QUADRATIC AND T-CUBIC SPLINE APPROXIMATIONS TO A PLANAR SPIRAL

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Abstract. We show that two point G^1-Hermite quadratic and T-cubic spline interpolations to a smooth spiral are spirals if the interpolation points are taken close enough. The use of spirals gives the designer an excellent and speedy control over the shape of curve that is produced because there are no internal curvature maxima, curvature minima, inflection points, loops and cusps in a spiral segment.

1 Introduction Smooth curve representation is required for visualization of the scientific data. Smoothness is one of the most important requirements for the visual pleasing display. Fair curves are also important in computer-aided design (CAD) and computer-aided geometric design (CAGD). Cubic splines, although smoother, are not always helpful since they might have *unwanted* inflection points and singularities (see [4], [5]). Spirals are visually pleasing curves of monotone curvature; and they have the advantage of not containing curvature maxima, curvatue minima, inflection points and singularities. Many authors have advocated their use in the design of fair curves (see [1]). These spirals are desirable for applications such as the design of highway or railway routes and trajectories of mobile robots. The benefit of using such curves in the design of surfaces, in particular surfaces of revolution and swept surfaces, is the control of unwanted flat spots and undulations (see [8]). Some advantages of spirals are that they are parametric curves, the arc length can be expressed as a polynomial function of the parameter, the curvature can be expressed as a rational function of the parameter and the offset curve is a rational function of the parameter. These last three properties result from the fact that the T-cubic (the Tschirnhausen cubic) has a Pythagorean hodograph and can be expressed as cubic NURBS for compatibility with existing computer aided design software. Meek & Walton has considered two-point Hermite interpolating spirals by joining T-cubic spirals and/or arc/T-cubic spirals (see [2]). The T-cubic spline has a simple representation for treatment of its curvature while the numerator of the derivative of the curvature is quintic and difficult to treat even for the cubic curve.

If a smooth curve is a spiral, it is desirable that its approximation also be a spiral. Meek & Walton([3]) have considered two-point G^1 Hermite biarc approximation (interpolation) to the smooth planar spiral. The biarc spline composed of two circular arcs joined in a G^1 manner passes from one given point to another such that its unit tangent vector matches given unit tangent vectors at two points. Their results are as follows:
(1) If the interpolation points on the spiral are taken close enough, the biarc spline produced from joining biarcs is a spiral.
(2) The accuracy of the biarc spline approximation to the spiral is $O(h^3)$ where h is the arc length of the spiral between the two interpolation points.
(3) The accuracy of the curvature approximation is $O(h)$.

The object of this note is to derive the similar results for quadratic and T-cubic splines

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under the same assumption:
(a) If the interpolation points on the smooth spiral are close enough, the quadratic spline and the T-cubic spline through the two points matching the two unit tangent vectors at those points are spirals.
(b) The accuracy of their approximations to the spiral is $O(h^2)$.
(c) The accuracy of their curvature approximations is $O(h^2)$.

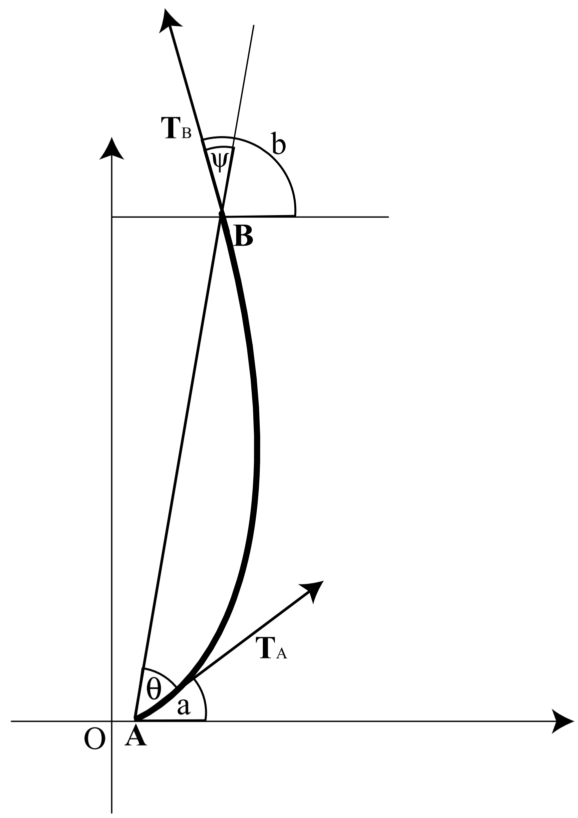


Figure 1: T-cubic spiral interpolating G^1 Hermite data taken from a spiral.

As in [3], let the spiral be

$$s(t)(=(x(t),y(t))) = \left(\int_o^t \frac{\cos(u)}{\kappa(u)}du, \int_o^t \frac{\sin(u)}{\kappa(u)}du \right) \quad (t \geq 0) \tag{1}$$

where t is the angle of the tangent vector with respect to the x- axis, and κ $(= 1/w)$ is a smooth non-negative and strictly monotone curvature of the spiral. Assume that the part of the spiral to be approximated is the part from $\boldsymbol{A} = s(a)$ to $\boldsymbol{B} = s(b)$ $(0 \leq a < b)$. Sections 2 and 3 deal with G^1 spiral (quadratic, T-cubic and cubic spline) approximations to the spiral. At the end of Section 3, we show that the usual cubic spline Hermite interpolation to the spiral is a spiral. We presented some examples of spiral approximations in Section 4. At the end, there is an Appendix of *Mathematica's* program for (26) and (27) in Section 5.

2 Quadratic spline approximation Consider a quadratic spline $z_2(t)(= (x(t), y(t)))$ of the form:

$$x(t) = u_0(1-t)^2 + 2u_1 t(1-t) + u_2 t^2$$
(2)
$$y(t) = v_0(1-t)^2 + 2v_1 t(1-t) + v_2 t^2$$

Match the unit tangent vectors at $\boldsymbol{A}$ and $\boldsymbol{B}$ to give

$$2u_0 + r_0 \cos a = 2u_2 - r_1 \cos b (= 2u_1)$$
(3)
$$2v_0 + r_0 \sin a = 2v_2 - r_1 \sin b (= 2v_1)$$

Conditions: $z_2(a)(= (u_0, v_0)) = s(a)$ and $z_2(b)(= (u_2, v_2)) = s(b)$ determine (r_0, r_1) as:

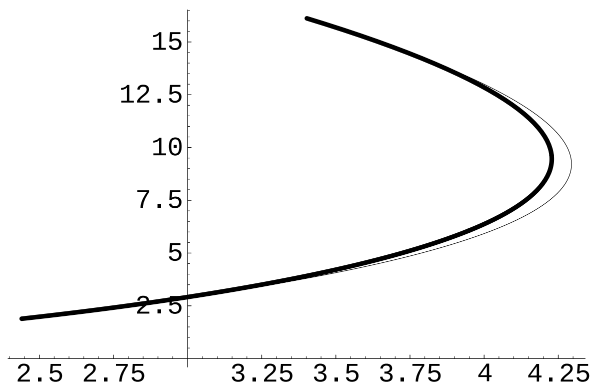
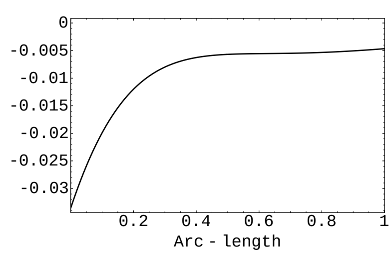


Figure 2: Quadratic spline approximation (thin mode) of spiral (thick mode) from (1).



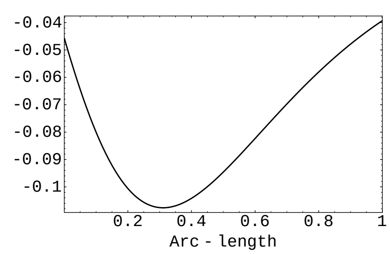


Figure 3: Graph of $\kappa_2(t) - \kappa(a(1-t)+bt)$ (left) and $\kappa_2'(t)$ (right) where $0 \le t \le 1$.

(4) $$r_0 = \frac{2}{\sin(b-a)} \int_a^b w(u)\sin(b-u)du, r_1 = \frac{2}{\sin(b-a)} \int_a^b w(u)\sin(u-a)du$$

Note that the curvature κ_2 of the quadratic spline z_2 is equal to

(5) $$\kappa_2(t) \left(= \frac{(\boldsymbol{z}' \times \boldsymbol{z}'')(t)}{\|\boldsymbol{z}'(t)\|^3} \right) = \frac{r_0 r_1 \sin(b-a)}{\{r_0^2(1-t)^2 + r_1^2 t^2 + 2r_0 r_1 t(1-t)\cos(a+b)\}^{3/2}}$$

where "$\times$" and $\|\bullet\|$ mean the cross product of the two vectors and the Euclidean norm, respectively. Note that the quadratic function of the denominator of κ_2 has its minimum at $t = t_m$:

$$t_m = \frac{r_0\{r_0 - r_1\cos(b-a)\}}{r_0^2 + r_1^2 - 2r_0r_1\cos(b-a)} \tag{6}$$

to obtain the spiral conditions: $t_m \le 0$ ($\Leftrightarrow$ the monotone decreasing curvature) and $t_m \ge 1$ ($\Leftrightarrow$the monotone increasing curvature) from which follows the spiral condition

$$r_1\cos(b-a) \ge r_0 \quad \text{or} \quad r_0\cos(b-a) \ge r_1 \tag{7}$$

Let $(a, b) = (c - d/2, c + d/2), d > 0$ and $c_i = w^i(c)$(the i-th derivative of $w(t)$ at $t = c$) to obtain

$$r_0 = c_0 d - \frac{c_1 d^2}{6} + O(d^3), \quad r_1 = c_0 d + \frac{c_1 d^2}{6} + O(d^3) \tag{8}$$

Since with $c_1 = -\kappa'(c)/\kappa^2(c)$

$$r_1\cos(b-a) - r_0 = \frac{c_1 d^2}{3} + O(d^3), \quad r_0\cos(b-a) - r_1 = -\frac{c_1 d^2}{3} + O(d^3), \tag{9}$$

the spiral condition (7) is valid for small d. In addition, we have the following asymptotic expansions, the proof of (b) and (c):

$$\begin{aligned} &(i) \quad z_2(t) - s(a(1-t)+bt) = \left\{\frac{c_1 t(1-t)}{3}\right\} d^2(\cos c, \sin c) + O(d^3) \\ &(ii) \quad \kappa_2(t) - \kappa(a(1-t)+bt) = \left\{\frac{3c_0(c_2 - 3c_0)(1-6t+6t^2) - 4c_1^2(1-3t+3t^2)}{36c_0^3}\right\} d^2 \\ &\qquad + O(d^3) \end{aligned} \tag{10}$$

where we note

$$h = \int_a^b w(u)du = c_0 d + O(d^3), \quad \text{i.e.,} \quad d = O(h) \tag{11}$$

3 T-cubic spline approximation We consider the following T-cubic spline $z_3(t)(= (x(t), y(t)))$ as

$$z_3'(t)(= (u(t)^2 - v(t)^2, 2u(t)v(t))) \quad (0 \le t \le 1) \tag{12}$$

with

$$\begin{aligned} u(t) &= r_0(1-t)\cos\frac{a}{2} + r_1 t\cos\frac{b}{2} \quad (r_0, r_1 > 0) \\ v(t) &= r_0(1-t)\sin\frac{a}{2} + r_1 t\sin\frac{b}{2} \end{aligned} \tag{13}$$

From (1) and (12), it is easy to check that the T-cubic spline matches the unit tangent vectors of the spiral at $\boldsymbol{A}$ and $\boldsymbol{B}$ as

$$z_3'(0) = s'(a)(= r_0^2(\cos a, \sin a)), \quad z_3'(1) = s'(b)(= r_1^2(\cos b, \sin b)) \tag{14}$$

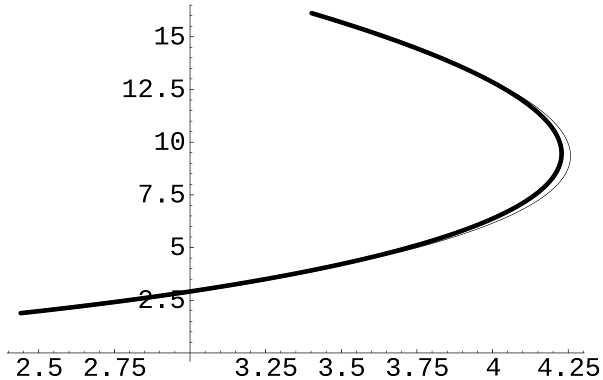


Figure 4: T-cubic spline approximation (thin mode) of spiral (thick mode) from (1).

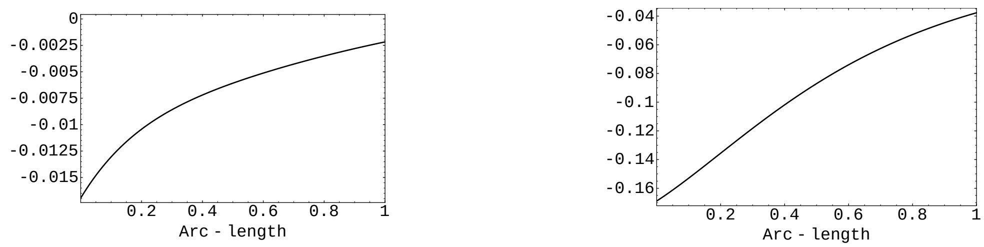


Figure 5: Graph of $\kappa_3(t) - \kappa(a(1-t)+bt)$ (left) and $\kappa_3'(t)$ (right) where $0 \le t \le 1$.

Then, the curvature κ_3 of the T-cubic spline z_3 is equal to

$$\kappa_3(t) = \frac{2r_0r_1\sin\frac{b-a}{2}}{\{r_0^2(1-t)^2 + r_1^2t^2 + 2r_0r_1t(1-t)\cos\frac{b-a}{2}\}^2} \tag{15}$$

Note that the quadratic function of the denominator of the above curvature has its minimum at $t = t_m$:

$$t_m = \frac{r_0(r_0 - r_1\cos\frac{b-a}{2})}{r_0^2 + r_1^2 - 2r_0r_1\cos\frac{b-a}{2}} \tag{16}$$

to obtain the spiral conditions: $t_m \le 0$ ($\Leftrightarrow$ the monotone decreasing curvature of the T-cubic spline) and $t_m \ge 1$ ($\Leftrightarrow$ the monotone increasing curvature of the T-cubic spline) from which follows the spiral condition ([2]):

$$r_1\cos\frac{b-a}{2} \ge r_0 \quad \text{or} \quad r_0\cos\frac{b-a}{2} \ge r_1 \tag{17}$$

Conditions $s(a) = z_3(a)$ and $s(b) = z_3(b)$ give a quadratic system of equations in (r_0, r_1):

$$r_0^2\cos a + r_0r_1\cos\frac{a+b}{2} + r_1^2\cos b = A_0 \left(= 3\int_a^b w(u)\cos u\,du \right)$$

$$r_0^2\sin a + r_0r_1\sin\frac{a+b}{2} + r_1^2\sin b = A_1 \left(= 3\int_a^b w(u)\sin u\,du \right)$$

Letting $r_1 = mr_0$, from above

$$\frac{A_0}{\cos a + m\cos\frac{a+b}{2} + m^2\cos b} = \frac{A_1}{\sin a + m\sin\frac{a+b}{2} + m^2\sin b}(= r_0^2) \tag{18}$$

Define the positive two quantities p, r and one more q that is positive or negative according to the monotone decreasing or increasing curvature κ of the spiral::

$$p = \int_a^b w(u)\sin(b-u)du,\ q = \int_a^b w(u)\sin\left(u - \frac{a+b}{2}\right)du,\ r = \int_a^b w(u)\sin(u-a)du \tag{19}$$

to obtain (if necessary, with help of *Mathematica*)

$$m = \frac{q+\sqrt{q^2+4pr}}{2p}\left(= 1 + \frac{c_1 d}{4c_0} + \frac{c_1^2 d^2}{32c_0^2} + O(d^3)\right) \tag{20}$$

Then, (18) gives

$$r_0^2 = c_0 d - \frac{c_1 d^2}{4} + O(d^3), \quad r_1^2 = c_0 d + \frac{c_1 d^2}{4} + O(d^3) \tag{21}$$

from which we get

$$r_1\cos\frac{b-a}{2} - r_0 = \frac{c_1 d^{\frac{3}{2}}}{4\sqrt{c_0}} + O(d^{\frac{5}{2}}), \quad r_0\cos\frac{b-a}{2} - r_1 = -\frac{c_1 d^{\frac{3}{2}}}{4\sqrt{c_0}} + O(d^{\frac{5}{2}}) \tag{22}$$

Hence, the spiral condition (17) is valid for small d. This completes the proof of (a).

Mathematica helps us give the asymptotic expansions, i.e., the proof of (b) and (c) as

$$(23)\quad \begin{aligned} &(i) \quad z_3(t) - s(a(1-t)+bt) = \left\{\frac{c_1 t(1-t)}{4}\right\} d^2(\cos c, \sin c) + O(d^3) \\ &(ii) \quad \kappa_3(t) - \kappa(a(1-t)+bt) = \left\{\frac{4c_0(c_2-c_0)(1-6t+6t^2) - c_1^2(5-18t+18t^2)}{48c_0^3}\right\} d^2 \\ &\qquad + O(d^3) \end{aligned}$$

The following results are not for the T-cubic spline but the usual cubic spline having two more degrees of freedom. Consider the general cubic spline $z(t)(= (x(t), y(t)))$ of the form:

$$(24)\quad \begin{aligned} x(t) &= u_0(1-t)^3 + 3u_1 t(1-t)^2 + 3u_2 t^2(1-t) + u_3 t^3 \\ y(t) &= v_0(1-t)^3 + 3v_1 t(1-t)^2 + 3v_2 t^2(1-t) + v_3 t^3 \end{aligned}$$

Now, require the following Hermite interpolation conditions

$$z(0) = s(a), z(1) = s(b); \quad z'(0) = (b-a)s'(a), z'(1) = (b-a)s'(b) \tag{25}$$

Then, note that all parameters $u_i, v_i, 0 \le i \le 3$ are uniquely determined. Letting $(r_0, r_1) = d(w(a), w(b))$, then *Mathematica* greatly helps us show for the curvature $k(t) = N(t)/\sqrt{D(t)^3}$ of the cubic spline $z(t)$

$$N'(t)D(t) - 3N(t)D'(t)/2 = -c_0^3 c_1 d^6 + O(d^7), \quad D(t) = c_0^2 d^2 + O(d^3) \tag{26}$$

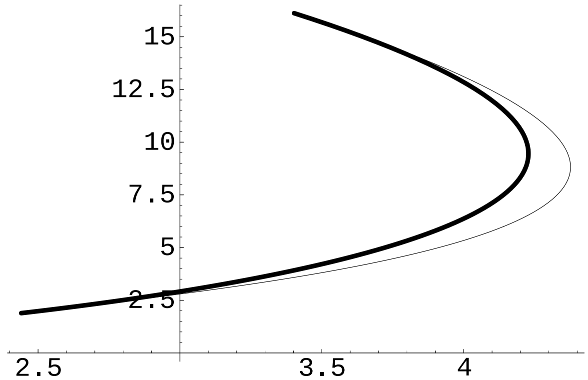


Figure 6: Usual cubic spline approximation (thin mode) of spiral (thick mode) from (1).

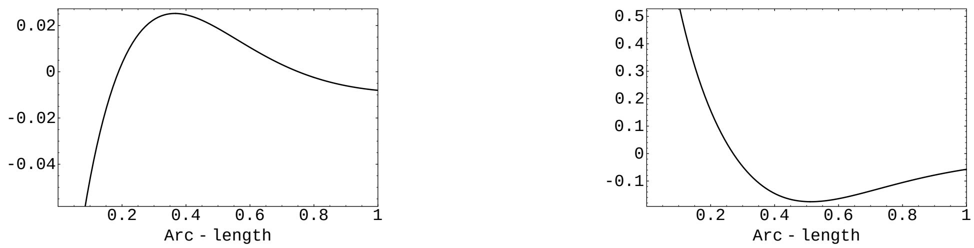


Figure 7: Graph of $k(t) - \kappa(a(1-t)+bt)$ (left) and $k'(t)$ (right) where $0 \le t \le 1$.

from which follows

$$k'(t) = -\frac{c_1 d}{c_0^2} + O(d^2) \tag{27}$$

To obtain the first expansion of (26), one should calculate the coefficients of $t^i, 0 \le i \le 5$ separately. Now, (27) implies that the cubic spline interpolation satisfying (25) is a spiral if the interpolation points on the smooth spiral are close enough. In addition,

$$\begin{aligned} &(i) \qquad z(t) - s(a(1-t)+bt) = \left\{ \frac{c_1 t^2 (1-t)^2}{24} \right\} d^4 \{(c_0 - 3c_2)(\sin c, \cos c) \\ &\qquad\qquad + (c_3 - 3c_1)(\cos c, -\sin c)\} + O(d^5) \\ &(ii) \qquad k(t) - \kappa(a(1-t)+bt) = \left\{ \frac{(c_0 - 3c_2)(1 - 6t + 6t^2)}{12 c_0^2} \right\} d^2 + O(d^3) \end{aligned} \tag{28}$$

Note that no useful sufficient spiral conditions for the usual cubic spline except the T-cubic one have ever derived since the numerator $N'(t)D(t) - 3N(t)D'(t)$ of $\kappa'(t)$ is a very complicated quintic polynomial whose real roots must be outside $[0, 1]$.

4 Demonstration and Analysis For all examples in figures 2-7, we take $a = 1, b = 1.8$ and $w(t) = e^{2t}$. Spiral from (1) is sketched in thick mode to compare it with the spiral approximations of quadratic and cubic splines from section 2 and 3 respectively. One can see that T-cubic spline approximation in figure 4 is the best approximation. Its accuracy can be seen in its curvature and derivative of curvature approximations in figure 5. Quadratic spline approximation in figure 2 is also reasonable but usual cubic spline case in figure 6 is poor. Curvature and derivative of curvature approximations of quadratic and usual cubic splines are given in figures 3 and 7 respectively.

We conclude that two point Hermite quadratic and T-cubic splines interpolations to a smooth spiral are also spirals. Fair curves can be designed interactively using quadratic and T-cubic splines. Due to simple algorithm, these spirals can be easily achieved and then implemented. Our future work directions are to revise this paper for G^2 case and investigate the existence and uniqueness of Pythagorean hodograph quintic spiral (see [6],[7],[8]) in a simple way and develop efficient algorithm for implementation.

5 Appendix

Here is *Mathematica's* program code for (26) and (27).

```
u1 = (3u0 + r0 Cos[a])/3; v1 = (3v0 + r0 Sin[a])/3;
u2 = (3u3 - r1 Cos[b])/3; v2 = (3v3 - r1 Sin[b])/3;
x[t_] := u0(1 - t)^3 + 3u1 t(1 - t)^2 + 3u2 t^2(1 - t) + u3 t^3
y[t_] := v0(1 - t)^3 + 3v1 t(1 - t)^2 + 3v2 t^2(1 - t) + v3 t^3
Nn[t_] := D[D[y[t], t], t]D[x[t], t] - D[D[x[t], t], t]D[y[t], t]
Dd[t_] := D[x[t], t]^2 + D[y[t], t]^2
R[t_] := D[Nn[t], t]Dd[t] - 3Nn[t]D[Dd[t], t]/2
u0 = Integrate[Cos[s]w[s], {s, 0, a}]; v0 = Integrate[Sin[s]w[s], {s, 0, a}];
u3 = Integrate[Cos[s]w[s], {s, 0, b}]; v3 = Integrate[Sin[s]w[s], {s, 0, b}];
a = c - d/2; b = c + d/2; r0 = d w[a]; r1 = d w[b];
eq1 = Simplify[Series[R[t], {d, 0, 6}]]
eq2 = Simplify[Series[Dd[t], {d, 0, 2}]]
Simplify[Together[eq1/eq2^(5/2)]]
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$-c_0^3 c_1 d^6 + O(d^7)$
$c_0^2 d^2 + O(d^3)$
$-\frac{c_1 d}{c_0^2} + O(d^2)$

References

[1] G. Farin. *Curves and Surfaces for Computer Aided Geometric Design - Practical Guide.* Academic Press, San Diego, 1993.
[2] D. S. Meek and D. J. Walton. Hermite interpolation with tschirnhausen cubic spirals. *Computer Aided Geometric Design*, 14:619–633, 1997.
[3] D. S. Meek and D. J. Walton. Spiral arc spline approximation to a planar spiral. *J. Comput. Appl. Math.*, 107:21–30, 1999.
[4] M. Sakai. Inflection points and singularities on planar rational cubic curve segments. *Computer Aided Geometric Design*, 16:149–156, 1999.
[5] M. Sakai. Osculatory interpolation. *Computer Aided Geometric Design*, 18:739–750, 2001.
[6] D. J. Walton and D. S. Meek. A pythagorean hodograph quintic spiral. *Computer Aided Design*, 28:943–950, 1996.
[7] D. J. Walton and D. S. Meek. G^2 curves composed of planar cubic and pythagorean hodograph quintic spirals. *Computer Aided Geometric Design*, 15:547–566, 1998.
[8] D. J. Walton and D. S. Meek. Planar G^2 curve design with spiral segments. *Computer Aided Design*, 30:529–538, 1998.

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