$$

with

$$\alpha_2 = \begin{cases} \tilde{\Delta}^{-1} & \text{if} \quad p < -1 \ \text{or} \ p > 2, \\ 1 & \text{if} \quad -1 \le p < 0 \ \text{or} \ 1 \le p \le 2, \\ \tilde{\Delta} & \text{if} \quad 0 < p < 1, \end{cases} \qquad \alpha_1 = \begin{cases} \tilde{\Delta} & \text{if} \quad p < 0 \ \text{or} \ p > 1, \\ 1 & \text{if} \quad 0 < p \le 1, \end{cases}$$

where

$$\begin{aligned} \tilde{\Delta} \equiv C(m, M; p) &= \frac{Mm^p - mM^p}{(1-p)(M-m)}\left(\frac{1-p}{p}\frac{M^p - m^p}{Mm^p - mM^p}\right)^p \\ &= \frac{\kappa^p - \kappa}{(p-1)(\kappa - 1)}\left(\frac{(p-1)(\kappa^p - 1)}{p(\kappa^p - \kappa)}\right)^p, \qquad \kappa = \frac{M}{m}. \end{aligned}$$

Remark 4. *Note that*

$$C(m, M; p) := \frac{Mm^p - mM^p}{(1-p)(M-m)}\left(\frac{1-p}{p}\frac{M^p - m^p}{Mm^p - mM^p}\right)^p \tag{6}$$

is called **Furuta's constant** *[7] when $p > 0$.*

Proof of Corollary 3. We first consider α_1. If $0 < p \le 1$, then the function $f(t) = t^p$ is operator concave and from the inequality (3) follows $\alpha_1 = 1$. But, if $p < 0$ or $p > 1$, then the function $f(t) = t^p$ is strictly convex (and $f > 0$). From the inequality (4) follows:

$$t_0 = \frac{p}{p-1}\frac{mM^p - Mm^p}{M^p - m^p} \qquad \text{and} \qquad \alpha_1 = \frac{m^p + \frac{M^p - m^p}{M-m}(t_0 - m)}{t_0^p} = \tilde{\Delta}.$$

Next, we consider α_2. If $0 < p < 1$, then the function $f(t) = t^p$ is strictly concave and it follows from inequality (4) that $\alpha_2 = \tilde{\Delta}$. If $-1 \le p < 0$ or $1 \le p \le 2$, then the function $f(t) = t^p$ is operator convex and from the inequality (3) follows $\alpha_2 = 1$. If $p < -1$ or $p > 2$, then the function $f(t) = t^p$ is strictly convex. Similar to Mond-Mond-Pečarić method, for any $s \in [m, M]$ we have $g_s(t) \equiv f(s) + f'(s)(t-s) \le f(t)$ for all $t \in [m, M]$. Then the following inequality holds (see [3, Remark 4.13]):

$$\sum_{j=1}^{k} \omega_j f(A_j) \ge \alpha_2\, f\left(\sum_{j=1}^{k} \omega_j A_j\right) \qquad \text{with} \qquad \alpha_2 = \max_{0 \le g_s \le f} \ \min_{m \le t \le M} \ \frac{g_s(t)}{f(t)}.$$

We choose s which is the unique solution of $\frac{g_s(m)}{f(m)} = \frac{g_s(M)}{f(M)}$. A simple calculation implies $\alpha_2 = \tilde{\Delta}^{-1}$.

Proof of Theorem 1. We prove this by a similar method as in [3, Theorem 5.7]. We shall consider only the case when $s \ne r$.

Suppose that $s \ge 1$. If $0 < r < 1$ then $m^r 1_H \le A_j^r \le M^r 1_H$ $(j = 1, \dots, k)$ implies $m^r 1_H \le \sum_{j=1}^{k} \omega_j A_j^r \le M^r 1_H$. Putting $p = \frac{s}{r}$ in Corollary 3 (for $1 < p \le 2$ or $p > 2$) and replaced A_j by A_j^r we have

$$\left(\sum_{j=1}^{k} \omega_j A_j^r\right)^{s/r} \le \sum_{j=1}^{k} \omega_j A_j^s \le C(m^r, M^r; \frac{s}{r}) \left(\sum_{j=1}^{k} \omega_j A_j^r\right)^{s/r}$$

if $s/2 \le r < 1$ or

$$C(m^r, M^r; \frac{s}{r})^{-1} \left(\sum_{j=1}^{k} \omega_j A_j^r\right)^{s/r} \le \sum_{j=1}^{k} \omega_j A_j^s \le C(m^r, M^r; \frac{s}{r}) \left(\sum_{j=1}^{k} \omega_j A_j^r\right)^{s/r}$$

if $0 < r < s/2$, where

$$\begin{aligned} C(m^r, M^r; \frac{s}{r}) &= \frac{m^r (M^r)^{\frac{s}{r}} - M^r (m^r)^{\frac{s}{r}}}{(\frac{s}{r} - 1)(M^r - m^r)} \left(\frac{(\frac{s}{r} - 1)((M^r)^{\frac{s}{r}} - (m^r)^{\frac{s}{r}})}{\frac{s}{r}(m^r (M^r)^{\frac{s}{r}} - M^r (m^r)^{\frac{s}{r}})}\right)^{\frac{s}{r}} \\ &= \frac{r(\kappa^s - \kappa^r)}{(s-r)(\kappa^r - 1)} \left(\frac{s(\kappa^r - \kappa^s)}{(r-s)(\kappa^s - 1)}\right)^{-\frac{s}{r}}. \end{aligned}$$

The function $f(t) = t^{\frac{1}{s}}$ is operator increasing if $s \ge 1$ and it follows that

$$\left(\sum_{j=1}^{k} \omega_j A_j^r\right)^{1/r} \le \left(\sum_{j=1}^{k} \omega_j A_j^s\right)^{1/s} \le C(m^r, M^r; \frac{s}{r})^{1/s} \left(\sum_{j=1}^{k} \omega_j A_j^r\right)^{1/r}$$

if $s/2 \le r < 1$ or

$$C(m^r, M^r; \frac{s}{r})^{-1/s} \left(\sum_{j=1}^{k} \omega_j A_j^r \right)^{1/r} \le \left(\sum_{j=1}^{k} \omega_j A_j^s \right)^{1/s} \le C(m^r, M^r; \frac{s}{r})^{1/s} \left(\sum_{j=1}^{k} \omega_j A_j^r \right)^{1/r}$$

if $0 < r < s/2$, where $C(m^r, M^r; \frac{s}{r})^{1/s} = \left\{ \frac{r(\kappa^s - \kappa^r)}{(s-r)(\kappa^r - 1)} \right\}^{\frac{1}{s}} \left\{ \frac{s(\kappa^r - \kappa^s)}{(r-s)(\kappa^s - 1)} \right\}^{-\frac{1}{r}} = \Delta^{-1}$.

Furthermore, consider the case of $s = 1$. Then for $1 \le 1/r \le 2$ we have $\left(\sum_{j=1}^{k} \omega_j A_j^r \right)^{1/r} \le \sum_{j=1}^{k} \omega_j A_j$, so for $s > 1$ we have

$$\left(\sum_{j=1}^{k} \omega_j A_j^r \right)^{1/r} \le \sum_{j=1}^{k} \omega_j A_j \le \left(\sum_{j=1}^{k} \omega_j A_j^s \right)^{1/s} \le \Delta^{-1} \left(\sum_{j=1}^{k} \omega_j A_j^r \right)^{1/r}$$

if $1/2 \le r < 1$ or

$$\Delta \left(\sum_{j=1}^{k} \omega_j A_j^r \right)^{1/r} \le \sum_{j=1}^{k} \omega_j A_j \le \left(\sum_{j=1}^{k} \omega_j A_j^s \right)^{1/s} \le \Delta^{-1} \left(\sum_{j=1}^{k} \omega_j A_j^r \right)^{1/r}$$

if $0 < r < 1/2$. Then we obtain desired inequalities for $1/2 \le r < 1$ or $0 < r < 1/2$.

If $r < 0$ then $M^r 1_H \le \sum_{j=1}^{k} \omega_j A_j^r \le m^r 1_H$ and Corollary 3 (for $-1 \le p < 0$ or $p < -1$), with the fact that the function $f(t) = t^{\frac{1}{s}}$ is operator increasing, gives

$$\left(\sum_{j=1}^{k} \omega_j A_j^r \right)^{1/r} \le \left(\sum_{j=1}^{k} \omega_j A_j^s \right)^{1/s} \le C(M^r, m^r; \frac{s}{r})^{1/s} \left(\sum_{j=1}^{k} \omega_j A_j^r \right)^{1/r}$$

if $r \le -s$ or

$$C(M^r, m^r; \frac{s}{r})^{-1/s} \left(\sum_{j=1}^{k} \omega_j A_j^r \right)^{1/r} \le \left(\sum_{j=1}^{k} \omega_j A_j^s \right)^{1/s} \le C(M^r, m^r; \frac{s}{r})^{1/s} \left(\sum_{j=1}^{k} \omega_j A_j^r \right)^{1/r}$$

if $-s < r < 0$, where $C(M^r, m^r; \frac{s}{r})^{1/s} = \left\{ \frac{r(\kappa^{-s} - \kappa^{-r})}{(s-r)(\kappa^{-r} - 1)} \right\}^{\frac{1}{s}} \left\{ \frac{s(\kappa^{-r} - \kappa^{-s})}{(r-s)(\kappa^{-s} - 1)} \right\}^{-\frac{1}{r}} = \Delta^{-1}$.

Therefore, similarly to above mentioned case $s = 1$ we have

$$\left(\sum_{j=1}^{k} \omega_j A_j^r \right)^{1/r} \le \sum_{j=1}^{k} \omega_j A_j \le \left(\sum_{j=1}^{k} \omega_j A_j^s \right)^{1/s} \le \Delta^{-1} \left(\sum_{j=1}^{k} \omega_j A_j^r \right)^{1/r}$$

if $r \le -1$ or

$$\Delta \left(\sum_{j=1}^{k} \omega_j A_j^r \right)^{1/r} \le \sum_{j=1}^{k} \omega_j A_j \le \left(\sum_{j=1}^{k} \omega_j A_j^s \right)^{1/s} \le \Delta^{-1} \left(\sum_{j=1}^{k} \omega_j A_j^r \right)^{1/r}$$

if $-1 < r < 0$. Then we obtain desired inequalities for $r \le -1$ or $-1 < r < 0$.

Next, suppose that $1 \le r < s$. In this case we put $p = \frac{r}{s}$. Then Corollary 3 (for $0 < p \le 1$), with the fact that the function $f(t) = t^{\frac{1}{r}}$ is operator increasing, gives

$$C(m^s, M^s; \frac{r}{s})^{1/r}\left(\sum_{j=1}^{k}\omega_j A_j^s\right)^{1/s} \le \left(\sum_{j=1}^{k}\omega_j A_j^r\right)^{1/r} \le \left(\sum_{j=1}^{k}\omega_j A_j^s\right)^{1/s},$$

where $C(m^s, M^s; \frac{r}{s})^{1/r} = \Delta$.

Therefore, we obtain the desired results on the intervals (ii), (iv) and the part of (i) in case $s \ge 1$ and $r \le s$.

Secondly, suppose that $s < 1$. Then it follows that $r \le -1$. Similarly, due to the mirror reflection direction $s = -r$, we obtain the desired results on the intervals (iii), (v) and the part of (i) in case $s < 1$ and $r \le s$.

3 The chaotic order among means.

In this section we discuss the chaotic order among power means (1).

Theorem 5. *Let $A_j \in \mathcal{B}_+(H)$ with $\mathsf{Sp}(A_j) \subseteq [m, M]$, $0 < m < M$, $(j = 1, \dots, k)$ and $\omega_j \in \mathbf{R}_+$ such that $\sum_{j=1}^k \omega_j = 1$. Denote $\kappa = \frac{M}{m}$. If $r, s \in \mathbf{R}$ then*

$$\Delta(\kappa; r, s) M_k^{[s]}(\mathbf{A}; w) \ll M_k^{[r]}(\mathbf{A}; w) \ll M_k^{[s]}(\mathbf{A}; w) \tag{7}$$

where

$$\Delta(\kappa; r, s) = \begin{cases} \left\{\frac{r(\kappa^s - \kappa^r)}{(s-r)(\kappa^r - 1)}\right\}^{-\frac{1}{s}} \left\{\frac{s(\kappa^r - \kappa^s)}{(r-s)(\kappa^s - 1)}\right\}^{\frac{1}{r}} & \text{if} \quad r < s, \quad r, s \ne 0, \\ \left(\frac{e \log \kappa^{\frac{p}{\kappa^p - 1}}}{\kappa^{\frac{p}{\kappa^p - 1}}}\right)^{\frac{sign(p)}{p}} & \text{if} \quad r = 0 < s = p \ \text{ or } \ r = p < s = 0. \end{cases} \tag{8}$$

Remark 6. *Note that $\Delta(\kappa; 0, 1)^{-1} \equiv M_\kappa(1) := \frac{\kappa^{\frac{1}{\kappa - 1}}}{e \log \kappa^{\frac{1}{\kappa-1}}}$, $(\kappa = \frac{M}{m})$ is called* **Specht's ratio** *and*

$$\Delta(\kappa; 0, s)^{-s} \equiv M_\kappa(s) := \frac{\kappa^{\frac{s}{\kappa^s - 1}}}{e \log \kappa^{\frac{s}{\kappa^s - 1}}} \tag{9}$$

is the generalized Specht's ratio [9, 8]. We remark that $M_{\kappa^r}(1) = M_\kappa(r)$.

Also, note that $\lim_{s \to 0} \Delta(\kappa; 0, s) = 1$ by the Yamazaki-Yanagida result [9, Lemma 12]: $\lim_{s\to 0}\{M_\kappa(s)\}^{\frac{1}{s}} = 1$.

For the proof of Theorem 5 we need two more results.

Lemma 7. *Let $A_j \in \mathcal{B}_+(H)$, $A_j > 0$ $(j = 1, \dots, k)$ and $\omega_j \in \mathbf{R}_+$ such that $\sum_{j=1}^k \omega_j = 1$. Then*

$$\mathbf{s} - \lim_{t \to 0} M_k^{[t]}(\mathbf{A}; w) = M_k^{[0]}(\mathbf{A}; w).$$

Proof. This limit was discussed in [1] for $\omega_j = 1/k$ and proved in [2, Lemma 2] for $k = 2$. As a matter of fact, applying the concavity of log-function and Krein's inequality we have

$$\sum_{j=1}^{k} \omega_j \log A_j \leq \frac{1}{t} \log \left(\sum_{j=1}^{k} \omega_j A_j^t \right) \to \sum_{j=1}^{k} \omega_j \log A_j \quad (t \to +0).$$

So $\mathbf{s} - \lim_{t\to+0} M_k^{[t]}(\mathbf{A}; w) = M_k^{[0]}(\mathbf{A}; w)$. Besides, for $t > 0$

$$M_k^{[-t]}(\mathbf{A}; w) = \left[\left(\sum_{j=1}^{k} \omega_j (A_j^{-1})^t \right)^{1/t} \right]^{-1} \to \left[\exp \left(\sum_{j=1}^{k} \omega_j \log(A_j^{-1}) \right) \right]^{-1} = M_k^{[0]}(\mathbf{A}; w).$$

So $\mathbf{s} - \lim_{t\to-0} M_k^{[t]}(\mathbf{A}; w) = M_k^{[0]}(\mathbf{A}; w)$.

Lemma 8. *Let $M > m > 0$ and $\Delta(\kappa; r, s)$ be defined by (8). Then*

$$\lim_{s\to 0} \Delta(\kappa; r, s) = \Delta(\kappa; r, 0) \qquad \text{and} \qquad \lim_{r\to 0} \Delta(\kappa; r, s) = \Delta(\kappa; 0, s).$$

For the proof of lemma 8 we need the following Yamazaki-Yanagida result [9, Proposition 14].

Lemma C (T.Yamazaki-M.Yanagida). *Let $C(m, M; p)$ and $M_\kappa(p)$ be defined by (6) and (9), respectively. Then for $p > 0$ and $M > m > 0$,*

$$\lim_{\delta\to+0} C(m^\delta, M^\delta; \frac{p}{\delta}) = M_\kappa(p),$$

where $\kappa = \frac{M}{m} > 1$.

Proof of Lemma 8. We have the the first limit putting $\delta = s$ and $p = r$ in Lemma C and applying the following relations:

$$C(m^s, M^s; \frac{r}{s})^{\frac{1}{r}} = \Delta(\kappa; r, s) \ \text{ if } s > 0, \qquad C(M^s, m^s; \frac{r}{s}) = C(m^s, M^s; \frac{r}{s}) \ \text{ if } s < 0,$$

and

$$M_\kappa(r)^{\frac{1}{r}} = \Delta(\kappa; r, 0).$$

Similarly, we obtain the second limit.

Proof of Theorem 5. We first show that for $r, s \in \mathbf{R} \setminus \{0\}$, $r < s$,

$$\log \left(\Delta(\kappa; r, s) M_k^{[s]}(\mathbf{A}; w) \right) \leq \log M_k^{[r]}(\mathbf{A}; w) \leq \log M_k^{[s]}(\mathbf{A}; w).$$

We assume $0 < r < s$. Then $m 1_H \leq A_j \leq M 1_H$ $(j = 1, \dots, k)$ implies $m^s 1_H \leq \sum_{j=1}^{k} \omega_j A_j^s \leq M^s 1_H$. Putting $p = \frac{r}{s}$ $(0 < p < 1)$ in Corollary 3 and replaced A_j by A_j^s, we have

$$C(m^s, M^s; \frac{r}{s}) \left(\sum_{j=1}^{k} \omega_j A_j^s \right)^{r/s} \leq \sum_{j=1}^{k} \omega_j A_j^r \leq \left(\sum_{j=1}^{k} \omega_j A_j^s \right)^{r/s},$$

where

$$C(m^s, M^s; \frac{r}{s}) = \frac{s(\kappa^r - \kappa^s)}{(r-s)(\kappa^s - 1)} \left(\frac{r(\kappa^s - \kappa^r)}{(s-r)(\kappa^r - 1)} \right)^{-\frac{r}{s}}.$$

As the function $f(t) = \log t$ is operator monotone on $\langle 0, \infty \rangle$ we have

$$r \log \left(C(m^s, M^s; \frac{r}{s})^{1/r} \left(\sum_{j=1}^{k} \omega_j A_j^s \right)^{1/s} \right) \leq \log \left(\sum_{j=1}^{k} \omega_j A_j^r \right) \leq r \log \left(\sum_{j=1}^{k} \omega_j A_j^s \right)^{1/s}$$

and so

$$\log \left(C(m^s, M^s; \frac{r}{s})^{1/r} M_k^{[s]}(\mathbf{A}; w) \right) \leq \log M_k^{[r]}(\mathbf{A}; w) \leq \log M_k^{[s]}(\mathbf{A}; w), \tag{10}$$

where $C(m^s, M^s; \frac{r}{s})^{1/r} = \Delta(\kappa; r, s)$.

Next, we assume $r < s < 0$. Then $M^r 1_H \leq A_j^r \leq m^r 1_H$, $(j = 1, \ldots, k)$ and so $M^r 1_H \leq \sum_{j=1}^{k} \omega_j A_j^r \leq m^r 1_H$. Putting $p = \frac{s}{r}$ $(0 < p < 1)$ in Corollary 3 and replaced A_j by A_j^r, we have

$$C(M^r, m^r; \frac{s}{r}) \left(\sum_{j=1}^{k} \omega_j A_j^r \right)^{s/r} \leq \sum_{j=1}^{k} \omega_j A_j^s \leq \left(\sum_{j=1}^{k} \omega_j A_j^r \right)^{s/r},$$

and so

$$\log \left(C(M^r, m^r; \frac{s}{r})^{1/s} M_k^{[r]}(\mathbf{A}; w) \right) \geq \log M_k^{[s]}(\mathbf{A}; w) \geq \log M_k^{[r]}(\mathbf{A}; w), \tag{11}$$

where $C(M^r, m^r; \frac{s}{r})^{1/s} = \Delta(\kappa; r, s)^{-1}$.

Next, we assume $r < 0 < s$. If $0 < -r < s$ or $0 < s < -r$, we put $p = \frac{r}{s}$ or $p = \frac{s}{r}$ in Corollary 3 $(-1 \leq p < 0)$, respectively. Then we have

$$\left(\sum_{j=1}^{k} \omega_j A_j^s \right)^{r/s} \leq \sum_{j=1}^{k} \omega_j A_j^r \leq C(m^s, M^s; \frac{r}{s}) \left(\sum_{j=1}^{k} \omega_j A_j^s \right)^{r/s}$$

or

$$\left(\sum_{j=1}^{k} \omega_j A_j^r \right)^{s/r} \leq \sum_{j=1}^{k} \omega_j A_j^s \leq C(M^r, m^r; \frac{s}{r}) \left(\sum_{j=1}^{k} \omega_j A_j^r \right)^{s/r}.$$

So

$$\log M_k^{[s]}(\mathbf{A}; w) \geq \log M_k^{[r]}(\mathbf{A}; w) \geq \log \left(C(m^s, M^s; \frac{r}{s})^{1/r} M_k^{[s]}(\mathbf{A}; w) \right), \tag{12}$$

with $C(m^s, M^s; \frac{r}{s})^{1/r} = \Delta(\kappa; r, s)$, or

$$\log M_k^{[r]}(\mathbf{A}; w) \leq \log M_k^{[s]}(\mathbf{A}; w) \leq \log \left(C(M^r, m^r; \frac{s}{r})^{1/s} M_k^{[r]}(\mathbf{A}; w) \right), \tag{13}$$

with $C(M^r, m^r; \frac{s}{r})^{1/s} = \Delta(\kappa; r, s)^{-1}$. Then the inequality (7) holds when $r < s$, $r, s \neq 0$.

In the end, if $r \to 0$ in (10) and (12), then

$$\Delta(\kappa; 0, s) \;\; M_k^{[s]}(\mathbf{A}; w) \ll M_k^{[0]}(\mathbf{A}; w) \ll M_k^{[s]}(\mathbf{A}; w)$$

by Lemma 8 and Lemma 7. Similarly, if $s \to 0$ in (11) and (13), then

$$M_k^{[0]}(\mathbf{A};w) \ll \Delta(\kappa;r,0)^{-1} \; M_k^{[r]}(\mathbf{A};w) \ll \Delta(\kappa;r,0)^{-1} \; M_k^{[0]}(\mathbf{A};w).$$

Then the inequality (7) holds when $r = 0 < s$ or $r < s = 0$.

Remark 9. *If we put $r = 0$ and $s = 1$ in Theorem 5, then we have the following inequality between arithmetic mean and geometric mean:*

$$\exp\left(\sum_{j=1}^{k} \omega_j \; \log A_j\right) \ll \sum_{j=1}^{k} \omega_j \; A_j \ll \frac{\kappa^{\frac{1}{\kappa-1}}}{e \log \kappa^{\frac{1}{\kappa-1}}} \quad \exp\left(\sum_{j=1}^{k} \omega_j \; \log A_j\right).$$

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