Fuzzy congruence on BCI-algebras

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ABSTRACT. In this paper we define fuzzy congruences on BCI-algebras and their quotient algebras, and prove some fundamental results :

1. There is a one to one correspondence between the set $FC(X)$ of all fuzzy closed ideals of X and the set $FCon_R(X)$ of all fuzzy regular congruences on X.
2. Let X, Y be BCI-algebras and $f : X \to Y$ be a BCI-homomorphism. If $\bar{A}$ is a fuzzy ideal of Y, then the quotient algebras $X/f^{-1}(\bar{A})$ and $f(X)/\bar{A}$ are BCI-algebras and $X/f^{-1}(\bar{A}) \cong f(X)/\bar{A}$

1 Introduction While there are many papers about fuzzy BCK/BCI-algebras and fuzzy ideals of those, we find few papers about *fuzzy congruences*. In the usual theory of crisp BCK/BCI-algebras, there exists a close relationship between ideals and congruences. It is a natural question to extend the relationship to the case of fuzzy BCK/BCI-algebras. In this paper we define fuzzy congruences on BCI-algebras and quotient fuzzy BCI-algebras by those and investigate their properties.

2 Preliminaries By a BCI-algebra we mean an algebraic structure $(X, *, 0)$ of type (2,0) satisfying the following conditions : For all $x, y, z \in X$,

1. $((x * y) * (x * z)) * (z * y) = 0$
2. $(x * (x * y)) * y = 0$
3. $x * x = 0$
4. $x * y = y * x = 0$ implies $x = y$

We define a relation "$\leq$" on X by $x \leq y$ if and only if $x * y = 0$. It is clear from definition that $\leq$ is a partial order on X. If a BCI-algebra X satisfies the extra condition $0 * x = 0$ for all $x \in X$, then it is called a BCK-algebra. In any BCI-algebra X, we have :

$(P1)$ $x * 0 = x$

$(P2)$ $x * y \leq x$

$(P3)$ $(x * y) * z = (x * z) * y$

$(P4)$ $(x * z) * (y * z) \leq x * y$

$(P5)$ $x \leq y$ implies $x * z \leq y * z$ and $z * y \leq z * x$

A non-empty subset A of a BCI-algebra X is said to be an *ideal* of X if

$(I1)$ $0 \in A$

$(I2)$ $x * y \in A$ and $y \in A$ imply $x \in A$.

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Moreover an ideal A is called *closed* if $x \in A$ implies $0 * x \in A$.

We denote by $C(X)$ the set of all closed ideals of X.

A binary relation θ on X is called a *congruence* on X if

$(C1)$ θ is an equivalence relation on X

$(C2)$ $(x, y) \in \theta$ implies $(x * z, y * z) \in \theta$ and $(z * x, z * y) \in \theta$ for all $x, y, z \in X$

Also a relation θ is called *regular* if

$$(R)\ (x * y, 0) \in \theta \text{ and } (y * x, 0) \in \theta \text{ imply } (x, y) \in \theta$$

By $Con_R(X)$ we mean the set of all regular congruences on X. We have the following result ([1, 2]) :

Proposition 1. *Let X be a BCI-algebra. Then $C(X)$ and $Con_R(X)$ are lattices with respect to set inclusion and they are isomorphic as lattices, that is, $C(X) \cong Con_R(X)$.*

Let X be a BCI-algebra. By a *fuzzy set* of X we mean a mapping from X to $[0, 1]$. A fuzzy set $\bar{A}$ of X (i.e. $\bar{A} : X \to [0, 1]$) is called a *fuzzy ideal* if, for all $x, y, z \in X$

(i) $\bar{A}(0) \geq \bar{A}(x)$

(ii) $\bar{A}(x) \geq \bar{A}(x * y) \wedge \bar{A}(y) (= min\{\bar{A}(x * y), \bar{A}(y)\})$

A fuzzy ideal $\bar{A}$ of X is called *closed* if $\bar{A}(0 * x) \geq \bar{A}(x)$ for every $x \in X$. It is easy to show the next result. So we omit the proof.

Lemma 1. *Let $\bar{A}$ be a fuzzy ideal of X. Then*
(1) If $x \leq y$ then $\bar{A}(x) \geq \bar{A}(y)$
(2) $\bar{A}(x * z) \geq \bar{A}(x * y) \wedge \bar{A}(y * z)$

We define a fuzzy congruence on a BCI-algebra X. A binary function θ from $X \times X$ to $[0, 1]$ is called a *fuzzy congruence* on X if it satisfies the conditions : For all $x, y, z \in X$,

1. $\bar{\theta}(0, 0) = \bar{\theta}(x, x)$
2. $\bar{\theta}(x, y) = \bar{\theta}(y, x)$
3. $\bar{\theta}(x, z) \geq \bar{\theta}(x, y) \wedge \bar{\theta}(y, z)$
4. $\bar{\theta}(x * u, y * u), \bar{\theta}(u * x, u * y) \geq \bar{\theta}(x, y)$

Lemma 2. *If $\bar{\theta}$ satisfies the conditions (2),(3), and (4) above, then (1) $\bar{\theta}(0, 0) = \bar{\theta}(x, x)$ if and only if (1)' $\bar{\theta}(0, 0) \geq \bar{\theta}(x, y)$, for all $x, y \in X$,*

Proof. Suppose that $\bar{\theta}(0, 0) = \bar{\theta}(x, x)$. Since $\bar{\theta}$ satisfies the conditions (2) and (3), we have $\bar{\theta}(0, 0) = \bar{\theta}(x, x) \geq \bar{\theta}(x, y) \wedge \bar{\theta}(y, x) = \bar{\theta}(x, y)$.

Conversely, it is sufficient to prove $\bar{\theta}(0, 0) \leq \bar{\theta}(x, x)$. From (4), we have $\bar{\theta}(0, 0) \leq \bar{\theta}(x * 0, x * 0) = \bar{\theta}(x, x)$. □

Theorem 1. *If $\bar{A}$ is a fuzzy ideal of X, then the fuzzy relation $\bar{\theta_A}(x, y)$ defined by $\bar{\theta_A}(x, y) = \bar{A}(x * y) \wedge \bar{A}(y * x)$ is a fuzzy congruence.*

Proof. We only show that $\bar{\theta_A}$ satisfies the conditions (3) and (4). For the case of (3), we have

$$\begin{aligned}\bar{\theta_A}(x,z) = \bar{A}(x*z)\wedge\bar{A}(z*x) &\geq \bar{A}(x*y)\wedge\bar{A}(y*z)\wedge\bar{A}(z*y)\wedge\bar{A}(y*x)\\ &= (\bar{A}(x*y)\wedge\bar{A}(y*x))\wedge(\bar{A}(y*z)\wedge\bar{A}(z*y))\\ &= \bar{\theta_A}(x,y)\wedge\bar{\theta_A}(y,z)\end{aligned}$$

For the case of (4), it follows from lemma 1 that

$$\begin{aligned}\bar{\theta}(x*u,y*u) &= \bar{A}((x*u)*(y*u))\wedge\bar{A}((y*u)*(x*u))\\ &\geq \bar{A}(x*y)\wedge\bar{A}(y*x)\\ &= \bar{\theta_A}(x,y)\end{aligned}$$

It is similar the case of $\bar{\theta_A}(u*x,u*y)\geq\bar{\theta_A}(x,y)$. □

Conversely,

Theorem 2. *If $\bar{\theta}$ is a fuzzy congruence, then the function $\bar{A}_\theta$ from X to [0,1] defined by $\bar{A}_\theta(x)=\bar{\theta}(x,0)$ is a fuzzy ideal of X.*

Proof. By lemma 2, $\bar{A}_\theta(0)=\bar{\theta}(0,0)\geq\bar{\theta}(x,0)=\bar{A}_\theta(x)$ and $\bar{A}_\theta(x)=\bar{\theta}(x,0)\geq\bar{\theta}(x,x*y)\wedge\bar{\theta}(x*y,0)\geq\bar{\theta}(0,y)\wedge\bar{\theta}(x*y,0)=\bar{A}_\theta(y)\wedge\bar{A}_\theta(x*y)$

Hence $\bar{A}_\theta$ is the fuzzy ideal of X. □

In general, for every fuzzy ideal $\bar{A}$ of X, we have $A_{\bar{\theta_A}}(x)=\bar{\theta_A}(x,0)=\bar{A}(x*0)\wedge\bar{A}(0*x)=\bar{A}(x)\wedge\bar{A}(0*x)\leq\bar{A}(x)$

In particular if X is a BCK-algebra then we have $A_{\bar{\theta_A}}=\bar{A}$ for every fuzzy BCK-ideal $\bar{A}$ of X.

Lemma 3. *If $\bar{A}$ is a fuzzy closed ideal, then we have $\bar{\theta_A}(x*y,0)\wedge\bar{\theta_A}(y*x,0)=\bar{\theta_A}(x,y)$, that is, $\bar{\theta_A}$ is a fuzzy regular congruence.*

Proof. Since $\bar{A}$ is closed, it follows that $\bar{\theta_A}(x*y,0)=\bar{A}(x*y)\wedge\bar{A}(0*(x*y))=\bar{A}(x*y)$ and similarly $\bar{\theta_A}(y*x,0)=\bar{A}(y*x)$. Hence $\bar{\theta_A}(x*y,0)\wedge\bar{\theta_A}(y*x,0)=\bar{A}(x*y)\wedge\bar{A}(y*x)=\bar{\theta_A}(x,y)$. This means that if $\bar{A}\in FC(X)$ then $\bar{\theta_A}\in FCon_R(X)$. □

Conversely we have

Lemma 4. *If $\bar{\theta}$ is a fuzzy regular congruence, then $\bar{A}_\theta$ is a fuzzy closed ideal.*

Proof. It follows from definition that

$$\begin{aligned}\bar{A}_\theta(0*x) &= \bar{\theta}(0*x,0)\\ &= \bar{\theta}(0*x,x*x)\\ &\geq \bar{\theta}(0,x)=\bar{\theta}(x,0)=\bar{A}_\theta(x)\end{aligned}$$

Thus $\bar{A}_\theta$ is closed. □

From the above we can conclude that

(1) For any fuzzy closed ideal $\bar{A}$ of X, $\bar{A}=A_{\bar{\theta_A}}$.

(2) For any fuzzy regular congruence $\bar{\theta}$ of X, $\bar{\theta}=\bar{\theta_A}$.

Because, for the case of (1), we have $\bar{A}_{\theta_A}(x) = \bar{\theta_A}(x,0) = \bar{A}(x*0) \wedge \bar{A}(0*x) = \bar{A}(x) \wedge \bar{A}(0*x) = \bar{A}(x)$, and for the case of (2), since $\bar{\theta}$ is regular, $\theta_{\bar{A}_\theta}(x,y) = \bar{A}_\theta(x*y) \wedge \bar{A}_\theta(y*x) = \bar{\theta}(x*y,0) \wedge \bar{\theta}(y*x,0) = \bar{\theta}(x,y)$.

Thus we get one of main theorems of the paper.

Theorem 3. *Let X be a BCI-algebra. Then we have $FC(X) \cong FCon_R(X)$*

Proof. We define a map ξ from $FC(X)$ to $FCon_R(X)$ by $\xi(\bar{A}) = \bar{\theta_A}$ for any fuzzy closed ideal $\bar{A}$ of X. It is clear from the above that ξ is an isomorphism. We note that $FC(X)$ and $FCon_R(X)$ are lattices with set inclusion orders, respectively. □

We can also show the next theorem, which is so-called the transfer principle ([3]).

Theorem 4. *If $\bar{\theta}$ is a fuzzy relation on X, then $\bar{\theta}$ is a fuzzy congruence if and only if for all $\alpha \in [0,1]$ if $U(\bar{\theta}:\alpha) \neq \emptyset$ then $U(\bar{\theta}:\alpha)$ is a congruence on X, where $U(\bar{\theta}:\alpha) = \{(x,y) \in X \times X \mid \bar{\theta}(x,y) \geq \alpha\}$*

Proof. ($\Longrightarrow$) Suppose that $\bar{\theta}$ is a fuzzy congruence on X. Take any $\alpha \in [0,1]$ such that $U(\bar{\theta}:\alpha)$ is not empty. It is sufficient to show that $U(\bar{\theta}:\alpha)$ is a congruence on X. Since $U(\bar{\theta}:\alpha)$ is not empty, there is an element $(u,v) \in X \times X$ such that $(u,v) \in U(\bar{\theta}:\alpha)$. This means that $\alpha \leq \bar{\theta}(u,v)$. Since $\bar{\theta}$ is the congruence, we have $\alpha \leq \bar{\theta}(u,v) \leq \bar{\theta}(0,0) = \bar{\theta}(x,x)$. That is, $(x,x) \in U(\bar{\theta}:\alpha)$.

Suppose that $(x,y),(y,z) \in U(\bar{\theta}:\alpha)$. Since $\alpha \leq \bar{\theta}(x,y), \bar{\theta}(y,z)$, we have $\alpha \leq \bar{\theta}(x,y) \wedge \bar{\theta}(y,z) \leq \bar{\theta}(x,z)$. Hence $(x,z) \in U(\bar{\theta}:\alpha)$.

At last we assume that $(x,y) \in U(\bar{\theta}:\alpha)$. Since $\alpha \leq \bar{\theta}(x,y) \leq \bar{\theta}(x*u,y*u), \bar{\theta}(u*x,u*y)$, we have $(x*u,y*u),(u*x,u*y) \in U(\bar{\theta}:\alpha)$.

Hence from the above we can conclude that $U(\bar{\theta}:\alpha)$ is the congruence on X if it is not empty.

($\Longleftarrow$) Conversely, suppose that for all $\alpha \in [0,1]$ if $U(\bar{\theta}:\alpha) \neq \emptyset$ then $U(\bar{\theta}:\alpha)$ is a congruence on X. We only show that $\bar{\theta}(x,z) \geq \bar{\theta}(x,y) \wedge \bar{\theta}(y,z)$. Take any $\alpha \in [0,1]$ such that $U(\bar{\theta}:\alpha)$ is not empty. Since the relation $U(\bar{\theta}:\alpha)$ is transitive, if $(x,y),(y,z) \in U(\bar{\theta}:\alpha)$ then $(x,z) \in U(\bar{\theta}:\alpha)$. This means that if $\bar{\theta}(x,y), \bar{\theta}(y,z) \geq \alpha$ then $\bar{\theta}(x,z) \geq \alpha$ for any α. Hence we have $\bar{\theta}(x,z) \geq \bar{\theta}(x,y) \wedge \bar{\theta}(y,z)$.

The other cases can be proved similarly. □

Now we will define a quotient algebra by a fuzzy ideal. Let X be a BCI-algebra and $\bar{A}$ be a fuzzy ideal of X. For any element $x,y \in X$, we define $x \sim_{\bar{A}} y$ by

$$\bar{A}(x*y) = \bar{A}(y*x) = \bar{A}(0),$$

that is, $\theta_{\bar{A}}(x,y) = \bar{A}(x)$. Then it is clear that

Lemma 5. *$\sim_{\bar{A}}$ is a congruence relation on X.*

We define $X/\bar{A} = \{x/\bar{A} | x \in X\}$ and $x/\bar{A} = \{y \in X | x \sim_{\bar{A}} y\}$. We note that these sets are not fuzzy sets but crisp ones. By a fuzzy congruent BCI-algebra induced by a fuzzy ideal $\bar{A}$, we mean a map ξ from $X/\bar{A}$ to [0,1] which is defined by $\xi(x/\bar{A}) = \bar{A}(x)$. It is obvious that the map ξ is well-defined. Now we consider the property of a crisp set $X/\bar{A}$. For any element $x/\bar{A}, y/\bar{A} \in X/\bar{A}$, we define $x/\bar{A} * y/\bar{A} = (x*y)/\bar{A}$. It is easy to show

Theorem 5. *For any BCI-algebra X and fuzzy ideal $\bar{A}$ of X, $X/\bar{A}$ is a BCI-algebra.*

Proof. We only show that $X/\bar{A}$ satisfies the condition (4) : $x/\bar{A} * y/\bar{A} = y/\bar{A} * x/\bar{A} = 0/\bar{A}$ implies $x/\bar{A} = y/\bar{A}$. Suppose that $x/\bar{A} * y/\bar{A} = y/\bar{A} * x/\bar{A} = 0/\bar{A}$. Since $x*y \sim_{\bar{A}} y*x \sim_{\bar{A}} 0$, it follows from definition that $\bar{A}(x*y) = \bar{A}(y*x) = \bar{A}(0)$ and hence $x \sim_{\bar{A}} y$, This means that $x/\bar{A} = y/\bar{A}$. □

We have some applications. A BCK-algebra X is called *commutative* when it satisfies $x*(x*y)=y*(y*x)$ for all $x,y\in X$. It is well-known that the condition is equivalent to the following : $x*y=0$ implies $x*(y*(y*x))=0$. For a fuzzy ideal $\bar{A}$ of a BCK-algebra X is called *fuzzy commutative* if it satisfies the condition $\bar{A}(x*(y*(y*x)))\geq\bar{A}(x*y)$ for all $x,y\in X$. In this case we have the following.

Theorem 6. *Let $\bar{A}$ be a fuzzy ideal of a BCK-algebra X. Then we have $\bar{A}$: fuzzy commutative ideal $\Longleftrightarrow X/\bar{A}$: commutative BCK-algebra.*

Proof. ($\Longrightarrow$) It is sufficient to prove that $x/\bar{A}*y/\bar{A}=0/\bar{A}$ implies $x/\bar{A}*(y/\bar{A}*(y/\bar{A}*x/\bar{A}))=0/\bar{A}$, that is, $x*y\sim 0$ implies $x*(y*(y*x))\sim 0$. Suppose that $x*y\sim 0$. It follows from definition that $\bar{A}(x*y)=\bar{A}(0)$. Since $\bar{A}$ is commutative, we have $\bar{A}(0)=\bar{A}(x*y)\leq\bar{A}(x*(y*(y*x)))$ and hence $\bar{A}(x*(y*(y*x)))=\bar{A}(0)$.

On the other hand, since X is the BCK-algebra, it follows that $\bar{A}(0*(x*(y*(y*x))))=\bar{A}(0)$. Hence we get that $x*(y*(y*x))\sim 0$.

($\Longleftarrow$) Suppose that $X/\bar{A}$ is a commutative BCK-algebra. Since $\bar{A}$ is a fuzzy ideal, we have $\bar{A}(x*(y*(y*x)))\geq\bar{A}((x*(y*(y*x)))*(x*y))\wedge\bar{A}(x*y)=\bar{A}((x*(x*y))*(y*(y*x)))\wedge\bar{A}(x*y)$. That $X/\bar{A}$ is the commutative BCK-algebra implies $x/\bar{A}*(x/\bar{A}*y/\bar{A})=y/\bar{A}*(y/\bar{A}*x/\bar{A})$, hence $x*(x*y)\sim y*(y*x)$. This means that $\bar{A}((x*(x*y))*(y*(y*x)))=\bar{A}(0)$. From the above we get $\bar{A}(x*(y*(y*x)))\geq\bar{A}(0)\wedge\bar{A}(x*y)=\bar{A}(x*y)$. Thus $\bar{A}$ is the fuzzy commutative ideal. □

For the other cases, we can show the similar result. For example, we can show the following for the positive implicative BCK-algebra. A BCK-algebra X is called *positive implicative* if $(x*y)*y=0$ implies $x*y=0$ for all $x,y\in X$. For a fuzzy ideal $\bar{A}$ of X, $\bar{A}$ is said to be *fuzzy positive implicative* if $\bar{A}(x*y)\geq\bar{A}((x*y)*y)$ for all $x,y\in X$. In this case, we can show the next. The proof is clear, so we omit it.

Theorem 7. *For any BCK-algebra X and a fuzzy ideal $\bar{A}$ of X, $X/\bar{A}$ is a positive implicative BCK-algebra if and only if $\bar{A}$ is a fuzzy positive implicative ideal of X.*

These results are extentions of the following results respectively : For any BCK-algebra X and ideal A of X,

(1) X/A : commutative BCK-algebra $\Longleftrightarrow A$: commutative ideal

(2) X/A : positive implicative BCK-algebra $\Longleftrightarrow A$: positive implicatice ideal

Let X,Y be BCI-algebras and f be a BCI-homomorphism, that is, a map satisfying $f(x*y)=f(x)*f(y)$ for all $x,y\in X$. If $\bar{B}$ is a fuzzy ideal of Y, then the map $f^{-1}(\bar{B})$ defined by $f^{-1}(\bar{B})(x)=\bar{B}(f(x))$ for all $x\in X$ is a fuzzy ideal of X ([4]).

In this case we can show the following result which is an extension of *homomorphism theorem.*

Theorem 8. *Let X,Y be BCI-algebras, f a BCI-homomorphism, and $\bar{B}$ a fuzzy ideal of Y. Then there is a bijective BCI-homomorphism from $X/f^{-1}(\bar{B})$ onto $f(X)/\bar{B}$, that is, $X/f^{-1}(\bar{B})\cong f(X)/\bar{B}$.*

Proof. We define a map h from $X/f^{-1}(\bar{B})$ to $f(X)/\bar{B}$ by $h(x/f^{-1}(\bar{B}))=f(x)/\bar{B}$ for all $x\in X$. The map h is well-defined. Because, if $x/f^{-1}(\bar{B})=y/f^{-1}(\bar{B})$, since $x\sim_{f^{-1}(\bar{B})}y$, then we have $f^{-1}(\bar{B})(x*y)=f^{-1}(\bar{B})(y*x)=f^{-1}(\bar{B})(0)$ and hence $\bar{B}(f(x)*f(y))=\bar{B}(f(y)*f(x))=\bar{B}(f(0))=\bar{B}(0')$ by definition of $f^{-1}(\bar{B})$. This means that $f(x)\sim_{\bar{B}}f(y)$, that is, $f(x)/\bar{B}=f(y)/\bar{B}$. Hence h is well-defined.

For injectiveness of h, we suppose that $h(x/f^{-1}(\bar{B})) = h(y/f^{-1}(\bar{B}))$, that is, $f(x)/\bar{B} = f(y)/\bar{B}$. Since $f(x) \sim_{\bar{B}} f(y)$, we have $\bar{B}(f(x) * f(y)) = \bar{B}(f(y) * f(x)) = \bar{B}(0')$. It follows from definition that $f^{-1}(\bar{B})(x * y) = f^{-1}(\bar{B})(y * x) = f^{-1}(\bar{B})(0)$ and hence that $x/f^{-1}(\bar{B}) = y/f^{-1}(\bar{B})$.

It is easy to show that h is a surjective BCI-homomorphism.

Thus we can conclude that $X/f^{-1}(\bar{B}) \cong f(X)/\bar{B}$. □

In particular, if f is surjective then we have $X/f^{-1}(\bar{B}) \cong Y/\bar{B}$.

From the above we can prove that two quotient algebras $X/f^{-1}(\bar{B})$ and $f(X)/\bar{B}$ are isomorphic as fuzzy quotient algebras, that is,

Theorem 9. *For two fuzzy quotient algebras ξ and η which are defined by*
$\xi : X/f^{-1}(\bar{B}) \to [0,1]$, $\xi(x/f^{-1}(\bar{B})) = f^{-1}(\bar{B})(x)$
$\eta : f(X)/\bar{B} \to [0,1]$, $\eta(f(x)/\bar{B}) = \bar{B}(f(x))$, *respectively,*
there exists a bijective map h from $X/f^{-1}(\bar{B})$ to $f(X)/\bar{B}$ such that $\eta \circ h = \xi$.

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