EFFICIENT SOLUTIONS OF MULTICRITERIA LOCATION PROBLEMS WITH THE POLYHEDRAL GAUGE

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Received April 10, 2002

Abstract. A multicriteria location problem with the polyhedral gauge on a plane is considered. We propose an algorithm to find all efficient solutions of the location problem.

1. Introduction. Everything will take place in $\mathbb{R}^2$, equipped with its Euclidean norm, denoted by $\|\cdot\|$ and its canonical inner product, denoted by $\langle\cdot,\cdot\rangle$. Given demand points in $\mathbb{R}^2$, a problem to locate a new facility in $\mathbb{R}^2$ is called a single facility location problem. The problem is usually formulated as a minimization problem with an objective function involving distances between the facility and demand points. It is assumed that $m(\geq 2)$ distinct demand points $\boldsymbol{d}_i \in \mathbb{R}^2$, $i \in M \equiv \{1, 2, \cdots, m\}$ and *a gauge* γ: $\mathbb{R}^2 \to \mathbb{R}$, in the sense of Minkowski, are given. Let $\boldsymbol{x} \in \mathbb{R}^2$ be the variable location of the facility. We put $D \equiv \{\boldsymbol{d}_1, \boldsymbol{d}_2, \cdots, \boldsymbol{d}_m\}$. Then a multicriteria location problem is formulated as follows:

$$\text{(P)} \qquad \min_{\boldsymbol{x}\in\mathbb{R}^2} \left(\gamma(\boldsymbol{x}-\boldsymbol{d}_1), \gamma(\boldsymbol{x}-\boldsymbol{d}_2), \cdots, \gamma(\boldsymbol{x}-\boldsymbol{d}_m)\right)^T.$$

(P) is a problem to find an efficient solution. A point $\boldsymbol{x}_0 \in \mathbb{R}^2$ is called *an efficient solution* of (P) if there is no $\boldsymbol{x} \in \mathbb{R}^2$ such that $\gamma(\boldsymbol{x}-\boldsymbol{d}_i) \leq \gamma(\boldsymbol{x}_0-\boldsymbol{d}_i)$ for all $i \in M$ and $\gamma(\boldsymbol{x}-\boldsymbol{d}_\ell) < \gamma(\boldsymbol{x}_0-\boldsymbol{d}_\ell)$ for some $\ell \in M$. Let $E(D)$ be the set of all efficient solutions of (P). By the above definition and the definition of γ, given in section 2, $D \subset E(D)$.

Various distances or norms are used in multicriteria location problems. For example, rectilinear distance in [9, 15], asymmetric rectilinear distance in [11], the block norm in [8, 13], the gauge in [3, 4]. In particular, the polyhedral gauge is used in [4]. These distances and norm are special cases of the guage. In particular, rectilinear distance, asymmetric rectilinear distance, the block norm and the A-distance [7, 12] are special cases of the polyhedral gauge. In [4], the procedure for finding all efficient solutions of (P) with the polyhedral gauge is given. In this article, a multicriteria location problem with the polyhedral gauge in $\mathbb{R}^2$ is considered. First, we characterize efficient solutions of (P). Next, we propose the Frame Generating Algorithm to find $E(D)$, which requires $O(m^3)$ computational time.

In section 2, we give some properties of the polyhedral gauge. In section 3, main results in [4] are given. In section 4, we give some properties of efficient solutions of (P). In section 5, we propose the Frame Generating Algorithm to find $E(D)$, which requires $O(m^3)$ computational time. Finally, some conclusions are given in section 6.

2. Preliminaries. In this section, we give some properties of the polyhedral gauge.

The gauge of $\boldsymbol{x} \in \mathbb{R}^2$ in the sense of Minkowski, $\gamma(\boldsymbol{x})$, is denoted by $\gamma(\boldsymbol{x}) \equiv \inf\{\mu > 0: \boldsymbol{x} \in \mu B\}$, where $B \subset \mathbb{R}^2$ is a closed bounded convex set having the origin in its interior. For $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^2$, *the distance from* $\boldsymbol{y}$ *to* $\boldsymbol{x}$ is denoted by $\gamma(\boldsymbol{x}-\boldsymbol{y})$. The set B is called *the*

2000 *Mathematics Subject Classification.* 90B85.
Key words and phrases. location problem, multicriteria, polyhedral gauge, efficiency.

unit ball associated with γ. The set $B^\circ \equiv \{\boldsymbol{p} \in \mathbb{R}^2\colon \langle \boldsymbol{p}, \boldsymbol{x}\rangle \leq 1, \forall \boldsymbol{x} \in B\}$ is called *the polar* of B. If B is a closed bounded convex set having the origin in its interior, then so is B° and $B^{\circ\circ} = B$ (see [14]). It can be checked that the subdifferentail of the gauge γ of B at $\boldsymbol{x} \in \mathbb{R}^2$, $\partial\gamma(\boldsymbol{x})$, is given by

$$(1) \qquad \partial\gamma(\boldsymbol{x}) = \begin{cases} B^\circ & \text{if } \boldsymbol{x} = \boldsymbol{0}, \\ \{\boldsymbol{p} \in B^\circ : \langle \boldsymbol{p}, \boldsymbol{x}\rangle = \gamma(\boldsymbol{x})\} & \text{if } \boldsymbol{x} \neq \boldsymbol{0}. \end{cases}$$

A guage γ is said to be *polyhedral* if its unit ball is a polytope, i.e. the convex hull of a finite number of points. Throughout this paper, a gauge γ is polyhedral. Moreover, we denote the set of all extreme points of B by Ext (B), and assume that if $\boldsymbol{e} \in$ Ext (B) then $-\mu\boldsymbol{e} \in$ Ext (B) for some $\mu > 0$. We put Ext $(B) = \{\boldsymbol{e}_1, \boldsymbol{e}_2, \cdots, \boldsymbol{e}_{2r}\}$, assuming that $\boldsymbol{e}_j = \|\boldsymbol{e}_j\|(\cos\theta_j, \sin\theta_j)^T$, $j \in \{1, 2, \cdots, 2r\}$, $0 \leq \theta_1 < \theta_2 < \cdots < \theta_r < \pi \leq \theta_{r+1} < \cdots < \theta_{2r} < 2\pi$. Note that for each $j \in \{1, 2, \cdots, r\}$, $\boldsymbol{e}_{r+j} = -\mu\boldsymbol{e}_j$ for some $\mu > 0$. For each $j \in \{1, 2, \cdots, 2r\}$ and each $n \in \mathbb{Z}$, we put $\boldsymbol{e}_{2nr+j} \equiv \boldsymbol{e}_j$, $\theta_{2nr+j} \equiv 2n\pi + \theta_j$, $K_{2nr+j} \equiv \mathcal{C}\{\boldsymbol{e}_{2nr+j}, \boldsymbol{e}_{2nr+j+1}\}$ and $L_{2nr+j} \equiv K_{2nr+j-1} \bigcap K_{2nr+j} = \{\mu\boldsymbol{e}_{2nr+j}\colon \mu \geq 0\}$, where $\mathbb{Z}$ is the set of all integers and $\mathcal{C}\{\boldsymbol{e}_{2nr+j}, \boldsymbol{e}_{2nr+j+1}\} \equiv \{\lambda\boldsymbol{e}_{2nr+j} + \mu\boldsymbol{e}_{2nr+j+1}\colon \lambda, \mu \geq 0\}$. Note that for each $j \in \{1, 2, \cdots, 2r\}$ and each $n \in \mathbb{Z}$, $K_{2nr+j} = K_j = -K_{r+j}$ and $L_{2nr+j} = L_j = -L_{r+j}$. We put $L \equiv \bigcup_{i=1}^m \bigcup_{j=1}^{2r}(\{\boldsymbol{d}_i\} + L_j)$.

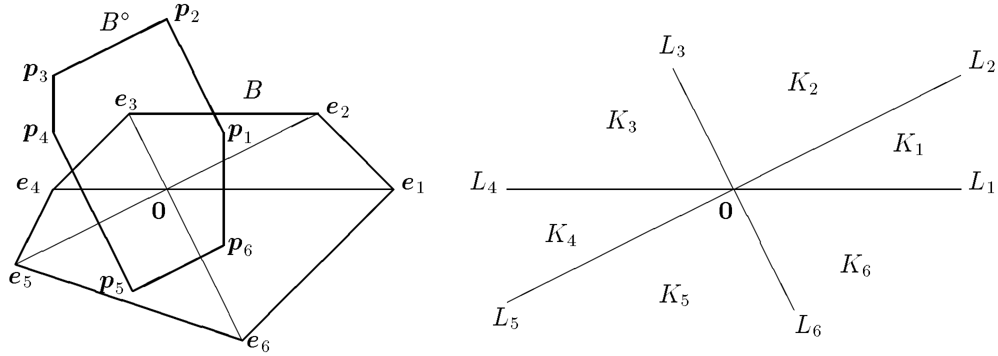

Figure 1. B, B° and $\boldsymbol{e}_j, \boldsymbol{p}_j, K_j, L_j$.

For $\boldsymbol{x} \in \mathbb{R}^2$, $\gamma(\boldsymbol{x})$ can be represented as follows (see [3]):

$$(2) \qquad \gamma(\boldsymbol{x}) = \min\left\{\sum_{j=1}^{2r} \mu_j : \boldsymbol{x} = \sum_{j=1}^{2r} \mu_j\boldsymbol{e}_j, \mu_j \geq 0, j \in \{1, 2, \cdots, 2r\}\right\}.$$

In other words, this means that the distance from $\boldsymbol{y}$ to $\boldsymbol{x}$ is the length of one of the shortest possible routes to travel from $\boldsymbol{y}$ to $\boldsymbol{x}$ by going only in the directions defined and oriented by the vectors $\boldsymbol{e}_1, \boldsymbol{e}_2, \cdots, \boldsymbol{e}_{2r}$. From (2), for each $j \in \{1, 2, \cdots, 2r\}$ and each $n \in \mathbb{Z}$ and each $\boldsymbol{x} = (x^1, x^2)^T \in \mathbb{R}^2$, if $\boldsymbol{x} \in K_{2nr+j}$, then $\gamma(\boldsymbol{x}) = ax^1 + bx^2$, where $\boldsymbol{p}_{2nr+j} \equiv (a, b)^T = 1/(e_j^1 e_{j+1}^2 - e_j^2 e_{j+1}^1)(e_{j+1}^2 - e_j^2, e_j^1 - e_{j+1}^1)^T$ and $\boldsymbol{e}_j \equiv (e_j^1, e_j^2)^T$, $\boldsymbol{e}_{j+1} \equiv (e_{j+1}^1, e_{j+1}^2)^T$. Note that $e_j^1 e_{j+1}^2 - e_j^2 e_{j+1}^1 \neq 0$ since $\boldsymbol{e}_j$ and $\boldsymbol{e}_{j+1}$ are linearly independent, and that $\boldsymbol{p}_{2nr+j} = \boldsymbol{p}_j$. It is assumed that $\boldsymbol{p}_j = \|\boldsymbol{p}_j\|(\cos\alpha_j, \sin\alpha_j)^T$, $j \in \{1, 2, \cdots, 2r\}$, $\alpha_1 < \alpha_2 < \cdots < \alpha_{2r}$, $\alpha_{2r} - \alpha_1 < 2\pi$. For each $j \in \{1, 2, \cdots, 2r\}$ and each $n \in \mathbb{Z}$, we put $\alpha_{2nr+j} \equiv 2n\pi + \alpha_j$. Note that $\alpha_{j+1} - \alpha_j < \pi$.

We denote by int (A), bd (A) and co (A), the interior, the boundary and the convex hull of a set $A \subset \mathbb{R}^2$ and by ri (C) the relative interior of a convex set $C \subset \mathbb{R}^2$.

From [2, Theorem 9.1, pp.57-58], we have

$$
\begin{aligned}
B^\circ &= \bigcap_{i=1}^{2r}\{\boldsymbol{x}\in\mathbb{R}^2 : \langle \boldsymbol{e}_i, \boldsymbol{x}\rangle \le 1\} = \mathrm{co}(\{\boldsymbol{p}_1, \boldsymbol{p}_2, \cdots, \boldsymbol{p}_{2r}\}),\\
\mathrm{Ext}(B^\circ) &= \{\boldsymbol{p}_1, \boldsymbol{p}_2, \cdots, \boldsymbol{p}_{2r}\},\\
\mathrm{bd}(B^\circ) &= \bigcup_{j=1}^{2r}[\boldsymbol{p}_j, \boldsymbol{p}_{j+1}]
\end{aligned}
$$

where $[\boldsymbol{p}_j, \boldsymbol{p}_{j+1}] \equiv \{\mu\boldsymbol{p}_j + (1-\mu)\boldsymbol{p}_{j+1}\colon\ 0 \le \mu \le 1\}$. From (1), for each $j \in \{1, 2, \cdots, 2r\}$ and each $n \in \mathbb{Z}$, we have

$$
\partial\gamma(\boldsymbol{x}) = \begin{cases} \{\boldsymbol{p}_{2nr+j}\} & \text{if } \boldsymbol{x} \in \mathrm{int}(K_{2nr+j}),\\ [\boldsymbol{p}_{2nr+j-1}, \boldsymbol{p}_{2nr+j}] & \text{if } \boldsymbol{x} \in L_{2nr+j} \setminus \{\boldsymbol{0}\},\\ \mathrm{co}(\{\boldsymbol{p}_1, \boldsymbol{p}_2, \cdots, \boldsymbol{p}_{2r}\}) & \text{if } \boldsymbol{x} = \boldsymbol{0}. \end{cases} \tag{3}
$$

3. Main results in [4].

In this section, main results in [4] are given.

We recall some notations and results in [4]. A finite family $\{C_1, C_2, \cdots, C_m\}$ of nonempty sets in $\mathbb{R}^2$ is said to be *suitably contained in a halfspace* if there exists a hyperplane containing the origin and such that one of its associated closed halfspaces contains all of the C_i's, with at least one of the C_i's contained in the corresponding open halfspace. In other words, a family $\{C_1, C_2, \cdots, C_m\}$ is suitably contained in a halfspace if and only if there exists $\boldsymbol{a} \ne \boldsymbol{0}$ such that, first, $\langle \boldsymbol{a}, \boldsymbol{x}\rangle \le 0$ for every $\boldsymbol{x}$ in $\bigcup_{i=1}^m C_i$, and second, $\langle \boldsymbol{a}, \boldsymbol{x}\rangle < 0$ for every $\boldsymbol{x}$ in some C_ℓ. For $\boldsymbol{x} \in \mathbb{R}^2$, we put $\Gamma(\boldsymbol{x}) \equiv \{\partial\gamma(\boldsymbol{x}-\boldsymbol{d}_1), \partial\gamma(\boldsymbol{x}-\boldsymbol{d}_2), \cdots, \partial\gamma(\boldsymbol{x}-\boldsymbol{d}_m)\}$.

Theorem 1.([4]) *The set $E(D)$ is the set of all $\boldsymbol{x} \in \mathbb{R}^2$ such that $\Gamma(\boldsymbol{x})$ is not suitably contained in a halfspace.*

A nonempty closed convex set $C \subset \mathbb{R}^2$ is called *an elementary convex set* with respect to D and γ if $C = \bigcap_{i=1}^m(\{\boldsymbol{d}_i\} + N(\boldsymbol{q}_i))$ for some $\boldsymbol{q}_i \in B^\circ$, $i \in M$, where

$$
N(\boldsymbol{p}) = \begin{cases} K_{2nr+j} & \text{if } \boldsymbol{p} = \boldsymbol{p}_{2nr+j},\\ L_{2nr+j+1} & \text{if } \boldsymbol{p} \in \mathrm{ri}([\boldsymbol{p}_{2nr+j}, \boldsymbol{p}_{2nr+j+1}]),\\ \{\boldsymbol{0}\} & \text{if } \boldsymbol{p} \in \mathrm{int}(B^\circ) \end{cases}
$$

for each $j \in \{1, 2, \cdots, 2r\}$ and each $n \in \mathbb{Z}$. For each $\boldsymbol{x} \in C$, we have $\boldsymbol{q}_i \in \partial\gamma(\boldsymbol{x} - \boldsymbol{d}_i)$.

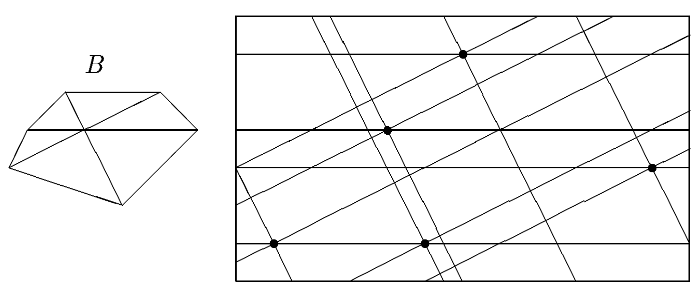

Figure 2. Elementary convex sets. (•: demand points)

Corollary 1.([4]) *If C is an elementary convex set, then either C is contained in $E(D)$ or else* ri (C) *and $E(D)$ are disjoint.*

Theorem 2.([4]) *The set $E(D)$ is a connected finite union of polytopes, each of which is an elementary convex set.*

In [4], practical rules with which the whole set $E(D)$ can be found are given. They are obvious consequences of Theorem 1 and Corollary 1, and described as follows:

Rule 1. *If $\boldsymbol{x} \notin D$ is such that the family $\Gamma(\boldsymbol{x})$ is suitably contained in a halfspace, then for every elementary convex set C containing $\boldsymbol{x}$* , ri (C) *and $E(D)$ are disjoint.* □

Rule 2. *If $\boldsymbol{x} \in \mathbb{R}^2$ is in the relative interior of an elementary convex set C and if the family $\Gamma(\boldsymbol{x})$ is not suitably contained in a halfspace, then C is contained in $E(D)$.* □

A point $\boldsymbol{x} \in \mathbb{R}^2$ is called *an intersection point* if $\boldsymbol{x}$ is an extreme point of some elementary convex set. Let I be the set of all intersection points. When I is known and it is possible to check whether $\Gamma(\boldsymbol{x})$ for $\boldsymbol{x} \in \mathbb{R}^2$ is suitably contained in a halfspace or not, the procedure for finding $E(D)$ can be described. First apply Rule 1 to every point of I. In this way, many elementary convex sets are eliminated. Then apply Rule 2 to every remaining elementary convex set, by considering first the elementary convex sets whose dimension is two, then one. This method is clearly finite. Implementing it efficiently, however, is a hard task.

4. Properties of efficient solutions. In this section, we give some properties of efficient solutions of (P).

Theorem 3. *Let C be a bounded elementary convex set such that* int $(C) \neq \emptyset$. *If* bd $(C) \subset E(D)$, *then $C \subset E(D)$.*

Proof. For $\boldsymbol{y} \in$ int (C), assume that $\boldsymbol{y} \notin E(D)$. From Theorem 1, $\Gamma(\boldsymbol{y})$ is suitably contained in a halfspace. For each $i \in M$, there exists $j_i \in \{1, 2, \cdots, 2r\}$ such that $\boldsymbol{y} \in \{\boldsymbol{d}_i\}$ + int (K_{j_i}). Then $C = \bigcap_{i=1}^m(\{\boldsymbol{d}_i\} + K_{j_i})$ and $\partial\gamma(\boldsymbol{y} - \boldsymbol{d}_i) = \{\boldsymbol{p}_{j_i}\}$, $i \in M$. Since $\Gamma(\boldsymbol{y})$ is suitably contained in a halfspace, $\bigcup_{i=1}^m K_{r+j_i} = -\bigcup_{i=1}^m K_{j_i} \neq \mathbb{R}^2$. Note that $\bigcup_{i=1}^m K_{j_i} \neq K_j$ for any $j \in \{1, 2, \cdots, 2r\}$ since C is bounded. We put $G(\boldsymbol{y}) \equiv \bigcup_{i=1}^m$ int $(\{\boldsymbol{y}\} + K_{r+j_i})$. Then we see that $D \subset G(\boldsymbol{y})$.

Without loss of generality, assume that $\alpha_{j_1} \leq \alpha_{j_2} \leq \cdots \leq \alpha_{j_m}$. For each $i \in M$ and each $n \in \mathbb{Z}$, we put $j_{nm+i} \equiv 2nr + j_i$. Note that $\boldsymbol{p}_{j_{nm+i}} = \boldsymbol{p}_{2nr+j_i} = \boldsymbol{p}_{j_i}$ and $\alpha_{j_{nm+i}} = \alpha_{2nr+j_i} = 2n\pi + \alpha_{j_i}$. Since $\Gamma(\boldsymbol{y})$ is suitably contained in a halfspace, one of the following conditions is satisfied.

(i) $0 < \alpha_{j_{m+k-1}} - \alpha_{j_k} < \pi$ for some $k \in M$.

(ii) $\alpha_{j_{m+k-1}} - \alpha_{j_k} = \pi$ for some $k \in M$, and $\alpha_{j_k} < \alpha_{j_\ell} < \alpha_{j_{m+k-1}}$ for some ℓ $(k < \ell < m + k - 1)$.

Case 1. First, assume that condition (i) is satisfied. We put $\boldsymbol{a} = -(\cos\frac{\alpha_{j_k}+\alpha_{j_{m+k-1}}}{2}, \sin\frac{\alpha_{j_k}+\alpha_{j_{m+k-1}}}{2})^T$. Then we see that $\langle \boldsymbol{a}, \boldsymbol{p}_{j_i} \rangle < 0$ for any $i \in M$, i.e. $\langle \boldsymbol{a}, \boldsymbol{x} \rangle < 0$ for any $\boldsymbol{x} \in \bigcup_{i=1}^m \partial\gamma(\boldsymbol{y} - \boldsymbol{d}_i)$. Since $\alpha_{j_{m+k-1}} - \alpha_{j_k} < \pi$, we have $R(\boldsymbol{y}) \equiv (\{\boldsymbol{y}\} + K_{r+j_k}) \bigcup (\{\boldsymbol{y}\} + K_{r+j_k+1}) \bigcup \cdots \bigcup (\{\boldsymbol{y}\} + K_{r+j_{m+k-1}}) \neq \mathbb{R}^2$. $R(\boldsymbol{y})$ is a cone with a vertex at $\boldsymbol{y}$. We put $P(\boldsymbol{y}) \equiv (\mathrm{int}(R(\boldsymbol{y})))^c$. Since $D \subset G(\boldsymbol{y}) \subset$ int $(R(\boldsymbol{y}))$, we have $D \bigcap P(\boldsymbol{y}) = \emptyset$. Moreover, $P(\boldsymbol{y}) = (\{\boldsymbol{y}\} + K_p) \bigcup (\{\boldsymbol{y}\} + K_{p+1}) \bigcup \cdots \bigcup (\{\boldsymbol{y}\} + K_{p+t})$ for some $p \in \{1, 2, \cdots, 2r\}$ and some $t \geq 0$, where $K_p = K_{r+j_{m+k-1}+1}$ and $K_{p+t} = K_{r+j_k-1}$. There exists $\boldsymbol{z} \in$ bd $(C) \bigcap P(\boldsymbol{y})$ such that $\boldsymbol{z}$ is not a vertex of C, i.e. $\boldsymbol{z}$ is a relative interior point of some edge of C. Let $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ be two end points of the edge containing $\boldsymbol{z}$. We put $Q \equiv \{\mu\boldsymbol{x}_1 + (1-\mu)\boldsymbol{x}_2\colon \mu \in \mathbb{R}\}$. Then $Q = \{\boldsymbol{z} + \mu\boldsymbol{e}_{j_0}\colon \mu \in \mathbb{R}\} = \{\boldsymbol{z}\} + L_{j_0} \bigcup L_{r+j_0}$ for some $j_0 \in \{1, 2, \cdots, 2r\}$. Since $\partial\gamma(\boldsymbol{z} - \boldsymbol{d}_i) = \partial\gamma(\boldsymbol{y} - \boldsymbol{d}_i) = \{\boldsymbol{p}_{j_i}\}$ for $\boldsymbol{d}_i \notin Q$, we see that $\langle \boldsymbol{a}, \boldsymbol{x}\rangle < 0$ for any $\boldsymbol{x} \in \partial\gamma(\boldsymbol{z} - \boldsymbol{d}_i)$. Let $\boldsymbol{d}_i$ be a demand point in Q. If $\boldsymbol{d}_i \in \{\boldsymbol{z}\} + L_{j_0}$, then $\boldsymbol{z} \in \{\boldsymbol{d}_i\} + L_{r+j_0}$, and $L_{r+j_0} = L_q$ for some $q \in \{j_k+1, \cdots, j_{m+k-1}\}$ since $D \bigcap P(\boldsymbol{z}) = \emptyset$. If $\boldsymbol{d}_i \in \{\boldsymbol{z}\} + L_{r+j_0}$, then $\boldsymbol{z} \in \{\boldsymbol{d}_i\} + L_{j_0}$, and $L_{j_0} = L_q$ for some $q \in \{j_k+1, \cdots, j_{m+k-1}\}$ since $D \bigcap P(\boldsymbol{z}) = \emptyset$. In either case, $\partial\gamma(\boldsymbol{z} - \boldsymbol{d}_i)$ can be represented as $\partial\gamma(\boldsymbol{z} - \boldsymbol{d}_i) = [\boldsymbol{p}_{q-1}, \boldsymbol{p}_q]$, and we have $\langle \boldsymbol{a}, \boldsymbol{x}\rangle < 0$ for any $\boldsymbol{x} \in \partial\gamma(\boldsymbol{z} - \boldsymbol{d}_i)$ since $j_k \leq q-1$, $q \leq j_{m+k-1}$. Thus, $\langle \boldsymbol{a}, \boldsymbol{x}\rangle < 0$ for any $\boldsymbol{x} \in \bigcup_{i=1}^m \partial\gamma(\boldsymbol{z} - \boldsymbol{d}_i)$. Therefore, $\boldsymbol{z} \notin E(D)$ from Theorem 1 since $\Gamma(\boldsymbol{z})$ is suitably contained in a halfspace. However, this contradicts our assumption that bd $(C) \subset E(D)$.

Case 2. Next, assume that condition (ii) is satisfied. Let $\boldsymbol{a}$, $R(\boldsymbol{y})$ and $P(\boldsymbol{y})$ be the same ones as in Case 1. In this case, $\langle \boldsymbol{a}, \boldsymbol{p}_{j_i}\rangle \leq 0$ for any $i \in M$ and $\langle \boldsymbol{a}, \boldsymbol{p}_{j_\ell}\rangle < 0$, i.e. $\langle \boldsymbol{a}, \boldsymbol{x}\rangle \leq 0$ for any $\boldsymbol{x} \in \bigcup_{i=1}^m \partial\gamma(\boldsymbol{y} - \boldsymbol{d}_i)$ and $\langle \boldsymbol{a}, \boldsymbol{x}\rangle < 0$ for any $\boldsymbol{x} \in \partial\gamma(\boldsymbol{y} - \boldsymbol{d}_\ell)$.

First, assume that $K_{j_\ell} \subset P(\boldsymbol{0})$ for some ℓ $(k < \ell < m+k-1)$ such that $\alpha_{j_k} < \alpha_{j_\ell} < \alpha_{j_{m+k-1}}$. There exists $\boldsymbol{z} \in$ bd $(C) \bigcap (\{\boldsymbol{y}\} + K_{j_\ell})$ such that $\boldsymbol{z}$ is not a vertex of C. Let $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, Q and j_0 be the same ones as in Case 1. By the similar argument in Case 1, we see that $\langle \boldsymbol{a}, \boldsymbol{x}\rangle \leq 0$ for any $\boldsymbol{x} \in \bigcup_{i=1}^m \partial\gamma(\boldsymbol{z} - \boldsymbol{d}_i)$. In this case, $\langle \boldsymbol{a}, \boldsymbol{x}\rangle < 0$ for any $\boldsymbol{x} \in \partial\gamma(\boldsymbol{z} - \boldsymbol{d}_\ell)$ since $\partial\gamma(\boldsymbol{z} - \boldsymbol{d}_\ell) = \partial\gamma(\boldsymbol{y} - \boldsymbol{d}_\ell) = \{\boldsymbol{p}_{j_\ell}\}$ by the definition of $\boldsymbol{z}$. Thus, $\boldsymbol{z} \notin E(D)$ from Theorem 1 since $\Gamma(\boldsymbol{z})$ is suitably contained in a halfspace. However, this contradicts our assumption that bd $(C) \subset E(D)$.

Next, assume that $K_{j_\ell} \not\subset P(\boldsymbol{0})$ for any ℓ $(k < \ell < m+k-1)$ such that $\alpha_{j_k} < \alpha_{j_\ell} < \alpha_{j_{m+k-1}}$. Then $K_{j_\ell} \neq K_j$, $j \in \{p, p+1, \cdots, p+t\}$ for any ℓ $(k < \ell < m+k-1)$ such that $\alpha_{j_k} < \alpha_{j_\ell} < \alpha_{j_{m+k-1}}$. And $0 < \theta_{p+t+1} - \theta_p < \pi$ since if $\theta_{p+t+1} - \theta_p \geq \pi$, then $K_{j_\ell} \subset P(\boldsymbol{0})$ for any ℓ $(k < \ell < m+k-1)$ such that $\alpha_{j_k} < \alpha_{j_\ell} < \alpha_{j_{m+k-1}}$. Moreover, we see that $D \bigcap P^-(\boldsymbol{y}) = \emptyset$, where $P^-(\boldsymbol{y}) \equiv (\{\boldsymbol{y}\} - K_p) \bigcup (\{\boldsymbol{y}\} - K_{p+1}) \bigcup \cdots \bigcup (\{\boldsymbol{y}\} - K_{p+t})$. If $D \bigcap P^-(\boldsymbol{y}) \neq \emptyset$, then there exists $\boldsymbol{d}_u \in D \bigcap P^-(\boldsymbol{y})$ such that $\alpha_{j_k} < \alpha_{j_{nm+u}} < \alpha_{j_{m+k-1}}$, $k < nm + u < m+k-1$ for some $n \in \mathbb{Z}$ and that $\boldsymbol{d}_u \in \{\boldsymbol{y}\} - K_q$ for some $q \in \{p, p+1, \cdots, p+t\}$. Since $\boldsymbol{d}_u \in \{\boldsymbol{y}\} - K_{j_{nm+u}}$, we have $K_{j_{nm+u}} = K_q \subset P(\boldsymbol{0})$.

Since $\theta_{p+t+1} - \theta_p < \pi$, $D \bigcap P^-(\boldsymbol{y}) = \emptyset$, we see that $P(\boldsymbol{0}) \subset \mathcal{C}\{\boldsymbol{e}_{j_k+1}, \boldsymbol{e}_{j_k+2}, \cdots, \boldsymbol{e}_{j_\ell}\}$ or $P(\boldsymbol{0}) \subset \mathcal{C}\{\boldsymbol{e}_{j_\ell+1}, \boldsymbol{e}_{j_\ell+2}, \cdots, \boldsymbol{e}_{j_{m+k-1}}\}$. It is sufficient to show the case $P(\boldsymbol{0}) \subset \mathcal{C}\{\boldsymbol{e}_{j_k+1}, \boldsymbol{e}_{j_k+2}, \cdots, \boldsymbol{e}_{j_\ell}\}$. It can be shown similarly the case $P(\boldsymbol{0}) \subset \mathcal{C}\{\boldsymbol{e}_{j_\ell+1}, \boldsymbol{e}_{j_\ell+2}, \cdots, \boldsymbol{e}_{j_{m+k-1}}\}$. Thus, we assume that $P(\boldsymbol{0}) \subset \mathcal{C}\{\boldsymbol{e}_{j_k+1}, \boldsymbol{e}_{j_k+2}, \cdots, \boldsymbol{e}_{j_\ell}\}$.

There exists $\boldsymbol{z} \in$ bd $(C) \bigcap P(\boldsymbol{y})$ such that $\boldsymbol{z}$ is not a vertex of C. Let $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, Q and j_0 be the same ones as in Case 1. By the similar argument in Case 1, we see that $\langle \boldsymbol{a}, \boldsymbol{x}\rangle \leq 0$ for any $\boldsymbol{x} \in \bigcup_{i=1}^m \partial\gamma(\boldsymbol{z} - \boldsymbol{d}_i)$. Now, choose any ℓ $(k < \ell < m+k-1)$ such that $\alpha_{j_k} < \alpha_{j_\ell} < \alpha_{j_{m+k-1}}$. If $\boldsymbol{z} \in \{\boldsymbol{d}_\ell\}$ + int (K_{j_ℓ}), then $\partial\gamma(\boldsymbol{z} - \boldsymbol{d}_\ell) = \partial\gamma(\boldsymbol{y} - \boldsymbol{d}_\ell) = \{\boldsymbol{p}_{j_\ell}\}$, and $\langle \boldsymbol{a}, \boldsymbol{x}\rangle < 0$ for any $\boldsymbol{x} \in \partial\gamma(\boldsymbol{z} - \boldsymbol{d}_\ell)$ since $\langle \boldsymbol{a}, \boldsymbol{p}_{j_\ell}\rangle < 0$. If $\boldsymbol{z} \notin$ int (K_{j_ℓ}), then we see that $\boldsymbol{z} \in \{\boldsymbol{d}_\ell\} + L_{j_\ell}$ since $P(\boldsymbol{0}) \subset \mathcal{C}\{\boldsymbol{e}_{j_k+1}, \boldsymbol{e}_{j_k+2}, \cdots, \boldsymbol{e}_{j_\ell}\}$, and that $\partial\gamma(\boldsymbol{z} - \boldsymbol{d}_\ell) = [\boldsymbol{p}_{j_\ell - 1}, \boldsymbol{p}_{j_\ell}]$. Since $\alpha_{j_k} < \alpha_{j_k+1} < \alpha_{j_\ell} < \alpha_{j_{m+k-1}}$, we have $\langle \boldsymbol{a}, \boldsymbol{x}\rangle < 0$ for any $\boldsymbol{x} \in \partial\gamma(\boldsymbol{z} - \boldsymbol{d}_\ell)$. Thus, $\boldsymbol{z} \notin E(D)$ from Theorem 1 since $\Gamma(\boldsymbol{z})$ is suitably contained in a halfspace. However, this contradicts our assumption that bd $(C) \subset E(D)$.

Therefore, it is proved that $C \subset E(D)$. □

For $\boldsymbol{x}_1, \boldsymbol{x}_2 \in I$, $\boldsymbol{x}_1$ is called *an adjacent intersection point to* $\boldsymbol{x}_2$ and $\boldsymbol{x}_2$ is called *an adjacent intersection point to* $\boldsymbol{x}_1$ if $\boldsymbol{x}_1 \neq \boldsymbol{x}_2$, $[\boldsymbol{x}_1, \boldsymbol{x}_2] \subset L$ and ri $([\boldsymbol{x}_1, \boldsymbol{x}_2]) \bigcap I = \emptyset$.

Theorem 4 *It is assumed that polytope B, which defines the polyhedral gauge, is symmetric around the origin, i.e. γ is a norm. For mutually adjacent intersection points $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$, if $\boldsymbol{x}_1$, $\boldsymbol{x}_2 \in E(D)$, then $[\boldsymbol{x}_1, \boldsymbol{x}_2] \subset E(D)$.*

Proof. For $\boldsymbol{z} \in$ ri $([\boldsymbol{x}_1, \boldsymbol{x}_2])$, we shall show that $\boldsymbol{z} \in E(D)$. When B is symmetric around the origin, $\boldsymbol{x}_0 \in E(D)$ if and only if $\boldsymbol{x}_0$ satisfies one of the following conditions (see [13, Proposition 2 and 3]):

(i) $D \bigcap (\{\boldsymbol{x}_0\} + \bigcup_{j=1}^{r-1} K_{\ell+j}) \neq \emptyset$ for any $\ell \in \{1, 2, \cdots, 2r\}$.

(ii) There exists $\ell \in \{1, 2, \cdots, 2r\}$ such that $D \bigcap (\{\boldsymbol{x}_0\} + \bigcup_{j=1}^{r-1} K_{\ell+j}) = \emptyset$, $D \bigcap \mathrm{int}\,(\{\boldsymbol{x}_0\} + \bigcup_{j=1}^{r-1} K_{r+\ell+j}) = \emptyset$, $D \bigcap (\{\boldsymbol{x}_0\} + K_\ell) \neq \emptyset$ and $D \bigcap (\{\boldsymbol{x}_0\} + K_{r+\ell}) \neq \emptyset$.

Without loss of generality, assume that $\boldsymbol{x}_2 - \boldsymbol{x}_1 = \mu \boldsymbol{e}_1$ for some $\mu > 0$. We put $U \equiv \bigcup_{j=2}^{r} (\mathrm{ri}([\boldsymbol{x}_1, \boldsymbol{x}_2]) + L_j \bigcup L_{r+j})$. Then $D \bigcap U = \emptyset$ since $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ are mutually adjacent intersection points.

Case 1. First, assume that $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ satisfy condition (i). Since $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ satisfy condition (i), $D \bigcap (\{\boldsymbol{z}\} + \bigcup_{j=1}^{r-1} K_{\ell+j}) \supset D \bigcap (\{\boldsymbol{x}_1\} + \bigcup_{j=1}^{r-1} K_{\ell+j}) \neq \emptyset$ for each $\ell \in \{1, 2, \cdots, r\}$, and $D \bigcap (\{\boldsymbol{z}\} + \bigcup_{j=1}^{r-1} K_{\ell+j}) \supset D \bigcap (\{\boldsymbol{x}_2\} + \bigcup_{j=1}^{r-1} K_{\ell+j}) \neq \emptyset$ for each $\ell \in \{r+1, \cdots, 2r\}$. Thus, $\boldsymbol{z} \in E(D)$ since $\boldsymbol{z}$ satisfies condition (i).

Case 2. Next, assume that $\boldsymbol{x}_1$ or $\boldsymbol{x}_2$ satisfies condition (ii). It is sufficient to show the case $\boldsymbol{x}_1$ satisfies condition (ii). It can be shown similarly the case $\boldsymbol{x}_2$ satisfies condition (ii). Thus, we assume that $\boldsymbol{x}_1$ satisfies condition (ii). In this case, $D \bigcap (\{\boldsymbol{x}_1\} + \mathrm{ri}\,(L_1)) \neq \emptyset$ or $D \bigcap (\{\boldsymbol{x}_1\} + \mathrm{ri}\,(L_{r+1})) \neq \emptyset$. We shall show only the case $D \bigcap (\{\boldsymbol{x}_1\} + \mathrm{ri}\,(L_1)) \neq \emptyset$. It can be shown similarly the case $D \bigcap (\{\boldsymbol{x}_1\} + \mathrm{ri}\,(L_{r+1})) \neq \emptyset$. In this case, we have $D \bigcap (\{\boldsymbol{z}\} + L_{r+1}) = \emptyset$. Because if $D \bigcap (\{\boldsymbol{z}\} + L_{r+1}) \neq \emptyset$, then $\boldsymbol{x}_1$ satisfies condition (i) and so $\boldsymbol{x}_1$ does not satisfy condition (ii). Since $\boldsymbol{x}_1$ satisfies condition (ii), it needs that ℓ in condition (ii) is 1 or r. Because $D \bigcap (\{\boldsymbol{x}_1\} + \bigcup_{j=1}^{r-1} K_{\ell+j}) \supset D \bigcap (\{\boldsymbol{x}_1\} + \mathrm{ri}\,(L_1)) \neq \emptyset$ for $\ell \in \{r+1, \cdots, 2r\}$, and $D \bigcap \mathrm{int}\,(\{\boldsymbol{x}_1\} + \bigcup_{j=1}^{r-1} K_{r+\ell+j}) \supset D \bigcap (\{\boldsymbol{x}_1\} + \mathrm{ri}\,(L_1)) \neq \emptyset$ for $\ell \in \{1, 2, \cdots, r\} \setminus \{1, r\}$. We shall show only the case ℓ in condition (ii) is 1. It can be shown similarly the case ℓ in condition (ii) is r. Since $D \bigcap (\{\boldsymbol{x}_1\} + \bigcup_{j=1}^{r-1} K_{1+j}) = \emptyset$, we see that $D \bigcap (\{\boldsymbol{z}\} + \bigcup_{j=1}^{r-1} K_{1+j}) = D \bigcap [(\{\boldsymbol{x}_1\} + \bigcup_{j=1}^{r-1} K_{1+j}) \bigcup ([\boldsymbol{x}_1, \boldsymbol{z}] \setminus \{\boldsymbol{x}_1\} + L_2)] \subset [D \bigcap (\{\boldsymbol{x}_1\} + \bigcup_{j=1}^{r-1} K_{1+j})] \bigcup (D \bigcap U) = \emptyset$. Since $D \bigcap \mathrm{int}\,(\{\boldsymbol{x}_1\} + \bigcup_{j=1}^{r-1} K_{r+1+j}) = \emptyset$, we see that $D \bigcap \mathrm{int}\,(\{\boldsymbol{z}\} + \bigcup_{j=1}^{r-1} K_{r+1+j}) \subset D \bigcap \mathrm{int}\,(\{\boldsymbol{x}_1\} + \bigcup_{j=1}^{r-1} K_{r+1+j}) = \emptyset$. We have $D \bigcap (\{\boldsymbol{z}\} + K_1) \neq \emptyset$ since $D \bigcap (\{\boldsymbol{x}_1\} + K_1) = D \bigcap [([\boldsymbol{x}_1, \boldsymbol{z}] + L_2) \bigcup (\{\boldsymbol{z}\} + K_1)] = [D \bigcap ([\boldsymbol{x}_1, \boldsymbol{z}] + L_2)] \bigcup [D \bigcap (\{\boldsymbol{z}\} + K_1)] \neq \emptyset$ and $D \bigcap ([\boldsymbol{x}_1, \boldsymbol{z}] + L_2) = D \bigcap [(\{\boldsymbol{x}_1\} + L_2) \bigcup ([\boldsymbol{x}_1, \boldsymbol{z}] \setminus \{\boldsymbol{x}_1\} + L_2)] = [D \bigcap (\{\boldsymbol{x}_1\} + L_2)] \bigcup [D \bigcap ([\boldsymbol{x}_1, \boldsymbol{z}] \setminus \{\boldsymbol{x}_1\} + L_2)] \subset [D \bigcap (\{\boldsymbol{x}_1\} + \bigcup_{j=1}^{r-1} K_{1+j})] \bigcup (D \bigcap U) = \emptyset$. Since $D \bigcap (\{\boldsymbol{x}_1\} + K_{r+1}) \neq \emptyset$, we see that $D \bigcap (\{\boldsymbol{z}\} + K_{r+1}) \supset D \bigcap (\{\boldsymbol{x}_1\} + K_{r+1}) \neq \emptyset$. Therefore, $\boldsymbol{z} \in E(D)$ since $\boldsymbol{z}$ satisfies condition (ii). □

Theorem 5. *It is assumed that polytope B, which defines the polyhedral gauge, is symmetric around the origin, i.e. γ is a norm. Let C be a bounded elementary convex set such that* int $(C) \neq \emptyset$. *If every extreme point of C is efficient solution of* (P), *then $C \subset E(D)$.*

Proof. From Theorem 4, bd $(C) \subset E(D)$. Thus, $C \subset E(D)$ from Theorem 3. □

Theorem 6 *It is assumed that $r = 2$ and that $D \subset \{\boldsymbol{x}_0\} + L_{j_0} \bigcup L_{r+j_0}$ for some $\boldsymbol{x}_0 \in \mathbb{R}^2$ and some $j_0 \in \{1, 2, \cdots, r\}$. Then $E(D) = \mathrm{co}\,(D)$.*

Proof. For $\boldsymbol{y} \notin \mathrm{co}\,(D)$, $\boldsymbol{y} \in \mathrm{ri}\,(C)$ for some unbounded elementary convex set C. Since C is unbounded, $C \not\subset E(D)$ from Theorem 2, and so $\mathrm{ri}\,(C) \bigcap E(D) = \emptyset$ from Corollary 1. Thus, we have $\boldsymbol{y} \notin E(D)$. We know $D \subset E(D)$. Without loss of generality, assume that $j_0 = 1$, $\theta_1 = 0$ and $d_1^1 < d_2^1 < \cdots < d_m^1$, where $\boldsymbol{d}_i \equiv (d_i^1, d_i^2)^T$, $i \in M$. For $\boldsymbol{y} \in \mathrm{co}\,(D) \setminus D = [\boldsymbol{d}_1, \boldsymbol{d}_m] \setminus D = \bigcup_{i=1}^{m-1} \mathrm{ri}\,([\boldsymbol{d}_i, \boldsymbol{d}_{i+1}])$, $\boldsymbol{y} \in \mathrm{ri}\,([\boldsymbol{d}_{i_0}, \boldsymbol{d}_{i_0+1}])$ for some $i_0 \in \{1, 2, \cdots, m-1\}$. Since $r = 2$, we have

$$\partial\gamma(\boldsymbol{y} - \boldsymbol{d}_i) = \begin{cases} [\boldsymbol{p}_4, \boldsymbol{p}_1] & \text{if } i \leq i_0, \\ [\boldsymbol{p}_2, \boldsymbol{p}_3] & \text{if } i > i_0. \end{cases}$$

Note that $[\boldsymbol{p}_4, \boldsymbol{p}_1]$ and $[\boldsymbol{p}_2, \boldsymbol{p}_3]$ are mutually opposite edges of the quadrangle $B^\circ = \text{co}\,(\{\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3, \boldsymbol{p}_4\})$. Since $\mathbf{0} \in \text{int}\,(B^\circ)$, $\Gamma(\boldsymbol{y})$ is not suitably contained in a halfspace. Thus, $\boldsymbol{y} \in E(D)$ from Theorem 1. Therefore, it is proved that $E(D) = \text{co}\,(D)$. □

5. Algorithm to find all efficient solutions. In this section, we propose the Frame Generating Algorithm to find $E(D)$, which requires $O(m^3)$ computational time.

Let $\boldsymbol{x}_1^*$ and $\boldsymbol{x}_2^*$ be any two efficient solutions of (P). From Corollary 1 and Theorem 2, there exists polygonal line in $E(D)$, which connects $\boldsymbol{x}_1^*$ and $\boldsymbol{x}_2^*$, i.e. there exists $\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_n \in E(D)$ such that $[\boldsymbol{x}_1^*, \boldsymbol{x}_1], [\boldsymbol{x}_1, \boldsymbol{x}_2], \cdots, [\boldsymbol{x}_n, \boldsymbol{x}_2^*] \subset E(D)$. In particular, if $\boldsymbol{x}_1^*$ and $\boldsymbol{x}_2^*$ are in L, then there exists polygonal line in $L \bigcap E(D)$, which connects $\boldsymbol{x}_1^*$ and $\boldsymbol{x}_2^*$. The set $L \bigcap E(D)$ is called *the frame* of $E(D)$. Note that the frame of $E(D)$ is the union of all one-dimensional elementary convex sets in $E(D)$. From Theorem 3, if the frame of $E(D)$ is determined, then $E(D)$ can be constructed. Thus, we give the Frame Generating Algorithm to find the frame of $E(D)$ in the following.

In the Frame Generating Algorithm, finding adjacent intersection points to an intersection point and checking that $\Gamma(\boldsymbol{x}_0)$ for $\boldsymbol{x}_0 \notin D$ is suitably contained in a halfplane or not are needed. First, adjacent intersection points to an intersection point can be found efficiently by using the method given in [7]. Next, we shall state how to check that $\Gamma(\boldsymbol{x}_0)$ for $\boldsymbol{x}_0 \notin D$ is suitably contained in a halfplane or not. For each $i \in M$, there exists $j_i \in \{1, 2, \cdots, 2r\}$ such that $\boldsymbol{x}_0 \in \{\boldsymbol{d}_i\} + \text{int}\,(K_{j_i})$ or $\boldsymbol{x}_0 \in \{\boldsymbol{d}_i\} + L_{j_i}$. From (3), we have

$$\partial\gamma(\boldsymbol{x}_0 - \boldsymbol{d}_i) = \begin{cases} \{\boldsymbol{p}_{j_i}\} & \text{if } \boldsymbol{x}_0 \in \{\boldsymbol{d}_i\} + \text{int}(K_{j_i}), \\ [\boldsymbol{p}_{j_i - 1}, \boldsymbol{p}_{j_i}] & \text{if } \boldsymbol{x}_0 \in \{\boldsymbol{d}_i\} + L_{j_i}. \end{cases}$$

For each $i \in M$, we put $\boldsymbol{q}_i^1 \equiv \boldsymbol{p}_{j_i}$, $\boldsymbol{q}_i^2 \equiv \boldsymbol{p}_{j_i}$, $\beta_i^1 \equiv \alpha_{j_i}$ and $\beta_i^2 \equiv \alpha_{j_i}$ if $\partial\gamma(\boldsymbol{x}_0 - \boldsymbol{d}_i) = \{\boldsymbol{p}_{j_i}\}$ and put $\boldsymbol{q}_i^1 \equiv \boldsymbol{p}_{j_i-1}$, $\boldsymbol{q}_i^2 \equiv \boldsymbol{p}_{j_i}$, $\beta_i^1 \equiv \alpha_{j_i-1}$ and $\beta_i^2 \equiv \alpha_{j_i}$ if $\partial\gamma(\boldsymbol{x}_0 - \boldsymbol{d}_i) = [\boldsymbol{p}_{j_i-1}, \boldsymbol{p}_{j_i}]$. Without loss of generality, we assume that $\beta_1^1 \leq \beta_2^1 \leq \cdots \leq \beta_m^1$ and that, for each $i \in \{1, 2, \cdots, m-1\}$, $\beta_i^2 \leq \beta_{i+1}^2$ if $\beta_i^1 = \beta_{i+1}^1$. For each $i \in M$ and each $n \in \mathbb{Z}$ and each $j \in \{1, 2\}$, we put $\boldsymbol{q}_{nm+i}^j \equiv \boldsymbol{q}_i^j$, $\beta_{nm+i}^j \equiv 2n\pi + \beta_i^j$. Then we see that $\Gamma(\boldsymbol{x}_0)$ is suitably contained in a halfspace if and only if one of the following conditions is satisfied.

(i) $\beta_{m+k-1}^2 - \beta_k^1 < \pi$ for some $k \in M$.

(ii) $\beta_{m+k-1}^2 - \beta_k^1 = \pi$ for some $k \in M$, and there exists $\ell \in M$ such that $\langle \boldsymbol{a}, \boldsymbol{x} \rangle < 0$ for any $\boldsymbol{x} \in \partial\gamma(\boldsymbol{x}_0 - \boldsymbol{d}_\ell)$, where $\boldsymbol{a} = -(\cos\frac{\beta_k^1 + \beta_{m+k-1}^2}{2}, \sin\frac{\beta_k^1 + \beta_{m+k-1}^2}{2})^T$.

When $\beta_{m+k-1}^2 - \beta_k^1 = \pi$ for some $k \in M$, if for $\boldsymbol{a} \neq \mathbf{0}$, $\langle \boldsymbol{a}, \boldsymbol{x} \rangle \leq 0$ for any $\boldsymbol{x} \in \bigcup_{i=1}^m \partial\gamma(\boldsymbol{x}_0 - \boldsymbol{d}_i)$, then $\boldsymbol{a} = -\mu(\cos\frac{\beta_k^1 + \beta_{m+k-1}^2}{2}, \sin\frac{\beta_k^1 + \beta_{m+k-1}^2}{2})^T$ for some $\mu > 0$. For such $\boldsymbol{a}$ and each $\ell \in M$, we see that $\langle \boldsymbol{a}, \boldsymbol{x} \rangle < 0$ for any $\boldsymbol{x} \in \partial\gamma(\boldsymbol{x}_0 - \boldsymbol{d}_\ell)$ if and only if $\langle \boldsymbol{a}, \boldsymbol{q}_\ell^1 \rangle < 0$ and $\langle \boldsymbol{a}, \boldsymbol{q}_\ell^2 \rangle < 0$. Now, it can be checked that $\Gamma(\boldsymbol{x}_0)$ is suitably contained in a halfspace or not, i.e. one of the above conditions is satisfied or not. From Theorem 1, it can be checked that $\boldsymbol{x}_0$ is an efficient solution of (P) or not by checking that $\Gamma(\boldsymbol{x}_0)$ is suitably contained in a halfspace or not.

Remark. In view of the fact that the frame of $E(D)$ is the union of all one-dimensional elementary convex sets in $E(D)$, which is connected, we can construct a connected graph $(I \bigcap E(D), E)$, where E is the set of arcs in the graph. Given $\boldsymbol{x}_1, \boldsymbol{x}_2 \in I \bigcap E(D)$, the arc $a(\boldsymbol{x}_1, \boldsymbol{x}_2)$ which connects $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ is in E if and only if $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ are mutually adjacent and $[\boldsymbol{x}_1, \boldsymbol{x}_2] \subset E(D)$. This concept will be guide for describing an algorithm to locate the frame of $E(D)$. It can be checked that $[\boldsymbol{x}_1, \boldsymbol{x}_2]$ is contained in $E(D)$ or not by checking $\Gamma(\boldsymbol{x}_0)$ for any one point $\boldsymbol{x}_0 \in \text{ri}\,([\boldsymbol{x}_1, \boldsymbol{x}_2])$ is suitably contained in a halfspace or not. If

$\Gamma(\boldsymbol{x}_0)$ is suitably contained in a halfspace, then $[\boldsymbol{x}_1, \boldsymbol{x}_2]$ is not contained in $E(D)$ from Theorem 1. If $\Gamma(\boldsymbol{x}_0)$ is not suitably contained in a halfspace, then $[\boldsymbol{x}_1, \boldsymbol{x}_2]$ is contained in $E(D)$ from Theorem 1 and Corollary 1.

The Frame Generating Algorithm finds one-dimensional elementary convex sets in the frame of $E(D)$, which are connected with some demand point. The set V is the set of checked intersection points which are connected with some demand point. The set $S \subset V$ is the set of intersection points which have been checked that one-dimensional elementary convex sets connected with them are contained in $E(D)$ or not. The set T is the union of one-dimensional elementary convex sets in $E(D)$ which have been checked before.

The Frame Generating Algorithm

Step 1. Set $V = D$, $S = \emptyset$ and $T = \emptyset$.

Step 2. If $V = S$, then stop. (The set T is the frame of $E(D)$.) Otherwise, choose any $\boldsymbol{x}_0 \in V \setminus S$ and set $S = S \bigcup \{\boldsymbol{x}_0\}$.

Step 3. Set W be the set of all adjacent intersection points to $\boldsymbol{x}_0$.

Step 4. If $W = \emptyset$, then go to Step 2, otherwise choose any $\boldsymbol{y}_0 \in W$.

Step 5. If $[\boldsymbol{x}_0, \boldsymbol{y}_0] \subset T$, then go to step 4. Otherwise, check $\Gamma(\boldsymbol{z}_0)$ for any one point $\boldsymbol{z}_0 \in$ ri $([\boldsymbol{x}_0, \boldsymbol{y}_0])$ is suitably contained in a halfspace or not. If $\Gamma(\boldsymbol{z}_0)$ is not suitably contained in a halfspace, then set $T = T \bigcup [\boldsymbol{x}_0, \boldsymbol{y}_0]$, and if $\boldsymbol{y}_0 \notin V$, then set $V = V \bigcup \{\boldsymbol{y}_0\}$. Go to Step 4.

In the Frame Generating Algorithm, the number of iterations is $O(m^2)$ since the number of intersection points is $O(m^2)$. In Step 3, determining all adjacent intersection points to $\boldsymbol{x}_0$ requires $O(1)$ computational time, assuming that $\{\boldsymbol{d}_i\} + L_j \bigcup L_{r+j}$, $i \in M$ for each $j \in \{1, 2, \cdots, r\}$ have been sorted according to their x-intercept or y-intercept, which requires $O(m \log m)$ computational time (see [7]). The number of intersection points adjacent to $\boldsymbol{x}_0$ is at most $2r$. In Step 5, checking that $\Gamma(\boldsymbol{z}_0)$ is suitably contained in a halfspace or not requires $O(m)$ computational time. Therefore, the Frame Generating Algorithm requires $O(m^3)$ computational time.

Finally, we consider an example problem for $\boldsymbol{d}_1 = (3\ ,4)^T$, $\boldsymbol{d}_2 = (7, 4)^T$, $\boldsymbol{d}_3 = (6, 7)^T$, $\boldsymbol{d}_4 = (8, 9)^T$ and $\boldsymbol{d}_5 = (13, 6)^T$, where $B = \mathrm{co}\,(\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_4, \boldsymbol{e}_5, \boldsymbol{e}_6\})$ and $\boldsymbol{e}_1 = (2, 0)^T$, $\boldsymbol{e}_2 = (\frac{4}{3}, \frac{2}{3})^T$, $\boldsymbol{e}_3 = (-\frac{1}{3}, \frac{2}{3})^T$, $\boldsymbol{e}_4 = (-1, 0)^T$, $\boldsymbol{e}_5 = (-\frac{4}{3}, \frac{2}{3})^T$, $\boldsymbol{e}_6 = (\frac{2}{3}, -\frac{4}{3})^T$ (see Figure 1). Applying the Frame Generating Algorithm for the multicriteria location problem (P), we have the frame of $E(D)$ illustrated in Figure 3.

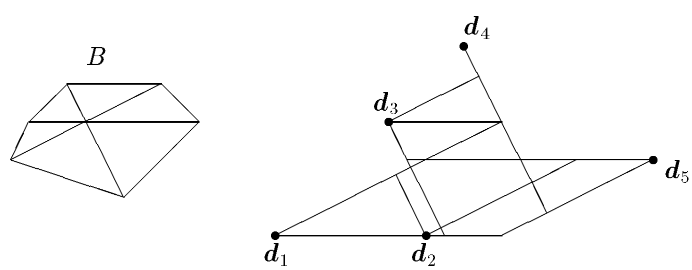

Figure 3. The frame of $E(D)$. (•: demand points)

6. Conclusions. We delt with a multicriteria location problem with the polyhedral gauge in $\mathbb{R}^2$. Our main interest was to find $E(D)$. First, we obtained characterizations of efficient solutions of (P) as Theorem 3-6 by using the concept of elementary convex sets. Next, we proposed the Frame Generating Algorithm to find the frame of $E(D)$. The Frame Generating Algorithm generates the frame of $E(D)$ by tracing one-dimensional elementary convex sets in $E(D)$. Furthermore, we gave the procedure for checking that a given point is an efficient solution of (P) or not.

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