APPROXIMATION OF FIXED POINTS OF NONEXPANSIVE NONSELF-MAPPINGS

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Abstract. Our purpose is to show two strong convergence theorems for nonexpansive nonself-mappings in a Hilbert space; these are generalizations of Wittmann's result[7], and are proved without any boundary conditions. For this purpose, a boundary condition, called *nowhere normal-outward* condition, is investigated and characterized.

1 Introduction Let H be a Hilbert space, let C be a nonempty closed convex subset of H, and let T be a nonexpansive nonself-mapping ¿from C into H such that the set $F(T)$ of all fixed points of T is nonempty. In 1992, Marino and Trombetta[2] defined two contraction mappings S_t and U_t as follows: For a given $u \in C$ and each $t \in (0,1)$,

$$S_t x = tPTx + (1-t)u \quad \text{for all} \quad x \in C \tag{1.1}$$

and

$$U_t x = P(tTx + (1-t)u) \quad \text{for all} \quad x \in C, \tag{1.2}$$

where P is the metric projection from H onto C. Then by the Banach contraction principle, there exists a unique element $x_t \in F(S_t)$(resp. $y_t \in F(U_t)$), i.e.

$$x_t = tPTx_t + (1-t)u \tag{1.3}$$

and

$$y_t = P(tTy_t + (1-t)u). \tag{1.4}$$

Recently, Xu and Yin[8] proved that if T is a nonexpansive nonself-mapping from C into H satisfying the weak inwardness condition, then $\{x_t\}$ (resp. $\{y_t\}$) defined by (1.3) (resp. (1.4)) converges strongly as $t \to 1$ to an element of $F(T)$ which is nearest to u in $F(T)$. This result was extended to a Banach space by Takahashi and Kim[6]. On the other hand, Wittmann[7] proved the following strong convergence theorem; see also [4]:

Theorem (Wittmann 1992).

Let H be a Hilbert space, let C be a nonempty closed convex subset of H, and let S be a nonexpansive mapping from C into itself. Let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \le \alpha_n \le 1$, $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Define a sequence $\{x_n\}$ as follows: $x_1 = x \in C$ and

$$x_{n+1} = \alpha_n x + (1-\alpha_n)Sx_n \ \textit{for}\ n \ge 1.$$

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If $F(S) \neq \emptyset$, then $\{x_n\}$ converges strongly to $Px \in F(S)$, where P is the metric projection from C onto $F(S)$.

In this paper, we extend the above Wittmann's result to nonexpansive nonself-mappings without any boundary conditions. For this purpose, we consider about a boundary condition in Section 2, which is called *nowhere normal-outward* condition. Also we show two propositions between the boundary condition and $F(T)$ when T is a nonexpansive nonself-mapping; the propositions play important roles in this paper. Finally, we introduce two iteration schemes for T by using the metric projection from H onto C, and show two strong convergence theorems, which are generalizations of the Wittmann's result in Section 3.

2 Preliminaries Throughout this paper, we denote the set of all positive integers by $\mathbf{N}$. Let H be a real Hilbert space with norm $\|\cdot\|$ and with inner product $\langle\cdot,\cdot\rangle$, let C be a closed convex subset of H, and let T be a nonself-mapping from C into H. We denote the set of all fixed points of T by $F(T)$. Then T is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\| \quad \text{for all} \quad x,\, y \in C.$$

For all $x \in H$, there exists a unique element Px of C satisfying

$$\|x - Px\| = \min_{y \in C} \|x - y\| \quad \text{for all} \quad x \in H.$$

This mapping P is said to be the metric projection from H onto C. We know that P is nonexpansive and for all $x \in H$, $z = Px$ if and only if $\langle x - z, y - z\rangle \leq 0$ for all $y \in C$. It is known that H satisfies Opial's condition [3]; see also [5]: if $\{x_n\}$ converges weakly to x, then

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

for all $y \neq x$.

Next, we introduce several boundary conditions upon the nonself-mapping.

(i) **Rothe's condition**: $T(\partial C) \subset C$, where ∂C is the boundary set of C;

(ii) **inwardness condition**[1]: $Tx \in I_c(x)$ for all $x \in C$, where

$$I_c(x) = \{y \in H \mid y = x + a(z - x) \text{ for some } z \in C \text{ and } a \geq 0\};$$

(iii) **weak inwardness condition**[1]: $Tx \in \mathrm{cl} I_c(x)$ for all $x \in C$, where cl denotes the norm-closure; and

(iv) **nowhere normal-outward condition**[1]: $Tx \in S_x^c$ for all $x \in C$, where P is the metric projection from H onto C, and

$$S_x = \{y \in H \mid y \neq x, Py = x\}.$$

It is easily seen that there hold implications: (i)$\Rightarrow$(ii)$\Rightarrow$(iii). It also holds that (iii)$\Rightarrow$(iv); see [1], p.354. To prove our results, we need the following propositions:

Proposition 2.1 *Let H be a Hilbert space, let C be a nonempty closed convex subset of H, let P be the metric projection from H onto C, and let T be a nonself-mapping from C into H satisfying the nowhere normal-outward condition. Then $F(T) = F(PT)$. Moreover, if C is bounded and T is nonexpansive, then T has a fixed point.*

Proof. At first we show $F(T) = F(PT)$. It is sufficient to prove that $F(PT)$ is a subset of $F(T)$. Let $x \in F(PT)$, that is $PTx = x$. Since $Tx \in S_x^c$, we obtain $Tx = x$. Next, suppose that C is bounded and T is nonexpansive. Then PT is a nonexpansive mapping from C into itself. Therefore $F(T) = F(PT) \neq \emptyset$, see [5]. □

Proposition 2.2 *Let H be a Hilbert space, let C be a nonempty closed convex subset of H, let T be a nonexpansive nonself-mapping from C into H. If $F(T) \neq \emptyset$, then T satisfies nowhere normal-outward condition.*

Proof. If there exists $x_0 \in C$ such that $Tx_0 \in S_{x_0}$, then $Tx_0 \neq x_0$ and $PTx_0 = x_0$, where P is the metric projection from H onto C. Let $z \in F(T)$, we have

$$\begin{aligned} \|Tx_0 - z\|^2 &= \|Tx_0 - x_0\|^2 + 2\langle Tx_0 - PTx_0, PTx_0 - z\rangle + \|PTx_0 - z\|^2 \\ &> \|x_0 - z\|^2. \end{aligned}$$

This contradicts that T is nonexpansive. Therefore, $Tx \in S_x^c$ for all $x \in C$.
□

Remark 2.1 By using Proposition 2.1 and Proposition 2.2, we can consider generalizations of fixed point theorems ¿from self-mappings to nonself-mappings. When T is a nonexpansive nonself-mapping, applying the fixed point theorems to self-mapping PT, we have some results with respect to nonself-mapping T. For example, we can show the following, which is a generalization result of Xu and Yin's result, see [8], and also note that it is proved without any boundary conditions:

Let H be a real Hilbert space, let C be a nonempty closed convex subset of H, let P be the metric projection from H onto C, and let T be a nonexpansive nonself-mapping from C into H. Let $\{x_t\}$ and $\{y_t\}$ be the nets defined by (1.3) and (1.4), respectively. If T satisfies nowhere normal-outward condition, then the following three conditions are equivalent:

- $F(T) \neq \emptyset$,
- *$\{x_t\}$ remains bounded as $t \to 1$,*
- *$\{y_t\}$ remains bounded as $t \to 1$.*

Also, if $F(T) \neq \phi$, then $\{x_t\}$ and $\{y_t\}$ converge strongly as $t \to 1$ to some fixed points of T.

In the next section, we can apply the idea to Theorem 3.1. However, we can not apply it to Theorem 3.2 simply; it is more complicated.

3 Main Results In this section, we prove two strong convergence theorems for nonexpansive nonself-mappings, which are generalizations of Wittmann's result[7], and also, which are not required any boundary conditions.

Theorem 3.1 *Let H be a Hilbert space, let C be a nonempty closed convex subset of H, let P_1 be the metric projection from H onto C, and let T be a nonexpansive nonself-mapping from C into H. Let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \leq \alpha_n \leq 1$, $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Define a sequence $\{x_n\}$ as follows: $x_1 = x \in C$ and*

$$x_{n+1} = \alpha_n x + (1 - \alpha_n) P_1 T x_n \text{ for } n \geq 1.$$

If $F(T) \neq \emptyset$, then $\{x_n\}$ converges strongly to $P_2 x \in F(T)$, where P_2 is the metric projection from C onto $F(T)$.

This theorem is proved easily by using Proposition 2.1 and Proposition 2.2, as shown in Remark 2.1.

Proof. Since P_1T is a nonexpansive mapping from C into itself, applying Wittmann's result, we obtain that $\{x_n\}$ converges strongly as $n \to \infty$ to a fixed point z of P_1T nearest to x. Using Proposition 2.1 and Proposition 2.2, we obtain $F(P_1T) = F(T)$. Hence $\{x_n\}$ converges strongly as $n \to \infty$ to a fixed point z of T nearest to x. □

Theorem 3.2 *Let H be a Hilbert space, let C be a nonempty closed convex subset of H, let P_1 be the metric projection from H onto C, and let T be a nonexpansive nonself-mapping from C into H. Let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \le \alpha_n \le 1$, $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Define a sequence $\{y_n\}$ as follows: $y_1 = y \in C$ and*

$$y_{n+1} = P_1(\alpha_n y + (1-\alpha_n)Ty_n) \text{ for } n \ge 1.$$

If $F(T) \neq \emptyset$, then $\{y_n\}$ converges strongly to $P_2 y \in F(T)$, where P_2 is the metric projection from C onto $F(T)$.

Proof. Let $z \in F(T)$. Then we have

$$\begin{aligned} \|y_2 - z\| &= \|P_1(\alpha_1 y + (1-\alpha_1)Ty_1) - P_1 z\| \\ &\le \|\alpha_1 y + (1-\alpha_1)Ty_1 - z\| \\ &\le \alpha_1\|y - z\| + (1-\alpha_1)\|y_1 - z\| \\ &= \|y - z\|. \end{aligned}$$

If $\|y_n - z\| \le \|y - z\|$ for some $n \in \mathbf{N}$, then we can show that $\|y_{n+1} - z\| \le \|y - z\|$ similarly. Therefore, by induction, we obtain $\|y_n - z\| \le \|y - z\|$ for all $n \in \mathbf{N}$ and hence $\{y_n\}$ and $\{Ty_n\}$ are bounded. Set $K = \sup\{\|Ty_n\| : n \in \mathbf{N}\}$. Then

$$\begin{aligned} \|y_{n+1} - y_n\| &= \|P_1(\alpha_n y + (1-\alpha_n)Ty_n) - P_1(\alpha_{n-1} y + (1-\alpha_{n-1})Ty_{n-1})\| \\ &\le \|\alpha_n y + (1-\alpha_n)Ty_n - \{\alpha_{n-1} y + (1-\alpha_{n-1})Ty_{n-1}\}\| \\ &= \|(\alpha_n - \alpha_{n-1})y + (1-\alpha_n)(Ty_n - Ty_{n-1}) + (\alpha_{n-1} - \alpha_n)Ty_{n-1}\| \\ &\le |\alpha_{n-1} - \alpha_n|\|y\| + (1-\alpha_n)\|y_n - y_{n-1}\| + |\alpha_{n-1} - \alpha_n|\|Ty_{n-1}\| \\ &\le |\alpha_{n-1} - \alpha_n|(\|y\| + K) + (1-\alpha_n)\|y_n - y_{n-1}\| \end{aligned}$$

for each $n \in \mathbf{N}$. By induction, we have

$$\|y_{n+m+1} - y_{n+m}\| \le \sum_{k=m}^{n+m-1} |\alpha_{k+1} - \alpha_k|(\|y\| + K) + \prod_{k=m}^{n+m-1} (1-\alpha_{k+1})\|y_{m+1} - y_m\|$$

for all $m, n \in \mathbf{N}$. By $\sum_{n=1}^{\infty} \alpha_n = \infty$, we have $\prod_{n=1}^{\infty}(1-\alpha_n) = 0$; see [4]. Hence we obtain

$$\limsup_{n\to\infty} \|y_{n+1} - y_n\| = \limsup_{n\to\infty} \|y_{n+m+1} - y_{n+m}\| \le \sum_{k=m}^{\infty} |\alpha_{k+1} - \alpha_k|(\|y\| + K)$$

for all $m \in \mathbf{N}$. By $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, we get $\lim_{n\to\infty} \|y_{n+1} - y_n\| = 0$. Also, from

$$\begin{aligned} \|y_n - P_1Ty_n\| &= \|P_1(\alpha_{n-1} y + (1-\alpha_{n-1})Ty_{n-1}) - P_1Ty_n\| \\ &\le \|\alpha_{n-1} y + (1-\alpha_{n-1})Ty_{n-1} - Ty_n\| \\ &\le \alpha_{n-1}\|y - Ty_n\| + (1-\alpha_{n-1})\|y_{n-1} - y_n\|, \end{aligned}$$

we obtain

$$\lim_{n\to\infty}\|y_n - P_1Ty_n\| = 0. \tag{3.5}$$

Next we prove

$$\limsup_{n\to\infty}\langle y_n - P_2y, y - P_2y\rangle \le 0. \tag{3.6}$$

Let $\{y_{n_k}\}$ be a subsequence of $\{y_n\}$ which satisfies

$$\lim_{k\to\infty}\langle y_{n_k} - P_2y, y - P_2y\rangle = \limsup_{n\to\infty}\langle y_n - P_2y, y - P_2y\rangle,$$

and which converges weakly as $k \to \infty$ to $y_0 \in C$. By (3.5) and Opial's condition, we obtain $y_0 \in F(P_1T)$. Applying Proposition 2.1 and Proposition 2.2, we conclude $y_0 \in F(T)$. Then we have

$$\begin{aligned}\limsup_{n\to\infty}\langle y_n - P_2y, y - P_2y\rangle &= \lim_{k\to\infty}\langle y_{n_k} - P_2y, y - P_2y\rangle\\ &= \langle y_0 - P_2y, y - P_2y\rangle \le 0.\end{aligned}$$

By (3.6), for any $\varepsilon > 0$, there exists $m \in \mathbf{N}$ such that

$$\langle y_n - P_2y, y - P_2y\rangle \le \varepsilon \tag{3.7}$$

for all $n \ge m$. On the other hand, from

$$P_1(\alpha_n y + (1-\alpha_n)Ty_n) - P_1(\alpha_n y + (1-\alpha_n)P_2y) = y_{n+1} - P_2y + \alpha_n(P_2y - y),$$

we have

$$\begin{aligned}&\|P_1(\alpha_n y + (1-\alpha_n)Ty_n) - P_1(\alpha_n y + (1-\alpha_n)P_2y)\|^2\\ &\qquad \ge \|y_{n+1} - P_2y\|^2 + 2\alpha_n\langle y_{n+1} - P_2y, P_2y - y\rangle.\end{aligned}$$

This implies

$$\|y_{n+1} - P_2y\|^2 \le (1-\alpha_n)^2\|Ty_n - P_2y\|^2 + 2\alpha_n\langle y_{n+1} - P_2y, y - P_2y\rangle$$

for all $n \in \mathbf{N}$. By (3.7), we have

$$\begin{aligned}\|y_{n+1} - P_2y\|^2 &\le 2\alpha_n\langle y_{n+1} - P_2x, y - P_2x\rangle + (1-\alpha_n)^2\|Ty_n - P_2y\|^2\\ &\le 2\alpha_n\varepsilon + (1-\alpha_n)\|Ty_n - P_2y\|^2 \le 2\alpha_n\varepsilon + (1-\alpha_n)\|y_n - P_2y\|^2\\ &= 2\varepsilon(1-(1-\alpha_n)) + (1-\alpha_n)\|y_n - P_2y\|^2\end{aligned}$$

for all $n \ge m$. This implies

$$\begin{aligned}\|y_{n+1} - P_2y\|^2 &\le 2\varepsilon\{1-(1-\alpha_n)\}\\ &\quad +2\varepsilon(1-\alpha_n)(1-(1-\alpha_{n-1}) + (1-\alpha_{n-1})\|y_{n-1} - P_2y\|^2)\\ &= 2\varepsilon\{1-(1-\alpha_n)(1-\alpha_{n-1})\} + (1-\alpha_n)(1-\alpha_{n-1})\|y_{n-1} - P_2y\|^2\end{aligned}$$

for all $n \ge m$. By induction, we obtain

$$\|y_{n+1} - P_2y\|^2 \le 2\varepsilon\left\{1 - \prod_{k=m}^{n}(1-\alpha_k)\right\} + \prod_{k=m}^{n}(1-\alpha_k)\|y_m - P_2y\|^2.$$

Therefore, from $\sum_{n=1}^{\infty}\alpha_n = \infty$, we obtain

$$\limsup_{n\to\infty}\|y_{n+1} - P_2y\|^2 \le 2\varepsilon.$$

Since ε is arbitrary, we can conclude that $\{y_n\}$ converges strongly to P_2y. □

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References

[1] B. R. Halpern and G. M. Bergman, *A fixed-point theorem for inward and outward maps*, Trans. Amer. Math. Soc., **130**, 353–358 (1968).

[2] G. Marino and G. Trombetta, *On approximating fixed points for nonexpansive maps*, Indian J. Math., **34**, 91–98 (1992).

[3] Z. Opial, *Weak convergence of the sequence of successive approximations for nonexpansive mappings*, Bull. Amer. Math. Soc., **73**, 591–597 (1967).

[4] W. Takahashi, *Convex Analysis and Approximation of Fixed Points (Japanese)*, Yokohama-Publishers, 2000.

[5] W. Takahashi, *Nonlinear Functional Analysis*, Yokohama-Publishers, 2000.

[6] W. Takahashi and G. E. Kim, *Strong convergence of approximants to fixed points of nonexpansive nonself-mappings in Banach spaces*, Nonlinear Analysis, **32**, 447–454 (1998).

[7] R. Wittmann, *Approximation of fixed points of nonexpansive mappings*, Arch. Math., **58**, 486–491 (1992).

[8] H. K. Xu and X. M. Yin, *Strong convergence theorems for nonexpansive nonself-mappings*, Nonlinear Analysis, **24**, 223–228 (1995).