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POLYNOMIAL HULLS OF GRAPHS OF ANTIHOLOMORPHIC FUNCTIONS

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Abstract. We consider the methods of determining polynomial hulls of graphs which are defined by restrictions to the unit torus or the unit sphere of antiholomorphic functions or maps. As an application we give the explicit descriptions of the polynomial hulls of graphs on the unit torus in $\mathbb{C}^2$ of the complex conjugate functions of homogeneous polynomials.

1. Introduction. Let X be a compact subset in $\mathbb{C}^N$ and $C(X)$ denote the Banach algebra of all continuous functions on X with sup-norm $\| \ \|_X$. Denote by $\hat{X}$ the polynomial hull of X, i.e.,

$$\hat{X} = \{z = (z_1, \cdots, z_N) \in \mathbb{C}^N : |p(z)| \le \|p\|_X, \ \textit{for every polynomial } p\}.$$

X is said to be polynomially convex if $\hat{X} = X$. For a subset S of $\mathbb{C}^n$ and continuous functions $f_1, \cdots, f_m$ on S, we denote by $G(f)$ or $G(f;S)$ the graph of $f = (f_1, \cdots, f_m)$ on S, i.e.,

$$G(f) = \{(z, f(z)) \in \mathbb{C}^{n+m} : z \in S\}.$$

When K is a compact subset in $\mathbb{C}^n$, for $f_1, \cdots, f_m \in C(K)$ we denote by $[f_1, \cdots, f_m; K]$ the uniform closure of all polynomials of $f_1, \cdots, f_m$ on K, and put $P(K) = [z_1, \cdots, z_n; K]$. The maximal ideal space of the uniform algebra $P(K)$ can be identified with the polynomial hull of K.

The following problems have been studied by several authors:

(1) When the graph $G(f)$ is polynomially convex ?

(2) Does $[z_1, \cdots, z_n, f_1, \cdots, f_m; K] = C(K)$ hold ?

(3) If $\widehat{G(f)} \ne G(f)$, what is the structure of the set $\widehat{G(f)} \setminus G(f)$? etc.

Let D be the open unit disk $\{\lambda \in \mathbb{C} : |\lambda| < 1\}$ and T its boundary ∂D. We denote by B the open unit ball $\{z = (z_1, \cdots, z_n) \in \mathbb{C}^n : \sum_{j=1}^n |z_j|^2 < 1\}$ and by ∂B the unit sphere. Let π be the projection of $\mathbb{C}^{n+m}$ to $\mathbb{C}^n$ defined by $\pi(z, z') = z$, for all $z \in \mathbb{C}^n$ and $z' \in \mathbb{C}^m$. We denote by D^n and T^n the unit polydisc and the torus in $\mathbb{C}^n$ respectively.

When $K = T$ and $f \in C(T) \setminus P(T)$, it follows from Wermer's maximality theorem that $[z, f; T] = C(T)$ and $\widehat{G(f)} = G(f)$ hold. When $K = \partial B$ and $f \in C(\partial B)$, Alexander [2] proved that $\pi(\widehat{G(f)}) = \bar{B}$ for the case $n > 1$, and moreover if f is the restriction to ∂B of a pluriharmonic function g on $\bar{B}$, then $\widehat{G(f)} = G(g; \bar{B})$.

Let K be a compact subset of $\mathbb{C}^n$ and $f_1, \cdots, f_m, m \ge n$, be functions in $C(K)$ which have C^1 extensions to some neighborhood of K. Put $E = \{z \in K : \mathrm{rank}(\frac{\partial f_j}{\partial \bar{z}_k}(z)) < n\}$ and $A = [z_1, \cdots, z_n, f_1, \cdots, f_m; K]$. Weinstock [10] proved that if the graph $G(f)$ of $f =$

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$(f_1, \cdots, f_m)$ is polynomially convex, then $A = \{f \in C(K) : f|_E \in A|_E\}$, where $A|_E$ denotes the algebra of restrictions $f|_E$ to E of functions f in A.

In this paper we treat with the cases that K is the torus T^2 or the sphere ∂B, and f is the restriction to K of an antiholomorphic function on $\overline{D^2}$ or an antiholomorphic map on $\bar{B}$. We wish to describe the polynomial hulls of graphs. In our setting the set $\widehat{G(f)} \setminus G(f)$ is contained in the zero set of a pluriharmonic function.

2. Known facts and lemmas. Let M be a C^∞ real submanifold of an open set U in $\mathbb{C}^N$. For a point $z \in M$ we denote by T_zM the tangent space of M at z. M is called totally real at z if T_zM contains no nontrivial complex subspaces. M is called totally real if it is totally real at every $z \in M$. We state the basic properties of totally real submanifolds.

Proposition 2.1. Let $\rho_j, j = 1, \cdots, n$, be C^∞ real functions on an open subset U of $\mathbb{C}^n$. Put $M = \{z \in U : \rho_j(z) = 0, \ j = 1, \cdots, n\}$. Assume that the complex Jacobian $\det\left(\dfrac{\partial \rho_j}{\partial z_k}(z)\right) \neq 0$ for all $z \in M$. Then

(1) M is a totally real submanifold of U.

(2) If g is a smooth map of U to $\mathbb{C}^m$, and f is the map restricted to M of g, then the graph $G(f)$ is also a totally real submanifold of $U \times \mathbb{C}^m$.

We need the following two theorems.

Theorem 2.2. ([5],[7]). Let M be a C^∞ totally real submanifold of U in $\mathbb{C}^N$.

(1) Assume that X is a compact polynomially convex subset of M, then $P(X) = C(X)$.

(2) For each point $z^0 \in M$ there exists a small ball B_0 centered at z^0 such that $\overline{B_0} \cap M$ is polynomially convex.

Theorem 2.3. (Rossi's local maximum modulus principle [8]). Let X be a connected compact subset of $\mathbb{C}^N$. If $z^0 \in \hat{X} \setminus X$ and V is an open neighborhood of z^0 in $\hat{X} \setminus X$, then for each polynomial p,

$$|p(z^0)| \leq \max_{z \in bV} |p(z)|$$

where bV is the boundary of V in $\hat{X}$.

It is known that if X is a connected compact subset of $\mathbb{C}^N$, then $\hat{X}$ is also connected.

Main Lemma. Let X be a compact connected subset of $\mathbb{C}^N$ and U an open subset of $\mathbb{C}^N$ with $U \cap X = \phi$. If $\hat{X} \cap U$ is contained in a totally real submanifold M of U, then we have $\hat{X} \cap U = \phi$.

Proof. Assume that $z^0 \in \hat{X} \cap U$. By Theorem 2.2, we can choose a small ball B_0 centered at z^0 so that $\bar{B}_0 \cap X = \phi$ and $P(\bar{B}_0 \cap M) = C(\bar{B}_0 \cap M)$. We put $V = M \cap \hat{X} \cap B_0$, and denote by bV the boundary of V in $\hat{X}$. If $bV = \phi$, then $\bar{V} = V$ in $\hat{X}$, and so $\bar{V} \cap (\hat{X} \setminus V) = \phi$. By the assumption, $\hat{X}$ is connected, so it does not occur. Thus we have $bV \neq \phi$. Then there exists a function $f \in C(\bar{V})$ such that $|f(z^0)| > 1$ and $\|f\|_{bV} < 1$. Since $P(\bar{V}) = C(\bar{V})$, we can choose a polynomial p such that $|p(z^0)| > 1$ and $\|p\|_{bV} < 1$. This contradicts to the local maximum modulus principle. Thus we have $\hat{X} \cap U = \phi$. $\square$

We also make use of the following lemmas.

Lemma 2.4. ([6]).Let E be a compact polynomially convex subset of $\mathbb{C}^n$ and K a closed subset of E. If $g_j, j = 1, \cdots, m$, are pluriharmonic on E and f is the map restricted to K of $g = (g_1, \cdots, g_m)$, then we have $\widehat{G(f)} \subset G(g; E)$.

Lemma 2.5. Suppose that every point of K is a peak point for $P(K)$ and f is a continuous map of K to $\mathbb{C}^m$. If $z^0 \in K$ and $(z^0, w^0) \in \widehat{G(f)}$, then we have $w^0 = f(z^0)$.

The proof is obtained by the same way as in Proposition 2.2 of Ahern and Rudin [1].

3. The graphs of continuous functions on T^2. @ We begin with some special cases. We denote by D_j and $T_j (j = 1, 2)$ the unit disks and the tori in z_j complex planes respectively.

Example 3.1. We consider a function $f_1 \in C(T_1) \setminus P(T_1)$ and put $f(z_1, z_2) = f_1(z_1)$ on T^2. Then we have

$$\widehat{G(f)} = G(f) \cup \bigcup_{z_1 \in T} \{(z_1, z_2, f_1(z_1)) : z_2 \in D\}.$$

Example 3.2. Let $g_j, j = 1, 2$, be non-constant inner functions in $P(\bar{D}_j)$. We consider $f(z_1, z_2) = \overline{g_1(z_1) g_2(z_2)}$ on T^2. Then we have $\widehat{G(f)} = G(f)$.

Proof. We put $\tilde{g}_j(z_1, z_2) = g_j(z_j), j = 1, 2$, on T^2. Since $g_j \bar{g}_j = 1$ on T_j, it follows from the properties of tensor products of uniform algebras that

$$\begin{aligned}
[z_1, z_2, f; T^2] &= [z_1, z_2, \overline{\tilde{g}_1}, \overline{\tilde{g}_2}; T^2] = [z_1, \bar{g}_1; T_1] \otimes [z_2, \bar{g}_2; T_2] \\
&= C(T_1) \otimes C(T_2) = C(T^2).
\end{aligned}$$

Thus the maximal ideal space of $[z_1, z_2, f; T^2]$ can be identified with T^2, and the proof is finished. □

4. Polynomial hulls of graphs on T^2 of antiholomolphic functions. @We consider cases that antiholomorphic functions are the complex conjugates of homogeneous polynomials. We need the following lemma.

Let U be an open subset of $\mathbb{C}^2$. Let g and h be holomorphic functions on U. We consider the set $N = \{z \in U : \bar{g}(z) = h(z)\}$. We put

$$\Delta(z) = \begin{vmatrix} \dfrac{\partial g}{\partial z_1}(z) & \dfrac{\partial g}{\partial z_2}(z) \\ \\ \dfrac{\partial h}{\partial z_1}(z) & \dfrac{\partial h}{\partial z_2}(z) \end{vmatrix}.$$

Lemma 4.1. If a point $z_0 \in N$ satisfies the condition $\Delta(z^0) \neq 0$, then there exists an open ball B_0 centered at z^0 such that $B_0 \cap N$ is a totally real submanifold in B_0.

Proof. We set $g = u + iv$, $h = U + iV (u, v, U, V$ are real), and $\rho_1 = U - u, \rho_2 = V + v$. Then $N = \{z \in U : \rho_j(z) = 0, j = 1, 2\}$. By the Cauchy-Riemann equation, we have

$$\frac{\partial v}{\partial z_j} = \frac{1}{2}(\frac{\partial v}{\partial x_j} - i\frac{\partial v}{\partial y_j}) = -\frac{1}{2}(\frac{\partial u}{\partial y_j} + i\frac{\partial u}{\partial x_j}) = -i\frac{\partial u}{\partial z_j}.$$

By a simple calculation, it follows that

$$\begin{aligned}
\det\left(\frac{\partial \rho_j}{\partial z_k}\right) &= -i\{(U_{z_1} - u_{z_1})(U_{z_2} + u_{z_2}) - (U_{z_2} - u_{z_2})(U_{z_1} + u_{z_1})\} \\
&= -2i(U_{z_1} u_{z_2} - u_{z_1} U_{z_2}) = \frac{i}{2}(h_{z_2} g_{z_1} - h_{z_1} g_{z_2}).
\end{aligned}$$

Since $\Delta(z) \neq 0$ for a small neighborhood of z_0 in N, the lemma is proved by Proposition 2.1. □

We already considered the case that f is the restriction to T^2 of a function $\overline{z_1^m}$ or $\overline{z_1^m z_2^n}$ for some positive integers m, n in Example 3.1 and 3.2. The general form of f is the restriction to T^2 of

$$\begin{aligned}\overline{g(z_1,z_2)} &= \overline{cz_1^m z_2^n(z_1^k + a_1 z_1^{k-1} z_2 + a_2 z_1^{k-2} z_2^2 + \cdots + a_k z_2^k)} \quad (a_k \neq 0) \\ &= \bar{c}(\bar{z}_1 - \bar{\lambda}_1 \bar{z}_2)(\bar{z}_1 - \bar{\lambda}_2 \bar{z}_2)\cdots(\bar{z}_1 - \bar{\lambda}_k \bar{z}_2)\bar{z}_1^m \bar{z}_2^n\end{aligned}$$

where k is a positive integer, m and n are nonnegative integers, and $c, \lambda_1, \lambda_2, \cdots, \lambda_k$ are some constant numbers with $c\lambda_1\lambda_2\cdots\lambda_k \neq 0$. We put

$$J = \{j \in \{1, 2, \cdots, k\} : |\lambda_j| = 1\}.$$

Our main result is the following

Theorem 4.2.
(1) If $J \neq \phi$, then $\widehat{G(f)} = G(f) \cup \bigcup_{j\in J}\{(z_1, z_2, 0) : z_1 - \lambda_j z_2 = 0, z_2 \in D\}$.
(2) If $J = \phi$, then $\widehat{G(f)} = G(f)$, moreover $[z_1, z_2, f; T^2] = C(T^2)$.

Proof. We can assume that $c = 1$. We put $D_j^* = \{(z_1, z_2, 0) : z_1 - \lambda_j z_2 = 0, z_2 \in D\}$ and $D_j = \{(z_1, z_2) : z_1 - \lambda_j z_2 = 0, z_2 \in D\}$.

(1)(a) Assume that $J \neq \phi$, and $j \in J$. Since $\{(z_1, z_2, 0) : z_1 - \lambda_j z_2 = 0,\ z_2 \in T\} \subset G(f)$, we have $D_j^* \subset \widehat{G(f)}$. Thus

$$\bigcup_{j\in J} D_j^* \subset \widehat{G(f)}.$$

To determine its polynomial hull $\widehat{G(f)}$ we put $a = \Pi_{j=1}^k(1 + |\lambda_j|)$. Then since $\|f\|_{T^2} \leq a$, we have $\widehat{G(f)} \subset \bar{D}^2 \times a\bar{D}$. If $(z_1, z_2) \in T^2$, then $z_1\bar{z}_1 = z_2\bar{z}_2 = 1$, and hence we have

$$f(z_1, z_2) = \frac{1}{z_1^{m+k} z_2^{n+k}}\ \Pi_{j=1}^k(z_2 - \bar{\lambda}_j z_1)$$

on T^2. We put

$$\begin{aligned}N = \{(z_1, z_2, z_3) \in \bar{D}^2 \times a\bar{D} : z_3 &= \bar{z}_1^m \bar{z}_2^n\ \Pi_{j=1}^k(\bar{z}_1 - \bar{\lambda}_j \bar{z}_2), \\ z_1^{m+k} z_2^{n+k} z_3 &-\ \Pi_{j=1}^k(z_2 - \bar{\lambda}_j z_1) = 0\}.\end{aligned}$$

By Lemma 2.4, we have that $\widehat{G(f)} \subset N$. Put $N' = N \setminus (\bigcup_{j\in J} D_j^* \cup G(f))$. The idea of the proof is to show that every point of N' near $\bigcup_{j\in J} D_j^* \cup G(f)$ is totally real. We consider the set

$$M = \{(z_1, z_2) \in \bar{D}^2 \setminus (T^2 \cup E) :\ \bar{z}_1^m \bar{z}_2^n\ \Pi_{j=1}^k(\bar{z}_1 - \bar{\lambda}_j \bar{z}_2) = \frac{1}{z_1^{m+k} z_2^{n+k}}\ \Pi_{j=1}^k(z_2 - \bar{\lambda}_j z_1)\},$$

where $E = \bar{D} \times \{0\} \cup \{0\} \times \bar{D}$. We denote by $h(z_1, z_2)$ the right member of defining equation of M, and then

$$h(z) = \frac{z_2^k + \bar{a}_1 z_2^{k-1} z_1 + \bar{a}_2 z_2^{n-2} z_1^2 + \cdots + \bar{a}_k z_1^k}{z_1^{m+k} z_2^{n+k}}.$$

Then $\Delta(z_1, z_2) = g_{z_1} h_{z_2} - g_{z_2} h_{z_1}$

$$= \frac{-1}{z_1^{k+1} z_2^{k+1}} \{k(k+m+n)\bar{a}_k z_1^{2k} + \cdots - k(k+m+n) a_k z_2^{2k}\}.$$

The numerator is the homogeneous polynomial of degree $2k$ and the coefficients of z_1^{2k} and z_2^{2k} are nonzero. Thus, if we put

$$\Lambda = \{(z_1, z_2) \in \bar{D}^2 \setminus (T^2 \cup E) : \Delta(z_1, z_2) = 0\},$$

then there exist nonzero complex numbers $\mu_i,\ i = 1, \cdots, 2k$, such that

$$\Lambda = \bigcup_{i=1}^{2k} \{z \in \bar{D}^2 \setminus (T^2 \cup E) : z_1 - \mu_i z_2 = 0\}.$$

For later use we put

$$\Lambda_i = \{z \in \bar{D}^2 \setminus (T^2 \cup E) : z_1 - \mu_i z_2 = 0\},$$

Assume $|\lambda_1| = 1$, and put $g = (z_1 - \lambda_1 z_2) G(z_1, z_2)$, $h = (z_1 - \lambda_1 z_2) H(z_1, z_2)$, where

$$G(z_1, z_2) = z_1^m z_2^n \ \Pi_{j=2}^k (z_1 - \lambda_j z_2),$$

$$H(z_1, z_2) = \frac{-\bar{\lambda}_1}{z_1^{m+k} z_2^{n+k}} \ \Pi_{j=2}^k (z_2 - \bar{\lambda}_j z_1).$$

If $(z_1, z_2) \in D_1 \setminus \{(0,0)\}$, then

$$\Delta(z_1, z_2) = \{G + (z_1 - \lambda_1 z_2) G_{z_1}\}\{-\lambda_1 H + (z_1 - \lambda_1 z_2) H_{z_2}\}$$
$$-\{-\lambda_1 G + (z_1 - \lambda_1 z_2) G_{z_2}\}\{H + (z_1 - \lambda_1 z_2) H_{z_1}\} = 0.$$

Thus we have $\bigcup_{j \in J} (D_j \setminus \{(0,0)\}) \subset \Lambda$.

(1)(b) When $J \neq \phi$, we may assume that $J = \{1, 2, \cdots, q\},\ 1 \le q \le k$. We set

$$I = \{i : \mu_i \neq \lambda_j, j = 1, \cdots, q\}.$$

We define the set $M_i (i \in I)$ by

$$M_i = (M \setminus \bigcup_{j \in J} D_j) \cap \Lambda_i.$$

For a point $(z_1, z_2) \in M_i$, substituting $z_1 = \mu_i z_2$ in the expression of defining M, then we have

$$|z_2|^{2(m+n+k)} = \frac{|1 - \bar{\lambda}_{q+1} \mu_i|}{|\bar{\mu}_i - \bar{\lambda}_{q+1}|} \cdots \frac{|1 - \bar{\lambda}_k \mu_i|}{|\bar{\mu}_i - \bar{\lambda}_k|} \frac{1}{|\mu_i|^{2m+k}}.$$

We put $\rho_i = (\text{ the right member})^{\frac{1}{2(m+n+k)}}$ and

$$\Omega_i = \{(z_1, z_2) \in \bar{D}^2 \setminus (T^2 \cup E) : z_1 - \mu_i z_2 = 0, |z_2| = \rho_i\}.$$

Then we have $M_i \subset \Omega_i$ (Ω_i or M_i may be empty). If $|z_2| = \rho_i > 1$ or $\rho_i = 0$, then evidently $\Omega_i = \phi$. If $|z_2| = \rho_i = 1$ and $|\mu_i| \ge 1$, then $|z_1| = |\mu_i| \ge 1$, thus $\Omega_i = \phi$. If $|z_2| = \rho_i = 1$ and $|\mu_i| < 1$, since $|z_1| < 1$ and $|z_1| \neq |z_2|$, we have $\Omega_i \cap (\bigcup_{j \in J} D_j \cup T^2) = \phi$. If $0 < \rho_i < 1$, and if $0 < |\mu_i| < 1$ or $1 < |\mu_i| \le 1/\rho_i$, then since $|\mu_i| \neq |\lambda_j| = 1$ for $j \in J$, we have $\Omega_i \cap (\bigcup_{j \in J} D_j \cup T^2) = \phi$. If $0 < \rho_i < 1$ and $|\mu_i| = 1$, then since $|z_1| = \rho_i$ and $z_2(\mu_i - \lambda_j) \neq 0$ for $j \in J$, we have the same conclusion. Thus in any case it follows that

$$\bigcup_{i \in I} \Omega_i \cap (\bigcup_{j \in J} D_j \cup T^2) = \phi.$$

Since $\bigcup_{i\in I} M_i \subset \bigcup_{i\in I} \Omega_i$, we can choose an open neighborhood V in $\mathbb{C}^2$ of $\cup_{j\in J}\bar{D}_j \cup T^2$ such that $V\cap(M\setminus\cup_{j\in J}\bar{D}_i)$ is totally real. By Proposition 2.1, the graph $G(\bar{g}, V\cap(M\setminus\cup_{j\in J}\bar{D}_j)$ is totally real.

(1)(c) If a point $w^* = (w_1, w_2, w_3)$ with $(w_1, w_2)\in E\backslash\{(0,0)\}$, we consider the polynomial $p(z_1, z_2, z_3) = z_1^{m+k}z_2^{n+k}z_3 - \Pi_{j=1}^k(z_2 - \bar{\lambda}_j z_1)$. Then we have $|p(w^*)| = \Pi_j|w_2 - \bar{\lambda}_j w_1| > \|p\|_{G(f)} = 0$. Thus $w^* \notin \widehat{G}(f)$. Since $\widehat{G(f)}$ is connected and the graph $G(\bar{g}, V\cap(M\setminus \cup_{j\in J}\bar{D}_j))$ is not contained in $\widehat{G}(f)$ by the main lemma, it follows that $N\setminus(\bigcup_{j\in J}\overline{D_j^*}\cup G(f))$ is not contained in $\widehat{G(f)}$. The proof of (1) is completed.

(2) When $J=\phi$, by the same argument it follows that there exists a neighborhood V of $G(f)$ in N such that $V\backslash G(f)$ is totally real. Thus by the main lemma we have $\widehat{G(f)} = G(f)$. Since $G(f)$ is totally real, by Theorem 2.2 we have $[z_1, z_2, z_3; G(f)] = C(G(f))$ and so $[z_1, z_2, f; T^2] = C(T^2)$. □

5. Polynomial hulls of graphs on the sphrere of antiholomorphic maps. We obtain the following proposition by Weinstock's result [10] and the main lemma mentioned above. Let $g_j, j = 1, \cdots, n$, be holomorphic functions on a neighborhood of $\bar{B}$ in $\mathbb{C}^n$. We denote by $G(f)$ the graph of the map $f = \bar{g}|_{\partial B}$ restricted to ∂B of $\bar{g} = (\overline{g_1}, \cdots, \overline{g_n})$.

Proposition 5.1. Let U be an open neighborhood of ∂B. Assume that $\det\left(\dfrac{\partial g_j}{\partial z_k}(z)\right)\neq 0$, for all $z\in B\cap U$. Then we have that

$$\widehat{G(f)} = G(f) \text{ and } [z_1, \cdots, z_n, f_1, \ldots, f_n; \partial B] = C(\partial B).$$

Proof. By Lemma 2.4 we have $\widehat{G(f)} \subset G(f, \bar{B})$. It follows from the assumptions that $\det(\frac{\partial g_j}{\partial z_k}(z)) \neq 0$, for all $z\in B$ and so the graph $G(\bar{g}, B)$ is totally real. Thus we have $G(\bar{g}; B)\cap\widehat{G(f)} = \phi$ by the main lemma. Thus the graph $G(f)$ is a polynomially convex set. If $E = \{z\in\partial B : \det(\frac{\partial g_j}{\partial z_k}(z)) = 0\}$, then E is an interpolation set for $P(\bar{B})$ i.e., $P(\bar{B})|_E = C(E)$. By Weinstock's result we have $[z_1, \cdots, z_n, f_1, \ldots, f_n; \partial B]$. □

Example 5.2. If $g_1 = z_1^2 + 2z_2 - z_2^2$ and $g_2 = 2z_1 + 2z_1z_2$, then $\det\left(\dfrac{\partial g_j}{\partial z_k}(z)\right) = 4(z_1^2 + z_2^2 - 1)\neq 0$ on B. Thus the graph $G(f)$ is polynomially convex and $[z_1, z_2, f_1, f_2; \partial B] = C(\partial B)$.

Example 5.3. If $g_1 = z_1^2$ and $g_2 = z_2^2$, then $\det\left(\dfrac{\partial g_j}{\partial z_k}(z)\right) = 4z_1z_2$. The polynomial hull of graph $G(f)$ on ∂B of $\bar{g} = (\bar{g}_1, \bar{g}_2)$ is contained in $G(\bar{g}, \bar{B})$. By the main lemma we have

$$\widehat{G(f)} \subset G(f)\cup\{(z_1, z_2, \bar{z}_1^2, \bar{z}_2^2) : \ (z_1, z_2)\in B,\ z_1z_2 = 0\}.$$

By using the main lemma once more, we have $\widehat{G(f)} \subset G(f)\cup\{(0,0,0,0)\}$. Since $\widehat{G(f)}$ is connected, it follows that $G(f)$ is polynomially convex. By the proposition we have $[z_1, z_2, f_1, f_2; \partial B] = C(\partial B)$.

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