A POISSON ARRIVAL SELECTION PROBLEM FOR GAMMA PRIOR INTENSITY WITH NATURAL NUMBER PARAMETER

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Abstract. This article studies a continuous-time generalization of the so-called secretary problem. A man seeks an apartment. Opportunities to inspect apartments arise according to a homogeneous Poisson process of unknown intensity λ having a Gamma prior density, $G(r, 1/a)$, where r is natural number. At any epoch he is able to rank a given apartment amongst all those inspected to date, where all permutations of ranks are equally likely and independent of the Poisson process.The objective to maximize the probability of selecting best apartment from those (if any) available in the interval $[0, T]$, where T is given. This problem is reformulated as the optimal stopping problem and it is shown to be the monotone case. The optimal strategy for the problem is solved to be a threshold rule.

1 Introduction. This article studies a continuous-time generalization of the so-called secretary problem which is as follows: A man has been allowed a fixed time T in which to find an apartment. Opportunities to inspect apartments occur at the epochs of a homogeneous Poisson process of unknown intensity λ. He inspects each apartment when the opportunity arises, and he must decide immediately whether to accept or not. At any epoch he is able to rank a given apartment amongst all those inspected to date, where all permutations of ranks are equally likely and independent of the Poisson process. The objective is to maximize the probability of selecting the best apartment from those (if any) available in the interval $[0, T]$. We show that if the prior density of the intensity is Gamma $G(r, 1/a)$ where r is natural number, then the optimal strategy for this problem can be described as follows; accept the jth option which arrives after time $s_j^{(r)*}$ if the option is the first relatively best option (if any), where $s_j^{(r)*}$ is nonincreasing sequence of j and is determined by the unique root of a certain equation, and $s_j^{(r)*} \to (T+a)/e - a$ as $j \to \infty$ for all r.

Bruss (1987) showed that if the prior density of the intensity of the Poisson process is exponential with parameter $a > 0$, $E(1/a)$ (note that this is Gamma distribution with parameters 1 and $1/a$, $G(1, 1/a)$), then the optimal strategy is to accept the first relatively best option (if any) after time $s^* = (T+a)/e - a$. Cowan and Zabczyk (1978) studied the problem where the intensity of the Poisson process is known. Therefore, Bruss' problem is an extension of their model. A different approach to the secretary problem with an unknown number of options had been developed by Presman and Sonin (1972). Ano (2001) studied multiple selections with the objective of maximizing the probability of selecting the overall best option for both the problems of Bruss and Presman and Sonin, and derived the optimal strategy for Bruss' multiple selection problem.

To find the optimal strategy for Bruss' problem, he directly calculated the maximum probability of selecting the best apartment when the current relatively best option is accepted, and the maximum probability of selecting the best one when the current relatively

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best option is rejected. In Section 2 of this paper, his problem is resolved from a different approach and it is shown that it is monotone in the sense of Chow, Robbins, and Siegmund (1971). This section has an instructive and preparative meaning of the next sections. For a good background of the Poisson process to the Section 2, see Chapter 7 of Karlin (1966). The following questions naturally arise: if the prior density of the intensity is Gamma $G(r, 1/a), r = 1, 2, 3, \cdots$, what is the optimal strategy? In Section 3 and 4, the problem of Gamma prior intensity with natural number parameter is studied in detail.

2 Resolution of Bruss' problem. The one-stage look-ahead stopping strategy is employed to resolve Bruss' problem. Let $S_1, S_2, \cdots$ denote the arrival times of the Poisson process, $\{N(t)\}_{t\geq 0}$. For unknown intensity λ, an exponential prior density $a\exp\{-a\lambda\}I(\lambda > 0)$ is assumed, where a is known and nonnegative. Then by Bayes' theorem, the conditional posterior density $f(\lambda|S_j = s)$ given $S_j = s$, is $f(\lambda|S_j = s) = (\lambda^j/j!)(s+a)^{j+1}\exp\{-(s+a)\lambda\}I(\lambda > 0), s \in [0, T]$. Bruss showed that the posterior distribution of $N(T)$ given $s_1, \cdots, s_j (= s)$ is equivalent to a Pascal distribution with parameters $(j, (s+a)/(T+a))$, that is,

$$\begin{aligned} P(N(T) = n|S_1 = t_1, \cdots, S_{j-1} &= t_{j-1}, S_j = s) \\ &= P(N(T) = n|S_j = s) \\ &= \binom{n}{j}\left(\frac{s+a}{T+a}\right)^{j+1}\left(1 - \frac{s+a}{T+a}\right)^{n-j}. \end{aligned}$$

Let (j, s) denote the state of the process, when option number j arrives at time s and that is the relatively best one. Define the relative rank of jth option by Y_j. $Y_j = 1$ represents the relative rank of jth option is best, $Y_j = 2$ does the one of jth option is second and so on. Define the ture rank of jth option by X_j. $X_j = 1$ means that X_j is the best option of all, so $X_j = \min(X_1, \cdots, X_{N(T)})$. Let $W_j(s)$ denote the maximum probability of obtaining the best option starting from state (j, s), that is,

$$W_j(s) = \sup_{\tau\in[j,N(T)]} P(X_\tau = 1|S_j = s, Y_j = 1).$$

Similarly, let $U_j(s)$ be the corresponding probability when we select the current relatively best option, that is,

$$\begin{aligned} U_j(s) &= \sum_{n\geq j} P(X_j = 1, N(T) = n|S_j = s, Y_j = 1) \\ &= \sum_{n\geq j}\left(\frac{j}{n}\right)P(N(T) = n|S_j = s) = \frac{s+a}{T+a}. \end{aligned}$$

Let $V_j(s)$ be the corresponding probability when we don't select the current relatively best option and proceed optimally thereafter, that is,

$$V_j(s) = \int_0^{T-s}\sum_{k\geq 1} p_{(j,s)}^{(k,u)} W_{j+k}(s+u)du,$$

where $p_{(j,s)}^{(k,u)}$ is the one-step transition probability from state (j, s) to state $(j+k, s+u)$ and is defined as

$$\begin{aligned} p_{(j,s)}^{(k,u)} &= \int_0^\infty P(S_{j+k} = s+u|S_j = s, \lambda) \\ &\quad \times P(Y_{j+k} = 1|Y_j = 1, S_j = s, S_{j+k} = s+u, \lambda)g(\lambda|S_j = s)d\lambda. \end{aligned}$$

Since

$$P(Y_{j+k}=1|Y_j=1,S_j=s,S_{j+k}=s+u,\lambda)=\frac{j}{(j+k-1)(j+k)},$$

$p_{(j,s)}^{(k,u)}$ can be derived from the equality $\int_0^\infty \lambda^{k+j}\exp\{-\lambda(s+a+u)\}d\lambda=\Gamma(k+j+1)/(s+a+u)^{k+j+1}$ as follows.

$$\begin{aligned} p_{(j,s)}^{(k,u)} &= \int_0^\infty \frac{\lambda e^{-\lambda u}(\lambda u)^{k-1}}{(k-1)!}\frac{j}{(j+k-1)(j+k)}\frac{e^{-\lambda(s+a)}\lambda^j(s+a)^{j+1}}{j!}d\lambda \\ &= \frac{s+a}{(s+a+u)^2}\binom{j+k-2}{k-1}\left(\frac{s+a}{s+a+u}\right)^j\left(\frac{u}{s+a+u}\right)^{k-1}. \end{aligned}$$

By the principle of optimality, we have the following optimal equation,

$$W_j(s)=\max\{U_j(s),V_j(s)\},\ j\ge 1, 0<s\le T,$$

with $W_j(T)=1$ for $j=1,2,\cdots$. Let B be the one-step look-ahead stopping region, that is, B is the set of state (j,s) for which selecting the current relatively best option is at least as good as waiting for the next relatively best option to appear and then selecting it. Therefore, B is given by

$$B=\{(j,s):U_j(s)-\int_0^{T-s}\sum_{k\ge 1}p_{(j,s)}^{(k,u)}U_{j+k}(s+u)du\ge 0\}.$$

Define $h_j(s)$ as $h_j(s)\equiv U_j(s)-\int_0^{T-s}\sum_{k\ge1}p_{(j,s)}^{(k,u)}U_{j+k}(s+u)du$. A stopping problem is defined as monotone if the events $G_j(s)=\{h_j(s)\ge 0\}$, are monotone non-decreasing in j and s, i.e., $G_0(s)\subset G_1(s)\subset\cdots$ a.s. and for $u>0$, $G_j(s)\subset G_j(s+u)\subset\cdots$ a.s. Now we have

$$\begin{aligned} h_j(s) &= \frac{s+a}{T+a}-\int_0^{T-s}\sum_{k\ge1}p_{(j,s)}^{(k,u)}\left(\frac{s+a+u}{T+a}\right)du \\ &= \frac{s+a}{T+a}-\int_0^{T-s}\sum_{k\ge1}\frac{s+a}{(s+a+u)^2}\binom{j+k-2}{k-1}\left(\frac{s+a}{s+a+u}\right)^j \\ &\qquad\qquad \times\left(\frac{u}{s+a+u}\right)^{k-1}\left(\frac{s+a+u}{T+a}\right)du \\ &= \frac{s+a}{T+a}-\int_0^{T-s}\frac{s+a}{T+a}\frac{1}{s+a+u}du \\ &= \frac{s+a}{T+a}\left(1+\ln\left(\frac{s+a}{T+a}\right)\right). \end{aligned}$$

This function $h_j(s)$ does not depend on j, so we write this $h(s)$. The third equality follows from the equalities, $p_{(j,s)}^{(k,u)}=(s+a)/(s+a+u)^2\times$ {Pascal distribution with parameters $(k,u/(s+a+u))$} and $\sum_{k\ge1}p_{(j,s)}^{(k,u)}=(s+a)/(s+a+u)^2$. Using $h(s)$, B reduces to

$$B=\{(j,s):h(s)\ge0\}=\{s:\ln\left(\frac{s+a}{T+a}\right)\ge -1\}=\{s:s\ge (T+a)/e-a\}.$$

Because $\ln((s+a)/(T+a))$ is nondecreasing in s, if $h(s) \geq 0$, then $h(s+u) \geq 0$, for $u > 0$. Therefore, for $0 < u < T - s$, we can see that

$$P(h(s+u) \geq 0 | h(s) \geq 0) = P(s+u \in B | s \in B) = 1,$$

or, equivalently,

$$G(s) \subset G(s+u) \subset \cdots \text{ a.s.}$$

Thus, the problem is monotone and B is "closed". It is known that if the problem is monotone, then B gives the optimal stopping region. Now we have reached the following result.

Theorem 1 (Bruss(1987)) *If the prior density of the intensity of the Poisson process is exponential with parameter $a > 0$, then the problem is monotone and the optimal strategy is to accept the first relatively best option (if any) after time $s^* = (T+a)/e - a$.*

3 Problem for Gamma prior intensity, $G(2, 1/a)$.

Suppose that the prior density of the intensity λ of the Poisson process is Gamma with parameters $r > 0, a > 0$, $G(r, 1/a)$. Then, the density of λ is given by

$$g(\lambda) = \frac{a^r}{\Gamma(r)} e^{-a\lambda} \lambda^{r-1}.$$

The posterior density of λ given $S_1 = s_1, \cdots, S_j = s$ can be computed and turns out to be Gamma, $G(r+j, 1/(a+s))$. We can see that the posterior distribution of $N(T)$ given $S_1 = s_1, \cdots, S_j = s$ is again a Pascal distribution.

Lemma 2 *The posterior distribution of $N(T)$ given $S_1 = s_1, \cdots, S_j = s (0 < s < T)$ only depends on the values of j and S_j, and is a Pascal distribution with parameters $r+j, (s+a)/(T+a)$. That is, for $n = j, j+1, \cdots$*

$$P(N(T) = n | S_1 = s_1, \cdots, S_j = s) = \frac{\Gamma(n+r)}{\Gamma(r+j)(n-j)!} \left(\frac{s+a}{T+a}\right)^{r+j} \left(\frac{T-s}{T+a}\right)^{n-j}.$$

This statement was found in Bruss (1987) without the proof. The proof for Lemma 2 above was derived in a similar manner as the proof of the Lemma in Section 2 of Bruss (1987) (in which the posterior distribution of $N(T)$ is given for the exponential prior intensity). The precise proof for Lemma 2 follows.

Proof. The arrival times $S_1, S_2, \cdots, S_{j-1}$ of the Poisson process, given $S_j = s$, are i.i.d. random variables with the uniform distribution on $(0, s)$. The joint distribution of the order statistics $S_1 < S_2 < \cdots < S_j = s$, therefore, only depends on the values j and S_j. Thus,

$$\begin{aligned} P(N(T) = n | S_1 = s_1, \cdots, S_j = s) &= P(N(T) = n | N(s) = j, S_j = s) \\ &= P(N(T) = n | S_j = s). \end{aligned}$$

Let $g(\lambda | S_j = s)$ be the conditional density of λ given $S_j = s$; then we get

$$P(N(T) = n | S_j = s) = \int_0^\infty P(N(T) = n | S_j = s, \lambda) g(\lambda | S_j = s) d\lambda. \tag{1}$$

Since $\{N(t)\}_{t\geq 0}$ has independent stationary increments, it follows that

$$P(N(T)=n|S_j=s,\lambda)=\frac{(\lambda(T-s))^{n-j}}{(n-j)!}e^{-\lambda(T-s)}. \tag{2}$$

The conditional density of S_j given λ equals

$$f(s|\lambda)=\frac{(\lambda s)^{j-1}}{(j-1)!}\lambda e^{-\lambda s}.$$

Using Bayes' theorem, we get

$$g(\lambda|S_j=s)=\frac{\lambda^{j+r-1}e^{-\lambda(s+a)}}{\int_0^\infty u^{j+r-1}e^{-u(s+a)}du}=\frac{\lambda^{j+r-1}e^{-\lambda(s+a)}}{\frac{1}{(s+a)^{j+r}}\Gamma(j+r)}. \tag{3}$$

We obtain from (1), (2) and (3)

$$\begin{aligned}
P(N(T)=n|S_j=s) &= \int_0^\infty \frac{(\lambda(T-s))^{n-j}}{(n-j)!}e^{-\lambda(T-s)}\frac{\lambda^{j+r-1}e^{-\lambda(s+a)}(s+a)^{j+r}}{\Gamma(j+r)}d\lambda \\
&= \frac{(T-s)^{n-j}(s+a)^{j+r}}{\Gamma(j+r)(n-j)!}\int_0^\infty \lambda^{n+r-1}e^{-\lambda(T+a)}d\lambda \\
&= \frac{(T-s)^{n-j}(s+a)^{j+r}}{\Gamma(j+r)(n-j)!}\frac{\Gamma(n+r)}{(T+a)^{n+r}} \\
&= \frac{\Gamma(n+r)}{\Gamma(j+r)(n-j)!}\left(\frac{s+a}{T+a}\right)^{j+r}\left(\frac{T-s}{T+a}\right)^{n-j}.
\end{aligned}$$

"#

For the problem with Gamma prior density of the intensity, let $U_j^{(r)}(s)$, $V_j^{(r)}(s)$ and $W_j^{(r)}(s)$ denote the same probabilities defined in Section 2. Using the formula

$$\frac{(n+r-1)!}{n}=(r-1)!\sum_{i=0}^{r-1}\frac{(n+i-1)!}{i!},\ \text{for } r=1,2,\cdots, \tag{4}$$

$U_j^{(r)}(s)$ becomes

$$\begin{aligned}
U_j^{(r)}(s)=E\left(\frac{j}{N(T)}\,\middle|\, S_j=s\right) &= \sum_{n\geq j}\left(\frac{j}{n}\right)P(N(T)=n|S_j=s) \\
&= \frac{j(r-1)!}{(j+r-1)!}\sum_{i=0}^{r-1}\frac{(j+i-1)!}{i!}\theta^{r-i},
\end{aligned}$$

where $\theta=(s+a)/(T+a)$. In particular, for $r=1,2,$ and 3, we have $U_j^{(r)}(s)=\theta,\theta(j+\theta)/(j+1)$, and $\theta(2\theta(\theta+j)+j(j+1))/((j+1)(j+2))$, respectively.

Since the distribution of the interarrival time of the Poisson process is the exponential distribution, the posterior distribution of λ, given $S_j=s$, is given by (3), and the conditional probability that the $(j+k)$th option is the first relatively best option among $(j+k)$ options after the jth option, which was the relatively best one among j options, is $j/((j+k-1)(j+k))$, the one-step transition probability $p_{(j,s)}^{(k,u)}$ is given by

$$
\begin{aligned}
(5)\qquad p_{(j,s)}^{(k,u)} &= \int_0^\infty \frac{\lambda e^{-\lambda u}(\lambda u)^{k-1}}{\Gamma(k)} \frac{j}{(j+k-1)(j+k)} \frac{\lambda^{j+r-1}e^{-\lambda(s+a)}(s+a)^{j+r}}{\Gamma(j+r)} d\lambda \\
&= \frac{j(s+a)^{j+r}u^{k-1}}{\Gamma(k)(j+k-1)(j+k)\Gamma(j+r)} \int_0^\infty \lambda^{j+r+k-1}e^{-\lambda(s+a+u)}d\lambda \\
&= \frac{\Gamma(j+k+r)}{\Gamma(k)\Gamma(j+r)} \frac{j}{(j+k-1)(j+k)} \frac{s+a}{(s+a+u)^2} \\
&\qquad \times \left(\frac{s+a}{s+a+u}\right)^{j+r-1} \left(\frac{u}{s+a+u}\right)^{k-1}.
\end{aligned}
$$

Then we get

$$V_j^{(r)}(s) = \int_0^{T-s} \sum_{k\geq 1} p_{(j,s)}^{(k,u)} W_{j+k}^{(r)}(s+u)du.$$

By the principle of optimality, we have

$$W_j^{(r)}(s) = \max\{U_j^{(r)}(s), V_j^{(r)}(s)\}, j = 1, 2, \cdots,\ 0 < s \leq T,$$

with $W_j^{(r)}(T) = 1$ for $j = 1, 2, \cdots$. The one-stage look-ahead stopping region B_r is given by

$$B_r = \{(j,s) : G_j^{(r)}(s) \geq 0\}$$

where

$$G_j^{(r)}(s) \equiv U_j^{(r)}(s) - \int_0^{T-s} \sum_{k\geq 1} p_{(j,s)}^{(k,u)} U_{j+k}^{(r)}(s+u)du.$$

When the parameter r of the Gamma prior is 2, that is, $G(2, 1/a)$, we have

$$U_j^{(2)}(s) = \frac{1}{j+1}\frac{s+a}{T+a}\left(j + \frac{s+a}{T+a}\right).$$

Define $h_j^{(2)}(s)$ as

$$h_j^{(2)}(s) \equiv \frac{(j+1)(T+a)}{s+a} G_j^{(2)}(s);$$

then B_2 can be rewritten as $B_2 = \{(j,s)\ :\ h_j^{(2)}(s) \geq 0\}$. Thus

$$
\begin{aligned}
h_j^{(2)}(s) &= j + \frac{s+a}{T+a} - \frac{(T+a)(j+1)}{s+a} \int_0^{T-s} \sum_{k\geq 1} p_{(j,s)}^{(k,u)} \frac{1}{j+k+1} \frac{s+a+u}{T+a} \\
(6)\qquad &\qquad \times \left\{ j+k-1 + \left(1 + \frac{s+a+u}{T+a}\right)\right\} du.
\end{aligned}
$$

Using (5), the second term of RHS in (6) becomes

$$\int_0^{T-s} \sum_{k\geq 1} \frac{(j+k-2)!}{(k-1)!j!} \frac{j}{s+a+u} \left(\frac{s+a}{s+a+u}\right)^{j+1} \left(\frac{u}{s+a+u}\right)^{k-1}$$

$$\times\left\{j+k-1+\left(1+\frac{s+a+u}{T+a}\right)\right\}du \tag{7}$$

$$= \int_0^{T-s}\frac{j}{s+a+u}\sum_{k\geq 1}\frac{(j+k-1)!}{(k-1)!j!}\left(\frac{s+a}{s+a+u}\right)^{j+1}\left(\frac{u}{s+a+u}\right)^{k-1}du$$

$$+\int_0^{T-s}\sum_{k\geq 1}\frac{(j+k-2)!}{(k-1)!j!}\frac{j}{s+a+u}\left(\frac{s+a}{s+a+u}\right)^{j+1}\left(\frac{u}{s+a+u}\right)^{k-1}$$

$$\times\left(1+\frac{s+a+u}{T+a}\right)du$$

$$= \int_0^{T-s}\frac{j}{s+a+u}du+\int_0^{T-s}\frac{s+a}{(s+a+u)^2}\left(1+\frac{s+a+u}{T+a}\right)du$$

$$= -\left(j+\frac{s+a}{T+a}\right)\ln\frac{s+a}{T+a}+1-\frac{s+a}{T+a},$$

where the third equality follows from the following formulas, for $0<p<1$

$$\sum_{k\geq 1}\binom{j+k-1}{k-1}p^{j+1}(1-p)^{k-1}=1, \qquad \sum_{k\geq 1}\binom{j+k-2}{k-1}p^{j+1}(1-p)^{k-1}=p.$$

¿From (6) and (7), we have after some simplification

$$h_j^{(2)}(s)=H_j^{(2)}(\theta)=j(1+\ln\theta)+\theta\ln\theta+2\theta-1, \tag{8}$$

where $\theta=(s+a)/(T+a)$. Now we can write the one-stage look-ahead stopping region B_2 as

$$B_2=\left\{(j,s):H_j^{(2)}\left(\frac{s+a}{T+a}\right)\geq 0\right\}.$$

Since $H_j^{(2)}(\theta)$ is strictly increasing in θ and has exactly one solution $t_j^{(2)*}=(s_j^{(2)*}+a)/(T+a)$. Then B_2 is written as $B_2=\{(j,s):s\geq s_j^{(2)*}\}$. In the case where $H_j^{(2)}(\theta)=0$, then $1+\ln\theta=-(\theta\ln\theta+2\theta-1)/j$. When $j\to\infty$, the RHS of the above equation tends to zero. Thus $\lim t_j^{(2)*}=1/e$ and then $\lim s_j^{(2)*}=(T+a)/e-a$. On the other hand, since $H_{j+1}^{(2)}(\theta)-H_j^{(2)}(\theta)=1+\ln\theta$ then $H_j^{(2)}(\theta)$ is decreasing in j for $\theta<1/e$ and increasing for $\theta\geq 1/e$. Moreover $H_j^{(2)}(1/e)=1/e-1<0$ and therefore $t_j^{(2)*}>1/e$ for every j. So, $H_j(\theta)\geq 0$ implies $H_{j+k}(\theta+u)$ for every k and $u\geq 0$, and we can see that for $h_j^{(2)}(s)=H_j^{(2)}((s+a)/(T+a))$

$$P(h_{j+k}^{(2)}(s+u)\geq 0|h_j^{(2)}(s)\geq 0)=P((j+k,s+u)\in B_2|(j,s)\in B_2)=1,$$

for $k=1,2,\cdots,\ u>0$. Therefore, B_2 is "closed" and the problem becomes monotone and B_2 becomes optimal stopping region. Now we have the following result.

Theorem 3 *Suppose that the prior density of the intensity of the Poisson process is Gamma with parameters* 2 *and* $a>0$. *Then the optimal strategy is to accept the* j*th option which arrives after time* $s_j^{(2)*}$ *if the option is the first relatively best option (if any), where* $s_j^{(2)*}$ *is nonincreasing sequence of* j *and is determined by the unique root of the equation* $h_j^{(2)}(s)=0$.

The values $t_j^{(2)*}(=(s_j^{(2)*}+a)/(T+a))$ for $j=1,\cdots,10,\cdots,50,100,1000,10000$ are shown in Table 1.

Table 1: The values of $t_j^{(2)*}$ for $r = 2$

j	1	2	3	4	5	6
$t_j^{(2)*}$	.509242	.458519	.433823	.419516	.410248	.403776
j	7	8	9	10	20	30
$t_j^{(2)*}$	.399008	.395352	.392462	.390120	.379257	.375521
j	40	50	100	1000	10000	∞
$t_j^{(2)*}$	.373621	.372491	.370195	.368112	.367902	$1/e$

4 Problem for $G(r, 1/a)$.

For $r = 1, 2, \cdots$, we get

$$\begin{aligned}
G_i^{(r)}(s) &\equiv U_i^{(r)}(s) - \int_0^{T-s} \sum_{k\geq 1} p_{(i,s)}^{(k,u)} U_{i+k}^{(r)}(s+u)du \\
&= \frac{i(r-1)\,!}{(i+r-1)\,!} \sum_{j=0}^{r-1} \frac{(i+j-1)\,!}{j\,!} \left(\frac{s+a}{T+a}\right)^{r-j} \\
&\quad - \int_0^{T-s} \sum_{k\geq 1} \frac{\Gamma(i+k+r)}{\Gamma(k)\Gamma(i+r)} \frac{i}{(i+k)(i+k-1)} \frac{s+a}{(s+a+u)^2} \left(\frac{s+a}{s+a+u}\right)^{i+r-1} \\
&\quad \times \left(\frac{u}{s+a+u}\right)^{k-1} \frac{(i+k)(r-1)\,!}{(i+k+r-1)\,!} \sum_{j=0}^{r-1} \frac{(i+k+j-1)\,!}{j\,!} \left(\frac{s+a+u}{T+a}\right)^{r-j} du \\
&= \frac{i(r-1)\,!}{(i+r-1)\,!} \sum_{j=0}^{r-1} \frac{(i+j-1)\,!}{j\,!} \left(\frac{s+a}{T+a}\right)^{r-j} \\
&\quad - \int_0^{T-s} \sum_{k\geq 1} \frac{(r-1)\,!}{(k-1)\,!\,(i+r-1)\,!} \frac{i}{i+k-1} \frac{s+a}{(s+a+u)^2} \left(\frac{s+a}{s+a+u}\right)^{i+r-1} \\
&\quad \times \left(\frac{u}{s+a+u}\right)^{k-1} \sum_{j=0}^{r-1} \frac{(i+k+j-1)\,!}{j\,!} \left(\frac{s+a+u}{T+a}\right)^{r-j} du.
\end{aligned}$$

Let

$$h_i^{(r)}(s) \equiv \frac{(i+r-1)\,!}{i\,!\,(r-1)\,!} \frac{T+a}{s+a} G_i^{(r)}(s),$$

then

$$\begin{aligned}
h_i^{(r)}(s) &= \frac{(i+r-1)\,!}{i\,!\,(r-1)\,!} \frac{T+a}{s+a} \frac{i(r-1)\,!}{(i+r-1)\,!} \sum_{j=0}^{r-1} \frac{(i+j-1)\,!}{j\,!} \left(\frac{s+a}{T+a}\right)^{r-j} \\
&\quad - \frac{(i+r-1)\,!}{i\,!\,(r-1)\,!} \frac{T+a}{s+a} \int_0^{T-s} \sum_{k\geq 1} \frac{(i+k+r-1)\,!}{(k-1)(i+r-1)} \frac{i}{(i+k)(i+k-1)} \\
&\quad \times \frac{s+a}{(s+a+u)^2} \left(\frac{s+a}{s+a+u}\right)^{i+r-1} \left(\frac{u}{s+a+u}\right)^{k-1} \frac{(i+k)(r-1)\,!}{(i+k+r-1)\,!} \\
&\quad \times \sum_{j=0}^{r-1} \frac{(i+k+j-1)\,!}{j\,!} \left(\frac{s+a+u}{T+a}\right)^{r-j} du
\end{aligned}$$

$$= \sum_{j=0}^{r-1} \frac{(i+j-1)\,!}{(i-1)\,!\,j\,!} \left(\frac{s+a}{T+a}\right)^{r-j-1} - \int_0^{T-s} \sum_{k\geq 1} \frac{1}{(i-1)\,!\,(k-1)\,!} \frac{1}{i+k-1}$$

$$\times \frac{1}{s+a+u} \left(\frac{s+a}{s+a+u}\right)^{i+r-1} \left(\frac{u}{s+a+u}\right)^{k-1}$$

$$\times \sum_{j=0}^{r-1} \frac{(i+k+j-1)\,!}{j\,!} \left(\frac{s+a+u}{T+a}\right)^{r-j-1} du. \tag{9}$$

Thus B_r can be rewritten as

$$B_r = \{(i,s) : h_i^{(r)}(s) \geq 0\}.$$

For the formula (4), let $n = i+k-1$ and $r = j+1$, then

$$\frac{(i+k+j-1)\,!}{i+k-1} = j\,! \sum_{l=0}^{j} \frac{(i+k+l-2)\,!}{l\,!}.$$

Let $\hat{\theta} = (s+a)/(s+a+u)$. Using this, the second term of RHS of (9) is obtained by

$$- \int_0^{T-s} \sum_{k\geq 1} \frac{1}{(i-1)\,!\,(k-1)\,!} \frac{1}{s+a+u} \hat{\theta}^{i+r-1} (1-\hat{\theta})^{k-1} \tag{10}$$

$$\times \sum_{j=0}^{r-1} \frac{1}{j\,!} j\,! \sum_{l=0}^{j} \frac{(i+k+l-2)\,!}{l\,!} \left(\frac{s+a+u}{T+a}\right)^{r-j-1} du$$

$$= - \int_0^{T-s} \sum_{j=0}^{r-1} \sum_{l=0}^{j} \sum_{k\geq 1} \frac{(i+k+l-2)\,!}{(k-1)\,!\,(i+l-1)\,!} \frac{(i+l-1)\,!}{(i-1)\,!\,l\,!} \hat{\theta}^{i+r-1} (1-\hat{\theta})^{k-1}$$

$$\times \frac{1}{s+a+u} \left(\frac{s+a+u}{T+a}\right)^{r-j-1} du$$

$$= - \int_0^{T-s} \sum_{j=0}^{r-1} \sum_{l=0}^{j} \sum_{k\geq 1} \binom{i+k+l-2}{k-1} \hat{\theta}^{i+l+(r-l-1)} (1-\hat{\theta})^{k-1}$$

$$\times \binom{i+l-1}{l} \frac{1}{s+a+u} \left(\frac{s+a+u}{T+a}\right)^{r-j-1} du$$

$$= - \int_0^{T-s} \sum_{j=0}^{r-1} \binom{i+j-1}{j} \left(\frac{s+a}{T+a}\right)^{r-j-1} \frac{du}{s+a+u}$$

$$- \int_0^{T-s} \sum_{j=0}^{r-1} \sum_{l=0}^{j-1} \binom{i+l-1}{l} \left(\frac{(s+a)^{r-l-1}}{(T+a)^{r-j-1}}\right) (s+a+u)^{l-j-1} du$$

$$= \sum_{j=0}^{r-1} \binom{i+j-1}{j} \left(\frac{s+a}{T+a}\right)^{r-j-1} \ln\left(\frac{s+a}{T+a}\right)$$

$$+ \sum_{j=1}^{r-1} \sum_{l=0}^{j-1} \binom{i+l-1}{l} \frac{1}{j-l} \left\{ \left(\frac{s+a}{T+a}\right)^{r-l-1} - \left(\frac{s+a}{T+a}\right)^{r-j-1} \right\} \tag{11}$$

where the forth equality follows, for $0 \le \hat{\theta} \le 1$,

$$\sum_{k \ge 1} \binom{i+k-1}{k-1} \hat{\theta}^{i+1}(1-\hat{\theta})^{k-1} = 1.$$

Finally we have for $\theta = (s+a)/(T+a)$,

$$\begin{aligned} h_i^{(r)}(s) = H_i^{(r)}(\theta) &= \sum_{j=0}^{r-1} \binom{i+j-1}{j} \theta^{r-j-1}(1+\ln\theta) \\ &\quad + \sum_{j=1}^{r-1} \sum_{l=0}^{j-1} \binom{i+l-1}{l} \frac{1}{j-l}(\theta^{r-l-1} - \theta^{r-j-1}). \end{aligned} \tag{12}$$

When $r = 2$, $U_i^{(2)}(s)$ and $H_i^{(2)}(\theta)$ are given as follows and these functions are equivalent to the ones found in Section 3;

$$U_i^{(2)}(s) = \frac{\theta(i+\theta)}{i+1},$$

$$H_i^{(2)}(\theta) = i(1+\ln\theta) + \theta\ln\theta + 2\theta - 1.$$

The optimal stopping rule for $G(r, 1/a)$, $r = 1, 2, \cdots,\ a > 0$ is given in Theorem 6. To prove Theorem 6, we need two lemmas.

Lemma 4

$$h_{i+1}^{(r)}(s) - h_i^{(r)}(s) = h_{i+1}^{(r-1)}(s)$$

Proof. Let (12) rewrite as

$$h_i^{(r)}(s) = \sum_{m=0}^{r-1} \binom{i+m-1}{m} C_m^{(r)},$$

where

$$C_m^{(r)} = \theta^{r-m-1}(1+\ln\theta) + \sum_{j=m+1}^{r-1} \frac{1}{j-m}(\theta^{r-m-1} - \theta^{r-j-1}), \quad m = 0, 1, \cdots, r-2$$

$$C_{r-1}^{(r)} = 1 + \ln\theta.$$

Note that $C_m^{(r)} = C_{m+1}^{(r+1)}$ holds.

Therefore

$$\begin{aligned} h_{i+1}^{(r)}(s) - h_i^{(r)}(s) &= \sum_{m=0}^{r-1} \left\{ \binom{i+m}{m} - \binom{i+m-1}{m} \right\} C_m^{(r)} \\ &= \sum_{m=0}^{r-1} \left(\frac{i+m}{i} - 1 \right) \binom{i+m-1}{m} C_m^{(r)} \\ &= \sum_{m=1}^{r-1} \binom{i+m-1}{m-1} C_m^{(r)} \\ &= \sum_{m=0}^{r-2} \binom{i+m}{m} C_{m-1}^{(r-1)} = h_{i+1}^{(r-1)}(s). \end{aligned}$$

"#

Let $s_i^{(r)*}$ be the unique root of the equation $h_i^{(r)}(s) = 0$.

Lemma 5

(i) $h_i^{(r)}(s) \geq 0 \Longrightarrow h_i^{(r)}(s+u) \geq 0,\ 0 < u < T - s.$

(ii) $h_i^{(r)}(s) \geq 0 \Longrightarrow h_{i+1}^{(r)}(s) \geq 0\,!\$\,i = 1, 2, \cdots.$

(iii) $s_i^{(r)*}$ *is nonincreasing in* i.

Proof.

(i) To show the statement, it is enough to prove that (a) $h_i^{(r)}(s)$ is increasing in $s \in [s_i^{(r-1)*}, T]$, (b) $h_i^{(r)}(s) < 0$ for $s \in (0, s_i^{(r-1)*})$ and (c) $h_i^{(r)}(s) = 0$ has a unique root $s_i^{(r)*} \in [s_i^{(r-1)*}, T]$ (Note that this is consequently obtained by (a) and (b)). These are shown by induction on r. Since $\theta (= (s+a)/(T+a))$ is increasing function of s and $h_i^{(1)}(s) = 1 + \ln\theta$, obviously $h_i^{(1)}(s)$ is increasing in s and $h_i^{(1)}(s) = 0$ has the unique root $1/e (= s_i^{(1)*})$. Thus for $r = 1$, (a), (b) and (c) hold.

Assume that (a) $h_i^{(r)}(s)$ is increasing in $s \in [s_i^{(r-1)*}, T]$, (b) $h_i^{(r)}(s) < 0$ for $s \in (0, s_i^{(r-1)*})$ and (c) $h_i^{(r)}(s) = 0$ has a unique root $s_i^{(r)*} \in [s_i^{(r-1)*}, T]$. $h_i^{(r+1)}(s)$ can be written as

$$\begin{aligned} h_i^{(r+1)}(s) &= \sum_{j=0}^{r} \binom{i+j-1}{j} \theta^{r-j}(1+\ln\theta) + \sum_{j=1}^{r}\sum_{l=0}^{j-1} \binom{i+l-1}{l} \frac{1}{j-l}(\theta^{r-l} - \theta^{r-j}) \\ &= \theta h_i^{(r)}(s) + \binom{i+r-1}{r}(1+\ln\theta) + \sum_{l=0}^{r-1} \binom{i+l-1}{l} \frac{1}{r-l}(\theta^{r-l} - 1). \end{aligned}$$

Note that θ, $\ln\theta$ and θ^{r-l} are increasing in s. Therefore if $h_i^{(r)}(s) \geq 0$, that is $s \in [s_i^{(r)*}, T]$, then $h_i^{(r+1)}(s)$ is increasing in $s \in [s_i^{(r)*}, T]$. Thus for $r+1$, (a) holds. To show that (b) holds for $r+1$, first we see that $h_i^{(r+1)}(s_i^{(r)*}) < 0$. Let $\theta^* = (s_i^{(r)*} + a)/(T+a)$.

$$\begin{aligned} & h_i^{(r+1)}(s_i^{(r)*}) \\ &= \theta^* h_i^{(r)}(s_i^{(r)*}) + \binom{i+r-1}{r}(1+\ln\theta^*) + \sum_{l=0}^{r-1} \binom{i+l-1}{l} \frac{1}{r-l}(\theta^{*r-l} - 1) \\ &= \binom{i+r-1}{r}(1+\ln\theta^*) + \sum_{l=0}^{r-1} \binom{i+l-1}{l} \frac{1}{r-l}(\theta^{*r-l} - 1). \end{aligned} \tag{13}$$

On the other hand, using another transformation of the formula (4), we have

$$\binom{n+r-1}{r-1} = \sum_{i=1}^{r-1} \binom{n+i-1}{i}.$$

By the above equation, we can see that

$$\begin{aligned} & h_i^{(r)}(s_i^{(r)*}) = 0 \\ \Longleftrightarrow\ & \binom{i+r-1}{r-1}(1+\ln\theta^*) + \sum_{j=1}^{r-1}\sum_{l=0}^{j-1} \binom{i+l-1}{l} \frac{1}{j-l}(\theta^{*j-l} - 1) = 0. \end{aligned} \tag{14}$$

¿From (13) and (14), $h_i^{(r+1)}(s_i^{(r)*})$ can be written as

$$\begin{aligned}
&h_i^{(r+1)}(s_i^{(r)*}) \\
&= -\frac{i}{r}\sum_{j=1}^{r-1}\sum_{l=0}^{j-1}\binom{i+l-1}{l}\frac{1}{j-l}(\theta^{*j-l}-1)+\sum_{l=0}^{r-1}\binom{i+l-1}{l}\frac{1}{r-l}(\theta^{*r-l}-1) \\
&= -\sum_{m=1}^{r-1}\frac{i}{r}\binom{i+r-m-1}{r-m-1}\frac{1}{m}(\theta^{*m}-1)+\sum_{m=1}^{r}\binom{i+r-m-1}{r-m}\frac{1}{m}(\theta^{*m}-1) \\
&= \frac{1}{r}\sum_{l=0}^{r-1}\binom{i+l-1}{l}(\theta^{*r-l}-1) \\
(15)\ &< 0.
\end{aligned}$$

For $s \in (0, s_i^{(r)*})$,

$$\begin{aligned}
h_i^{(r+1)}(s) &= \theta h_i^{(r)}(s)+\binom{i+r-1}{r}(1+\ln\theta)+\sum_{l=0}^{r-1}\binom{i+l-1}{l}\frac{1}{r-l}(\theta^{r-l}-1) \\
&< \theta^* h_i^{(r)}(s_i^{(r)*})+\binom{i+r-1}{r}(1+\ln\theta^*)+\sum_{l=0}^{r-1}\binom{i+l-1}{l}\frac{1}{r-l}(\theta^{*r-l}-1) \\
&= h_i^{(r+1)}(s_i^{(r)*}) \\
&< 0 \quad \text{(from (15))}.
\end{aligned}$$

So (b) holds for $r+1$. Therefore (c) holds for $r+1$. The proof is completed.

(ii) To show

$$h_i^{(r)}(s) \geq 0 \Longrightarrow h_{i+1}^{(r)}(s) \geq 0, \quad i = 1, 2, \cdots,$$

from Lemma 4 it is sufficient to show

$$h_i^{(r)}(s) \geq 0 \Longrightarrow h_{i+1}^{(r-1)}(s) \geq 0. \tag{16}$$

To show this, it is sufficient to show from (i)

$$s_{i+1}^{(r-1)*} < s_i^{(r)*}.$$

That is, it must be shown

$$h_i^{(r)}(s_{i+1}^{(r-1)*}) < 0.$$

This is true because from Lemma 4 and (15)

$$h_i^{(r)}(s_{i+1}^{(r-1)*}) = h_{i+1}^{(r)}(s_{i+1}^{(r-1)*}) < 0.$$

(iii) ¿From (i), it is shown that $h_i^{(r)}(s) = 0$ has unique root $s_i^{(r)*}$ and one-step look-ahead stopping region B_r can be written as $B_r = \{(i,s) : s \geq s_i^{(r)*}\}$. ¿From (16), for $s \in [s_i^{(r)*}, T]$

$$h_i^{(r)}(s) \geq 0 \Longrightarrow h_{i+1}^{(r-1)}(s) \geq 0.$$

¿From Lemma 4, for $s \in [s_i^{(r)*}, T]$

$$h_i^{(r)}(s) \geq 0 \Longrightarrow h_{i+1}^{(r)}(s) \geq h_i^{(r)}(s).$$

Thus

$$h_{i+1}^{(r)}(s_i^{(r)*}) \geq h_i^{(r)}(s_i^{(r)*}) = 0.$$

Finally we get

$$s_{i+1}^{(r)*} \leq s_i^{(r)*}.$$

"#

Theorem 6 *Suppose that the prior density of the intensity of the Poisson process is Gamma with parameters $r = 1, 2, \cdots$ and $a > 0$. The optimal stopping rule τ_r^* is*

$$\tau_r^* = \min\{s_i \in [\, s_i^{(r)*}, T\,] : X_i = 1\},$$

where X_i is the relative rank of the ith option. That is, it is to accept the ith option which arrives after time $s_i^{(r)^*}$ if the option is the first relatively best option (if any), where $s_i^{(r)^*}$ is nonincreasing sequence of i and is determined by the unique root in $(0, T]$ of the equation $h_i^{(r)}(s) = 0$, that is

$$\sum_{j=0}^{r-1} \binom{i+j-1}{j} \theta^{r-j-1}(1 + \ln\theta) + \sum_{j=1}^{r-1}\sum_{l=0}^{j-1} \binom{i+l-1}{l} \frac{1}{j-l}(\theta^{r-l-1} - \theta^{r-j-1}) = 0,$$

where $\theta = (s+a)/(T+a)$.

Proof. ¿From Lemma 5 (i),

$$h_i^{(r)}(s) \geq 0 \Longrightarrow h_i^{(r)}(s+u) \geq 0, \quad 0 < u \leq T - s.$$

¿From Lemma 5 (ii),

$$h_i^{(r)}(s) \geq 0 \Longrightarrow h_{i+1}^{(r)}(s) \geq 0.$$

Then we have

$$h_i^{(r)}(s) \geq 0 \Longrightarrow h_{i+k}^{(r)}(s+u) \geq 0, \quad 0 < u \leq T - s, \quad k = 1, 2, \cdots.$$

Therefore one-step look-ahead stopping region B_r is "closed" and is optimal stopping region. So the first hitting time to B_r, $\tau_r^* = \min\{s \geq s_i^{(r)*} : (i, s) \in B_r\} = \min\{s \in [\, s_i^{(r)*}, T\,] : X_i = 1\}$ is optimal. Finally, Lemma 5 (iii) shows that $s_i^{(r)*}$ is nonincreasing in i. "#

The last theorem states the behaviour of the asymptotic value of $s_i^{(r)*}$.

Theorem 7

$$\lim_{i\to\infty} s_i^{(r)*} = \left[\frac{T+a}{e} - a\right]^+$$

Proof.

$$h_i^{(r)}(s) = 0 \quad \Longleftrightarrow \quad 1 + \ln\theta = -\sum_{m=0}^{r-2} \binom{i+m-1}{m} C_m^{(r)} \bigg/ \binom{i+r-2}{r-1}$$

When $i \to \infty$, RHS$\to 0$. Since $s_i^{(r)*}$ is uniqe root of the equation $h_i^{(r)}(s) = 0$, the statement holds. "#

It is well-known that the threshold value of the optimal stopping rule for large number of options n in the classical secretary problem is n/e. It is interesting to compare the threshold value n/e with T/e when $a=0$ in the Theorem 7 for all $r=1,2,\cdots$.

Table 2 gives the values of $t_i^{(r)*} \equiv (s_i^{(r)*}+a)/(T+a)$ for $G(r,1/a)$, $r=3,4,5,10$.

Table 2: The values of $t_i^{(r)*}$ for $r=3,4,5,10$

i	1	2	3	4	5	6	7	8
$t_i^{(3)*}$	.600198	.526948	.487532	.463460	.447350	.435847	.427234	.420549
$t_i^{(4)*}$	.663076	.580234	.532041	.501277	.480094	.464665	.452941	.443739
$t_i^{(5)*}$	.709002	.622828	.569494	.534148	.509195	.490694	.476449	.465150
$t_i^{(10)*}$	.827215	.750056	.692738	.649759	.616621	.590372	.569095	.551512

i	9	10	50
$t_i^{(3)*}$	.415212	.410854	.377036
$t_i^{(4)*}$	.436326	.430229	.381515
$t_i^{(5)*}$	.455973	.448374	.385930
$t_i^{(10)*}$	.536744	.524167	.407095

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