CHARACTERIZATIONS OF PSEUDO-BCK ALGEBRAS

YOUNG BAE JUN

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ABSTRACT. A characterization of a pseudo-BCK algebra is provided. Some properties of a pseudo-BCK algebra are investigated. Conditions for a pseudo-BCK algebra to be $\wedge$-semi-lattice ordered (resp. $\cap$-semi-lattice ordered) are given.

1. INTRODUCTION

The study of BCK-algebras was initiated by K. Iséki in 1966 as a generalization of the concept of set-theoretic difference and propositional calculus. G. Georgescu and A. Iorgulescu [1] introduced the notion of a pseudo-BCK algebra as an extension of BCK-algebra. In this paper, we give a characterization of pseudo-BCK algebra, and investigate some properties. We provide conditions for a pseudo-BCK algebra to be $\wedge$-semi-lattice ordered (resp. $\cap$-semi-lattice ordered).

2. PRELIMINARIES

For further details of BCK-algebras we refer to [3]. The notion of pseudo-BCK algebras is introduced by Georgescu and Iorgulescu [1] as follows:

Definition 2.1. A *pseudo-BCK algebra* is a structure $\mathfrak{X} = (X, \leq, *, \diamond, 0)$, where "$\leq$" is a binary relation on X, "$*$" and "$\diamond$" are binary operations on X and "0" is an element of X, verifying the axioms: for all $x, y, z \in X$,

(a1) $(x * y) \diamond (x * z) \leq z * y$, $(x \diamond y) * (x \diamond z) \leq z \diamond y$,
(a2) $x * (x \diamond y) \leq y$, $x \diamond (x * y) \leq y$,
(a3) $x \leq x$,
(a4) $0 \leq x$,
(a5) $x \leq y, \quad y \leq x \Longrightarrow x = y$,
(a6) $x \leq y \Longleftrightarrow x * y = 0 \Longleftrightarrow x \diamond y = 0$.

Remark 2.2. ([1, Remark 1.2]) If $\mathfrak{X}$ is a pseudo-BCK algebra satisfying $x * y = x \diamond y$ for all $x, y \in X$, then $\mathfrak{X}$ is a BCK- algebra.

In a pseudo-BCK algebra we have (see [1])

(p1) $x \leq y \Longrightarrow z * y \leq z * x, \quad z \diamond y \leq z \diamond x$.
(p2) $x \leq y, \quad y \leq z \Longrightarrow x \leq z$.
(p3) $(x * y) \diamond z = (x \diamond z) * y$.
(p4) $x * y \leq z \Longleftrightarrow x \diamond z \leq y$.
(p5) $x * y \leq x, \quad x \diamond y \leq x$.
(p6) $x * 0 = x = x \diamond 0$.
(p7) $x \leq y \Longrightarrow x * z \leq y * z, \quad x \diamond z \leq y \diamond z$.

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(p8) $x \wedge y$ (and $y \wedge x$) is a lower bound for $\{x, y\}$, where $x \wedge y := y \diamond (y * x)$ (and $y \wedge x := x \diamond (x * y)$).
(p9) $x \cap y$ (and $y \cap x$) is a lower bound for $\{x, y\}$ where $x \cap y := y * (y \diamond x)$ (and $y \cap x := x * (x \diamond y)$).

Definition 2.3. ([1, Definition 1.2]) We say that the pseudo-BCK algebra $\mathfrak{X}$ is

- $\wedge$-*semi-lattice ordered* if $x \wedge y = y \wedge x$ for all $x, y \in X$, that is, it satisfies the equality:
$$y \diamond (y * x) = x \diamond (x * y),\ \forall x, y \in X,$$
- $\cap$-*semi-lattice ordered* if $x \cap y = y \cap x$ for all $x, y \in X$, that is, it satisfies the equality:
$$y * (y \diamond x) = x * (x \diamond y),\ \forall x, y \in X,$$
- inf-*semi-lattice ordered* if it is both $\wedge$-semi-lattice ordered and $\cap$-semi-lattice ordered.

3. Characterizations of pseudo-BCK algebras

For any element x of a pseudo-BCK algebra $\mathfrak{X}$, the *initial section* of x is defined to be the set
$$\downarrow x := \{y \in X \mid y \leq x\}.$$

Proposition 3.1. *Let $\mathfrak{X}$ be a pseudo-BCK algebra. For any $x, y \in X$, we have*
$$\downarrow(x \wedge y) \subset \downarrow x \ \cap \downarrow y \text{ and } \downarrow(x \cap y) \subset \downarrow x \ \cap \downarrow y.$$

Proof. If $z \in \downarrow(x \wedge y)$, then $z \leq x \wedge y$. Since $x \wedge y$ is a lower bound for $\{x, y\}$, it follows from (a5) that $z \leq x$ and $z \leq y$ so that $z \in \downarrow x$ and $z \in \downarrow y$, that is, $z \in \downarrow x \ \cap \downarrow y$. Let $w \in \downarrow(x \cap y)$. Then $w \leq x \cap y$. Since $x \cap y$ is a lower bound for $\{x, y\}$, it follows from (a5) that $w \leq x$ and $w \leq y$. Hence $w \in \downarrow x$ and $w \in \downarrow y$, and thus $w \in \downarrow x \ \cap \downarrow y$. This completes the proof. □

Lemma 3.2. ([1, Proposition 1.15]) *Let $\mathfrak{X}$ be a pseudo-BCK algebra.*
(i) *If $\mathfrak{X}$ is $\wedge$-semi-lattice ordered, then $x \wedge y$ is the g.l.b. of $\{x, y\}$ for all $x, y \in X$.*
(ii) *If $\mathfrak{X}$ is $\cap$-semi-lattice ordered, then $x \cap y$ is the g.l.b. of $\{x, y\}$ for all $x, y \in X$.*

Proposition 3.3. *Let $\mathfrak{X}$ be a pseudo-BCK algebra. If $\mathfrak{X}$ is $\wedge$-semi-lattice ordered, then $\downarrow(x \wedge y) = \downarrow x \ \cap \downarrow y$.*

Proof. Let $z \in \downarrow x \ \cap \downarrow y$. Then $z \leq x$ and $z \leq y$. Hence $z \leq x \wedge y$ since $x \wedge y$ is the g.l.b. of $\{x, y\}$ by Lemma 3.2. This implies $z \in \downarrow(x \wedge y)$. Thus $\downarrow x \ \cap \downarrow y \subset \downarrow(x \wedge y)$. Since the reverse inclusion is by Proposition 3.1, we conclude that $\downarrow(x \wedge y) = \downarrow x \ \cap \downarrow y$. □

Proposition 3.4. *Let $\mathfrak{X}$ be a pseudo-BCK algebra such that*
$$\downarrow(x \wedge y) = \downarrow x \ \cap \downarrow y \text{ for all } x, y \in X.$$
Then $\mathfrak{X}$ is $\wedge$-semi-lattice ordered.

Proof. For any $x, y \in X$, we have
$$\downarrow(x \wedge y) = \downarrow x \ \cap \downarrow y = \downarrow y \ \cap \downarrow x = \downarrow(x \wedge y).$$
Hence $x \wedge y \in \downarrow(y \wedge x)$ and $y \wedge x \in \downarrow(x \wedge y)$. Therefore $x \wedge y \leq y \wedge x$ and $y \wedge x \leq x \wedge y$. It follows from (a5) that $x \wedge y = y \wedge x$. This completes the proof. □

Proposition 3.5. *Let $\mathfrak{X}$ be a pseudo-BCK algebra which is $\cap$-semi-lattice ordered. Then $\downarrow(x \cap y) = \downarrow x \ \cap \downarrow y$.*

Proof. Let $w \in \downarrow x \ \cap \downarrow y$. Then $w \leq x$ and $w \leq y$. Since $x \cap y$ is the g.l.b. of $\{x, y\}$, we have $w \leq x \cap y$, that is, $w \in \downarrow(x \cap y)$. Hence $\downarrow x \ \cap \downarrow y \subset \downarrow(x \cap y)$. This completes the proof. □

Proposition 3.6. *Let $\mathfrak{X}$ be a pseudo-BCK algebra. If $\mathfrak{X}$ satisfies the equality*

$$\downarrow(x \cap y) = \downarrow x \ \cap \ \downarrow y \text{ for all } x, y \in X,$$

then $\mathfrak{X}$ is $\cap$-semi-lattice ordered.

Proof. Let $x, y \in X$. Then $\downarrow(x \cap y) = \downarrow x \ \cap \ \downarrow y = \downarrow y \ \cap \ \downarrow x = \downarrow(y \cap x)$, and so $x \cap y \in \downarrow(y \cap x)$ and $y \cap x \in \downarrow(x \cap y)$. Hence $x \cap y \leq y \cap x$ and $y \cap x \leq x \cap y$. Using (a5), we get $x \cap y = y \cap x$. Consequently, $\mathfrak{X}$ is $\cap$-semi-lattice ordered. □

Proposition 3.7. *In any pseudo-BCK algebra we have*

$$x * (y \wedge x) = x * y \text{ and } x \diamond (y \cap x) = x \diamond y.$$

Proof. Note that $x * (y \wedge x) = x * \big(x \diamond (x * y)\big) \leq x * y$ by (a2). Since $y \wedge x \leq y$, it follows from (p1) that $x * y \leq x * (y \wedge x)$. Hence, by (a5), we have $x * (y \wedge x) = x * y$. Now using (a2), we obtain

$$x \diamond (y \cap x) = x \diamond \big(x * (x \diamond y)\big) \leq x \diamond y.$$

The inequality $y \cap x \leq y$ and the condition (p1) imply $x \diamond y \leq x \diamond (y \cap x)$. Therefore $x \diamond y = x \diamond (y \cap x)$ by (a5). This completes the proof. □

We now provide a characterization of a pseudo-BCK algebra.

Theorem 3.8. *A structure $\mathfrak{X} = (X, \leq, *, \diamond, 0)$ is a pseudo-BCK algebra if and only if it satisfies* (a1)*,* (a5)*,* (a6) *and*

(b1) $x * (0 \diamond y) = x = x \diamond (0 * y)$.

Proof. Assume that $\mathfrak{X}$ is a pseudo-BCK algebra. Then $x * (0 \diamond y) \leq x$ and $x \diamond (0 * y) \leq x$. Now $x \diamond \big(x * (0 \diamond y)\big) \leq 0 \diamond y = 0$ and $x * \big(x \diamond (0 * y)\big) \leq 0 * y = 0$, which imply that $x \diamond \big(x * (0 \diamond y)\big) = 0$ and $x * \big(x \diamond (0 * y)\big) = 0$, that is, $x \leq x * (0 \diamond y)$ and $x \leq x \diamond (0 * y)$. Hence, by (a5), we conclude that $x * (0 \diamond y) = x = x \diamond (0 * y)$. Conversely, let $\mathfrak{X} = (X, \leq, *, \diamond, 0)$ be a structure satisfying (a1), (a5), (a6) and (b1). Putting $x = z = 0$ in (a1), we have $(0 * y) \diamond (0 * 0) \leq 0 * y$ and $(0 \diamond y) * (0 \diamond 0) \leq 0 \diamond y$. It follows from (a6) and (b1) that

$$0 = \big((0 * y) \diamond (0 * 0)\big) \diamond (0 * y) = (0 * y) \diamond (0 * 0) = 0 * y \tag{3.1}$$

and

$$0 = \big((0 \diamond y) * (0 \diamond 0)\big) * (0 \diamond y) = (0 \diamond y) * (0 \diamond 0) = 0 \diamond y \tag{3.2}$$

so from (a6) that $0 \leq y$. Combining (3.1), (3.2) and (b1) implies

$$x \diamond 0 = x \diamond (0 * y) = x = x * (0 \diamond y) = x * 0. \tag{3.3}$$

Substituting 0 for y and z in (a1) and using (3.1), (3.2) and (3.3), we obtain

$$x \diamond x = (x * 0) \diamond (x * 0) \leq 0 * 0 = 0$$

and

$$x * x = (x \diamond 0) * (x \diamond 0) \leq 0 \diamond 0 = 0.$$

Since $0 \leq x$ for all $x \in X$, it follows from (a6) that $x \diamond x = 0 = x * x$, that is, $x \leq x$. Replacing y by 0 in (a1) and using (3.3), we get

$$x \diamond (x * z) = (x * 0) \diamond (x * z) \leq z * 0 = z$$

and

$$x * (x \diamond z) = (x \diamond 0) * (x \diamond z) \leq z \diamond 0 = z.$$

Hence the structure $\mathfrak{X}$ is a pseudo-BCK algebra. □

Proposition 3.9. *In any pseudo-BCK algebra $\mathfrak{X}$, we have*

(b2) $(y \wedge x) \diamond (y * x) \leq x \diamond \big(x * (x \wedge y)\big)$.

(b3) $(y \cap x) * (y \diamond x) \leq x * \big(x \diamond (x \cap y)\big)$.

Proof. (b2) For any $x, y \in X$, we have

$$\begin{aligned}
& \big((y \wedge x) \diamond (y * x)\big) * \big(x \diamond (x * (x \wedge y))\big) \\
= \ & \big((x \diamond (x * y)) \diamond (y * x)\big) * \big(x \diamond (x * (y \diamond (y * x)))\big) \\
= \ & \big((x * (x \diamond (x * (y \diamond (y * x))))) \diamond (x * y)\big) \diamond (y * x) \\
= \ & \big((x * (y \diamond (y * x))) \diamond (x * y)\big) \diamond (y * x) \\
\leq \ & \big(y * (y \diamond (y * x)\big) \diamond (y * x) \\
= \ & (y * x) \diamond (y * x) = 0.
\end{aligned}$$

It follows from (a4) and (a5) that

$$\big((y \wedge x) \diamond (y * x)\big) * \big(x \diamond \big(x * (x \wedge y)\big)\big) = 0,$$

that is, $(y \wedge x) \diamond (y * x) \leq x \diamond \big(x * (x \wedge y)\big)$.

(b3) Let $x, y \in X$. Then

$$\begin{aligned}
& \big((y \cap x) * (y \diamond x)\big) \diamond \big(x * (x \diamond (x \cap y))\big) \\
= \ & \big((x * (x \diamond y)) * (y \diamond x)\big) \diamond \big(x * (x \diamond (y * (y \diamond x)))\big) \\
= \ & \big((x \diamond (x * (x \diamond (y * (y \diamond x))))) * (x \diamond y)\big) * (y \diamond x) \\
= \ & \big((x \diamond (y * (y \diamond x))) * (x \diamond y)\big) * (y \diamond x) \\
\leq \ & \big(y \diamond (y * (y \diamond x)\big) * (y \diamond x) \\
= \ & (y \diamond x) * (y \diamond x) = 0.
\end{aligned}$$

Since $0 \leq x$ for all $x \in X$, it follows from (a5) that

$$\big((y \cap x) * (y \diamond x)\big) \diamond \big(x * (x \diamond (x \cap y))\big) = 0$$

so that $(y \cap x) * (y \diamond x) \leq x * \big(x \diamond (x \cap y)\big)$. This completes the proof. □

Definition 3.10. A pseudo-BCK algebra $\mathfrak{X}$ is said to be *positive implicative* if it satisfies

(a7) $(x * z) \diamond (y * z) = (x \diamond y) * z, \ \forall x, y, z \in X$,

(a8) $(x \diamond z) * (y \diamond z) = (x * y) \diamond z, \ \forall x, y, z \in X$,

Proposition 3.11. *If $\mathfrak{X}$ is a positive implicative pseudo-BCK algebra, then $x * y = x \diamond y$ for all $x, y \in X$.*

Proof. For any $x, y \in X$, we have

$$\begin{aligned}
x * y \ &= \ (x * y) \diamond 0 = (x * y) \diamond (y * y) = (x \diamond y) * y \\
&= \ (x * y) \diamond y = (x \diamond y) * (y \diamond y) = (x \diamond y) * 0 = x \diamond y,
\end{aligned}$$

which completes the proof. □

Note from Remark 2.2 and Proposition 3.11 that every positive implicative pseudo-BCK algebra is a positive implicative BCK-algebra. That is, there is no positive implicative pseudo-BCK algebras which are not positive implicative BCK-algebras.

Proposition 3.12. *If $\mathfrak{X}$ is a pseudo-BCK algebra satisfying the following implication*

$$x \leq y \implies x = x \wedge y \quad (\text{resp. } x = x \cap y), \tag{3.4}$$

then $\mathfrak{X}$ is $\wedge$-semi-lattice ordered (resp. $\cap$-semi-lattice ordered).

Proof. Since $x \wedge y \leq x$ for all $x, y \in X$, it follows from (3.4) that $x \wedge y = (x \wedge y) \wedge x$, that is, $y \diamond (y * x) = x \diamond \big(x * (y \diamond (y * x))\big)$ so from (p3), Proposition 3.7 and (a1) that

$$\begin{aligned}
\big(y \diamond (y * x)\big) * \big(x \diamond (x * y)\big) \ &= \ \big(x \diamond (x * (y \diamond (y * x)))\big) * \big(x \diamond (x * y)\big) \\
&= \ \big(x * (x \diamond (x * y))\big) \diamond \big(x * (y \diamond (y * x))\big) \\
&= \ (x * y) \diamond \big(x * (y \diamond (y * x))\big) \\
&\leq \ \big(y \diamond (y * x)\big) * y = 0.
\end{aligned}$$

Hence $y \diamond (y * x) \leq x \diamond (x * y)$ by (a4) and (a5). Since x and y are arbitrarily, we get $y \diamond (y * x) = x \diamond (x * y)$ for all $x, y \in X$. Therefore $\mathfrak{X}$ is $\wedge$-semi-lattice ordered. Next, note that $x \cap y \leq x$ for all $x, y \in X$. Hence, by (3.4), we have $x \cap y = (x \cap y) \cap x$, that is, $y * (y \diamond x) = x * \big(x \diamond (y * (y \diamond x))\big)$. It follows that

$$\begin{aligned}\big(y * (y \diamond x)\big) \diamond \big(x * (x \diamond y)\big) &= \big(x * (x \diamond (y * (y \diamond x)))\big) \diamond \big(x * (x \diamond y)\big)\\ &= \big(x \diamond (x * (x \diamond y))\big) * \big(x \diamond (y * (y \diamond x))\big)\\ &= (x \diamond y) * \big(x \diamond (y * (y \diamond x))\big)\\ &\leq \big(y * (y \diamond x)\big) \diamond y = 0\end{aligned}$$

so that $y * (y \diamond x) \leq x * (x \diamond y)$. The reverse inequality is also valid, because x and y are arbitrarily. Hence $y * (y \diamond x) = x * (x \diamond y)$, that is, $\mathfrak{X}$ is $\cap$-semi-lattice ordered. □

Corollary 3.13. *If $\mathfrak{X}$ is a pseudo-BCK algebra satisfying the following implication*

$$x \leq y \implies x \wedge y = x = x \cap y, \tag{3.5}$$

then $\mathfrak{X}$ is inf*-semi-lattice ordered.*

Proposition 3.14. *If a pseudo-BCK algebra $\mathfrak{X}$ is $\wedge$-semi-lattice ordered, then*

$$x \leq z,\ z * y \leq z * x \implies x \leq y.$$

Proof. Let $x, y, z \in X$ be such that $x \leq z$ and $z * y \leq z * x$. Then $x * z = 0$ and $(z * y) \diamond (z * x) = 0$, and so

$$\begin{aligned}x * y &= (x \diamond 0) * y = \big(x \diamond (x * z)\big) * y\\ &= \big(z \diamond (z * x)\big) * y = (z * y) \diamond (z * x) = 0.\end{aligned}$$

Hence $x \leq y$, ending the proof. □

Proposition 3.15. *If a pseudo-BCK algebra $\mathfrak{X}$ is $\cap$-semi-lattice ordered, then*

$$x \leq z,\ z \diamond y \leq z \diamond x \implies x \leq y.$$

Proof. Let $x, y, z \in X$ be such that $x \leq z$ and $z \diamond y \leq z \diamond x$. Then $x \diamond z = 0$ and $(z \diamond y) * (z \diamond x) = 0$. It follows that

$$\begin{aligned}x \diamond y &= (x * 0) \diamond y = \big(x * (x \diamond z)\big) \diamond y\\ &= \big(z * (z \diamond x)\big) \diamond y = (z \diamond y) * (z \diamond x) = 0\end{aligned}$$

so that $x \leq y$. This completes the proof. □

Proposition 3.16. *If a pseudo-BCK algebra $\mathfrak{X}$ satisfies*

$$x, y \leq z,\ z \diamond y \leq z \diamond x \implies x \leq y, \tag{3.6}$$

then $u = v * (v \diamond u)$ for all $u, v \in X$ with $u \leq v$.

Proof. Let $u, v \in X$ be such that $u \leq v$. Then $v * (v \diamond u) \leq v$ by (p5). Moreover, $v \diamond \big(v * (v \diamond u)\big) \leq v \diamond u$ by (a2). It follows from (3.6) that $u \leq v * (v \diamond u)$. Since $v * (v \diamond u) \leq u$ by (a2), we conclude that $u = v * (v \diamond u)$. □

Proposition 3.17. *Let $\mathfrak{X}$ be a pseudo-BCK algebra such that*

$$x, y \leq z,\ z * y \leq z * x \implies x \leq y. \tag{3.7}$$

Then $u = v \diamond (v * u)$ for all $u, v \in X$ with $u \leq v$.

Proof. Let $u, v \in X$ be such that $u \leq v$. Note from (p5) that $v \diamond (v * u) \leq v$. Since $v * \big(v \diamond (v * u)\big) \leq v * u$ by (a2), it follows from (3.7) that $u \leq v \diamond (v * u)$. Recall that $v \diamond (v * u) \leq u$ by (a2). Hence, by (a5), we have $u = v \diamond (v * u)$. □

Theorem 3.18. *A pseudo-BCK algebra $\mathfrak{X}$ is $\wedge$-semi-lattice ordered if and only if*

$$y \wedge x = y \diamond \big(y * (y \wedge x)\big),\ \forall x, y \in X.$$

Proof. Since $y \wedge x \leq y$ for all $x, y \in X$, the necessity is by Propositions 3.14 and 3.17. Let $\mathfrak{X}$ be a pseudo-BCK algebra which satisfies

$$y \wedge x = y \diamond \big(y * (y \wedge x)\big),\ \forall x, y \in X.$$

For any $x, y \in X$ with $x \leq y$, we have

$$x = x \diamond 0 = x \diamond (x * y) = y \diamond \big(y * (x \diamond (x * y))\big) = y \diamond (y * x) = x \wedge y,$$

and so $\mathfrak{X}$ is $\wedge$-semi-lattice ordered by Proposition 3.12. □

Theorem 3.19. *A pseudo-BCK algebra $\mathfrak{X}$ is $\cap$-semi-lattice ordered if and only if*

$$y \cap x = y * \big(y \diamond (y \cap x)\big),\ \forall x, y \in X. \tag{3.8}$$

Proof. Let $\mathfrak{X}$ be a $\cap$-semi-lattice ordered pseudo-BCK algebra. Using Propositions 3.15 and 3.16, we know that $y \cap x = y * \big(y \diamond (y \cap x)\big)$ for all $x, y \in X$. Conversely, assume that a pseudo-BCK algebra $\mathfrak{X}$ satisfies the condition (3.8). Let $x, y \in X$ be such that $x \leq y$. Then

$$\begin{aligned} x &= x * 0 = x * (x \diamond y) = y \cap x = y * \big(y \diamond (y \cap x)\big) \\ &= y * \big(y \diamond (x * (x \diamond y))\big) = y * (y \diamond x) = x \cap y, \end{aligned}$$

and so $\mathfrak{X}$ is $\cap$-semi-lattice ordered by Proposition 3.12. This completes the proof. □

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Department of Mathematics Education
Gyeongsang National University
Chinju (Jinju) 660-701, Korea

Email address: ybjun@nongae.gsnu.ac.kr