Scientiae Mathematicae Japonicae Online, Vol.**7** (2002), 231–239

THE QUADRATIC MOMENT MATRIX $E(1)$

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Abstract. For the quadratic moment problem $\gamma_{ij} = \int \bar{z}^i z^j \, d\mu(z)$ $(0 \le i + j \le 2, |i - j| \le 1)$, we showed that the necessary and sufficient condition for the existence of representing measure μ is $E(1) \ge 0$. In particular, we also obtained the equivalent conditions for the existence of representing measure supported in the unit circle $\mathbb{T}$ and in the closed unit disc $\mathbb{D}$.

1. Introduction and preliminaries

Given a collection of complex numbers

$$\gamma \equiv \gamma^{(2n)} : \gamma_{00}, \gamma_{01}, \gamma_{10}, \cdots, \gamma_{0,2n}, \gamma_{1,2n-1}, \cdots, \gamma_{2n-1,1}, \gamma_{2n,0} \tag{1}$$

with $\gamma_{00} > 0$ and $\gamma_{ji} = \overline{\gamma_{ij}}$. The *truncated complex moment problem* entails finding a positive Borel measure μ supported in the complex plane $\mathbb{C}$ such that

$$\gamma_{ij} = \int \bar{z}^i z^j \, d\mu(z) \qquad (0 \le i + j \le 2n);$$

and μ is called a *representing measure* for γ. This truncated complex moment problem was first considered by R. Curto and L. Fialkow and has been well-established (cf. [5], [6], [7], [8], [9], [10], [11], [13], [14], [16], [18]).

We recall first some notation from [7] and [8]. For $n \in \mathbb{N}$, let $m \equiv m(n) = (n+1)(n+2)/2$. For $A \in \mathcal{M}_m(\mathbb{C})$ (the $m \times m$ complex matrices), we denote the successive rows and columns according to the following lexicographic-functional ordering:

$$\mathit{1}, Z, \bar{Z}, Z^2, \bar{Z}Z, \bar{Z}^2, \cdots, Z^n, \bar{Z}Z^{n-1}, \cdots, \bar{Z}^{n-1}Z, \bar{Z}^n.$$

The authors in [7] defined the moment matrix $M(n) := M(n)(\gamma) \in \mathcal{M}_m(\mathbb{C})$ as follows: for $0 \le k + l \le n, 0 \le i + j \le n$, the entry in row $\bar{Z}^k Z^l$ and column $\bar{Z}^i Z^j$ is $M(n)_{(k,l)(i,j)} = \gamma_{l+i,j+k}$. These matrices come from Bram-Halmos characterization for a cyclic operator T satisfying $\gamma_{ij} = (T^{*i} T^j x_0, x_0)$, where x_0 is a cyclic vector for T ([3] or [4]). Recently, the authors in [15] considered moment matrices corresponded by Embry characterization for subnormality of such operators ([12]). We will write such matrices by $E(n)$.

Given a collection of complex numbers $\gamma \equiv \{\gamma_{ij}\}$ $(0 \le i + j \le 2n, \ |i - j| \le n)$, with $\gamma_{00} > 0$ and $\gamma_{ji} = \overline{\gamma_{ij}}$. The *truncated complex moment problem* which considered in [15] entails finding a positive Borel measure μ supported in the complex plane $\mathbb{C}$ such that

$$\gamma_{ij} = \int \bar{z}^i z^j \, d\mu(z) \qquad (0 \le i + j \le 2n, \ |i - j| \le n); \tag{2}$$

Key words and phrases. truncated complex moment problem, the quadratic moment problem, representing measure, double flat extension.

2000 Mathematics Subject Classification 44A60, 47B20.

*This research is partially supported by Grant-in-Aid Scientific Research (No.14540190).

μ is called a *representing measure* for γ.

We recall some notation from [15]. For $n \in \mathbb{N}$, we let

$$m = m[n] = \left(\left[\frac{n}{2}\right]+1\right)\left(\left[\frac{n+1}{2}\right]+1\right).$$

For a matrix $A \in \mathcal{M}_m(\mathbb{C})$, we first introduce the following order on the rows and columns of

$$A : 1, Z, Z^2, \bar{Z}Z, Z^3, \bar{Z}Z^2, Z^4, \bar{Z}Z^3, \bar{Z}^2Z^2, Z^5, \cdots.$$

We denote the entry of A in row $\bar{Z}^kZ^l$ and column $\bar{Z}^iZ^j$ by $A_{(k,l)(i,j)}$. If $n = 2k$, $k = 1, 2, \cdots$, let

$$\mathcal{SP}_n = \{p(z,\bar{z}) = a_{00} + a_{01}z + a_{02}z^2 + a_{11}\bar{z}z + a_{03}z^3 + a_{12}\bar{z}z^2 + \cdots + a_{kk}\bar{z}^kz^k\};$$

if $n = 2k+1, k = 0, 1, 2, \cdots$, let

$$\mathcal{SP}_n = \{p(z,\bar{z}) = a_{00} + a_{01}z + a_{02}z^2 + a_{11}\bar{z}z + a_{03}z^3 + a_{12}\bar{z}z^2 + \cdots + a_{k,k+1}\bar{z}^kz^{k+1}\},$$

where $a_{ij} \in \mathbb{C}$. It is clear that $\mathcal{SP}_n$ is a subspace of $\mathcal{P}_n$, the vector space of all complex polynomials in $z, \bar{z}$ of total degree $\leq n$. For $p \in \mathcal{SP}_n$, let $\widehat{p} = (a_{00}, a_{01}, \cdots, a_{kk})^T$ (which means the transposed) or $(a_{00}, a_{01}, \cdots, a_{k,k+1})^T$ in $\mathbb{C}^m$. We define a sesquilinear form $\langle \cdot, \cdot \rangle_A$ on $\mathcal{SP}_n$ by

$$\langle p, q \rangle_A := \langle A\widehat{p}, \widehat{q} \rangle \quad (p, q \in \mathcal{SP}_n).$$

In particular, $\langle \bar{z}^iz^j, \bar{z}^kz^l \rangle_A = A_{(k,l)(i,j)}$, for $0 \leq i+j \leq n, i \leq j$ and $0 \leq k+l \leq n, k \leq l$.

For the truncated complex moment problem (2), we define the moment matrix $E(n) \equiv E(n)(\gamma) \in \mathcal{M}_m(\mathbb{C})$ as follows: $E(n)_{(k,l)(i,j)} := \gamma_{l+i,j+k}$. In $E(n)$, for $0 \leq i+j \leq n, \bar{Z}^iZ^j$ denotes the unique column whose initial element is γ_{ij}. For example, if $n = 1$, i.e., γ : $\gamma_{00}, \gamma_{01}, \gamma_{10}, \gamma_{11}$, the *quadratic* moment matrix is

$$E(1) = \begin{bmatrix} \gamma_{00} & \gamma_{01} \\ \gamma_{10} & \gamma_{11} \end{bmatrix}.$$

If $E(n)$ is a moment matrix with representing measure μ for γ, then by direct computation we have that

$$\langle E(n)\widehat{p}, \widehat{p} \rangle = \int |p(z,\bar{z})|^2 d\mu \quad \text{for } p(z,\bar{z}) \in \mathcal{SP}_n.$$

Thus, we have the following

Theorem 1.1. *If γ admits a representing measure μ, then $E(n) \geq 0$.*

But the converse implication is not always true (cf. [15, Example 3.2]).

For $p \in \mathcal{SP}_n$, let $\mathcal{Z}(p) = \{z \in \mathbb{C} : p(z,\bar{z}) = 0\}$.

Lemma 1.2. ([15, Lemma 3.1]) *Let $\gamma \equiv \{\gamma_{ij}\}$ $(0 \leq i+j \leq 2n,\ |i-j| \leq n)$. Assume that γ has a representing measure μ. For $p \in \mathcal{SP}_n$, supp $\mu \subseteq \mathcal{Z}(p) \Longleftrightarrow p(Z,\bar{Z}) = 0$.*

For example, if we consider the moment problem for

$$E(2) := \begin{bmatrix} 1 & 0 & i & 1 \\ 0 & 1 & 1+i & 1-i \\ -i & 1-i & 3 & -3i \\ 1 & 1+i & 3i & 3 \end{bmatrix}.$$

We note that $Z^2 = i1 + (1+i)Z$ and $\bar{Z}Z = 1 + (1-i)Z$. Thus, if there exists a representing measure μ, then Lemma 1.2 shows that $z^2 = i + (1+i)z, |z|^2 = 1 + (1-i)z$, on supp μ, that is, the atoms are $z_0 = \frac{(1-\sqrt{3})(1+i)}{2}, z_1 = \frac{(1+\sqrt{3})(1+i)}{2}$. It is easy to check that the representing measure $\mu = \frac{(1+\sqrt{3})}{2\sqrt{3}}\delta_{z_0} + \frac{(-1+\sqrt{3})}{2\sqrt{3}}\delta_{z_1}$.

The following conjecture is a core problem in [15].

Conjecture 1.3. ([15, Conjecture 1.2]) Let $\gamma \equiv \{\gamma_{ij}\}$ $(0 \le i+j \le 2n,\ |i-j| \le n)$ be a truncated moment sequence. The following statements are equivalent.

(i) γ has a rank $E(n)$-atomic representing measure;

(ii) $E(n) \ge 0$ and $E(n)$ admits a flat (i.e., rank-preserving) extension $E(n+1)$.

The conjecture doesn't hold in general but for even number n (cf. [15, Example 3.7]). Thus, we give the following theorem in sharpness.

Theorem 1.4. ([15, Theorem 4.1]) *The truncated complex moment sequence* $\gamma \equiv \{\gamma_{ij}\}$ $(0 \le i+j \le 2n, |i-j| \le n)$ *has a rank* $E(n)$*-atomic representing measure if and only if* $E(n) \ge 0$ *and* $E(n)$ *admits a double flat extension* $E(n+2)$, *i.e., rank* $E(n) =$ *rank* $E(n+2)$.

In this paper we answer to the quadratic moment problem, and consider it on the unit circle $\mathbb{T}$ and closed unit disc $\mathbb{D}$.

Acknowledgment. The first-named author is grateful to Research Institute for Mathematical Sciences at Kyoto University and Kanagawa University for their hospitality and support during his visit.

2. The quadratic moment problem

Let $\gamma : \gamma_{00}, \gamma_{01}, \gamma_{10}, \gamma_{11}$ with $\gamma_{00} > 0, \gamma_{10} = \overline{\gamma_{01}}$ and $\gamma_{11} \in \mathbb{R}$. The *quadratic moment problem* entails finding a positive Borel measure μ supported in the complex plane $\mathbb{C}$ such that

$$\gamma_{ij} = \int \bar{z}^i z^j d\mu(z), \qquad (0 \le i+j \le 2,\ |i-j| \le 1).$$

As in section 1, we can obtain the moment matrix

$$E(1) = \begin{bmatrix} \gamma_{00} & \gamma_{01} \\ \gamma_{10} & \gamma_{11} \end{bmatrix}.$$

Let $r = \text{rank } E(1)$. We can obtain the following results similar to [7, Theorem 6.1].

Theorem 2.1. *The following statements are equivalent.*

i) γ *has a representing measure;*

ii) γ *has an* r*-atomic representing measure;*

iii) $E(1) \ge 0$.

In this case, if $r = 1$, *there exists a unique representing measure* $\mu = \gamma_{00}\delta_{\frac{\gamma_{01}}{\gamma_{00}}}$; *if* $r = 2$, *the* 2*-atomic representing measures contain a sub-parameter by a circle.*

Proof. i) $\Rightarrow$ iii): From Theorem 1.1.

ii) $\Rightarrow$ i): Trivial.

iii) $\Rightarrow$ ii): First, if $r=1$, i.e., $\det E(1)=\gamma_{00}\gamma_{11}-\gamma_{01}\gamma_{10}=0$, we claim that $\mu:=\gamma_{00}\delta_{\frac{\gamma_{01}}{\gamma_{00}}}$ is the unique representing measure of γ. In fact,

$$\begin{aligned}
&\int 1d\mu=\gamma_{00};\\
&\int zd\mu=\gamma_{00}\left(\frac{\gamma_{01}}{\gamma_{00}}\right)=\gamma_{01};\\
&\int \bar{z}d\mu=\gamma_{00}\left(\frac{\gamma_{10}}{\gamma_{00}}\right)=\gamma_{10};\\
&\int \bar{z}zd\mu=\gamma_{00}\left(\frac{\gamma_{01}}{\gamma_{00}}\right)\left(\frac{\gamma_{10}}{\gamma_{00}}\right)=\frac{\gamma_{01}\gamma_{10}}{\gamma_{00}}=\gamma_{11}.
\end{aligned}$$

Thus, μ is an 1-atomic representing measure for γ. If ν is any representing measure for γ, then the relation $Z=\alpha\mathit{1}$ and Lemma 1.2 imply supp $\nu=\{\frac{\gamma_{01}}{\gamma_{00}}\}$, whence $\nu=\mu$.

If $r=2$, i.e., $E(1)$ is positive and invertible, then $\delta:=\gamma_{00}\gamma_{11}-\gamma_{01}\gamma_{10}>0$. If a flat extension $E(2)$ of $E(1)$ can be obtained, by [15, Lemma 3.6], $E(2)$ admits a flat extension of the form $E(4)$. Then, by Theorem 1.4, γ has a rank $E(2)$-atomic (i.e., $E(1)$-atomic) representing measure. Therefore we construct a flat extension $E(2)$ of $E(1)$. Let

$$E(2):=\begin{bmatrix}\gamma_{00}&\gamma_{01}&\gamma_{02}&\gamma_{11}\\ \gamma_{10}&\gamma_{11}&\gamma_{12}&\gamma_{21}\\ \gamma_{20}&\gamma_{21}&c_{11}&c_{12}\\ \gamma_{11}&\gamma_{12}&c_{21}&c_{22}\end{bmatrix},$$

where $\gamma_{20}=\overline{\gamma_{02}},\gamma_{21}=\overline{\gamma_{12}},c_{21}=\overline{c_{12}}$. Then by Smul'jan's result (cf. [7, Proposition 2.2]), rank $E(2)=$ rank $E(1)$ if and only if

$$\begin{bmatrix}c_{11}&c_{12}\\ c_{21}&c_{22}\end{bmatrix}=W^*E(1)W,$$

where

$$W=E(1)^{-1}\begin{bmatrix}\gamma_{02}&\gamma_{11}\\ \gamma_{12}&\gamma_{21}\end{bmatrix}.$$

Let

$$\begin{aligned}
\begin{bmatrix}\alpha\\ \beta\end{bmatrix}&=E(1)^{-1}\begin{bmatrix}\gamma_{02}\\ \gamma_{12}\end{bmatrix}=\frac{1}{\delta}\begin{bmatrix}\gamma_{11}\gamma_{02}-\gamma_{01}\gamma_{12}\\ \gamma_{00}\gamma_{12}-\gamma_{10}\gamma_{02}\end{bmatrix},\\
\begin{bmatrix}a\\ b\end{bmatrix}&=E(1)^{-1}\begin{bmatrix}\gamma_{11}\\ \gamma_{21}\end{bmatrix}=\frac{1}{\delta}\begin{bmatrix}\gamma_{11}^2-\gamma_{01}\gamma_{21}\\ \gamma_{00}\gamma_{21}-\gamma_{10}\gamma_{11}\end{bmatrix}.
\end{aligned}$$

Then $E(2)$ will be of the form of a moment matrix if and only if $c_{11}=c_{22}$. That is,

$$\alpha\gamma_{20}+\beta\gamma_{21}=a\gamma_{11}+b\gamma_{12}.$$

It is equivalent to

$$\gamma_{11}^3-\gamma_{11}|\gamma_{02}|^2+2\mathrm{Re}(\gamma_{01}\gamma_{12}\gamma_{20}-\gamma_{01}\gamma_{11}\gamma_{21})=0.$$

Let $\gamma_{12}=0$. Then we have $|\gamma_{02}|=\gamma_{11}$. Therefore, $E(1)$ has a flat extension $E(2)$.

Now we construct a representing measure. Since $Z^2 = \alpha 1 + \beta Z$ and $\bar{Z}Z = a1 + bZ$, Lemma 1.2 implies that the two atoms z_0, z_1 of representing measure are the roots of

$$z^2 - (\alpha + \beta z) = 0, \quad \text{and} \quad \bar{z}z = a + bz.$$

We first show that $z_0 \neq z_1$. If, $z_0 = z_1$, i.e., $Z = z_0 1$, whence rank $E(1) = 1$. This is a contradiction.

Define

$$\mu := \rho_0 \delta_{z_0} + \rho_1 \delta_{z_1},$$

where

$$\rho_0 = \frac{z_1 \gamma_{00} - \gamma_{01}}{z_1 - z_0}, \quad \rho_1 = \frac{\gamma_{01} - z_0 \gamma_{00}}{z_1 - z_0}.$$

Under the assumption of $\rho_0, \rho_1 \in \mathbb{R}$, we next check the moments of μ :

$$\begin{aligned}
\int 1 d\mu &= \rho_0 + \rho_1 = \gamma_{00}; \\
\int z d\mu &= \rho_0 z_0 + \rho_1 z_1 = \frac{z_1 \gamma_{00} - \gamma_{01}}{z_1 - z_0} z_0 + \frac{\gamma_{01} - z_0 \gamma_{00}}{z_1 - z_0} z_1 \\
&= \frac{(z_1 - z_0)\gamma_{01}}{z_1 - z_0} = \gamma_{01}; \\
\int \bar{z} d\mu &= \rho_0 \bar{z}_0 + \rho_1 \bar{z}_1 = \overline{(\rho_0 z_0 + \rho_1 z_1)} = \bar{\gamma}_{01} = \gamma_{10}; \\
\int \bar{z} z d\mu &= \rho_0 |z_0|^2 + \rho_1 |z_1|^2 = \rho_0 (a + b z_0) + \rho_1 (a + b z_1) \\
&= a(\rho_0 + \rho_1) + b(\rho_0 z_0 + \rho_1 z_1) \\
&= a\gamma_{00} + b\gamma_{01} = \gamma_{11}.
\end{aligned}$$

Let

$$f_0(z) = \frac{z - z_1}{z_0 - z_1}, \qquad f_1(z) = \frac{z - z_0}{z_1 - z_0}.$$

Then $f_0(z_0) = 1$, $f_0(z_1) = 0$ and $f_1(z_0) = 0$, $f_1(z_1) = 1$. Since $E(1)$ is positive and invertible,

$$\begin{aligned}
0 < \langle E(1)\widehat{f}_0, \widehat{f}_0 \rangle &= \frac{\gamma_{00}|z_1|^2 - \gamma_{01}\overline{z_1} + \gamma_{11} - z_1 \gamma_{10}}{|z_0 - z_1|^2} \\
&= \int \frac{|z|^2 - z_1 \bar{z} - \overline{z_1} z + |z_1|^2}{|z_0 - z_1|^2} d\mu \\
&= \int |f_0|^2 d\mu = \rho_0.
\end{aligned}$$

Similarly, we have

$$0 < \langle E(1)\widehat{f}_1, \widehat{f}_1 \rangle = \int |f_1|^2 d\mu = \rho_1.$$

Thus μ is a 2-atomic representing measure. □

3. On the unit circle $\mathbb{T}$

In this section, we consider the quadratic moment problem on the unit circle $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ as in [10, Theorem 3.1].

Theorem 3.1. (Unit Circle) *Let $r = rank\ E(1)$. The following statements are equivalent for γ.*

(i) *There exists a representing measure supported in $\mathbb{T}$;*

(ii) *There exists a rank r-atomic representing measure supported in $\mathbb{T}$;*

(iii) $E(1) \geq 0$ *and* $\gamma_{11} = \gamma_{00}$.

Proof. (i) $\Rightarrow$ (iii): From Theorem 1.1 or Theorem 2.1, we have $E(1) \geq 0$, and

$$\gamma_{11} = \int_{\mathbb{T}} \bar{z} z d\mu = \int_{\mathbb{T}} 1 d\mu = \gamma_{00}.$$

(ii) $\Rightarrow$ (i): Trivial.

(iii) $\Rightarrow$ (ii): By Theorem 2.1, we have that there exists an r-atomic representing measure for γ. Therefore our goal is construct an r-atomic representing measure supported in $\mathbb{T}$. Without loss of generality, we let $\gamma_{00} = 1$. .

If $r = 1$, then $|\gamma_{01}|^2 = \gamma_{11} = \gamma_{00} = 1$. So $\mu = \delta_{\gamma_{01}}$ and supp $\mu \subseteq \mathbb{T}$.

If $r = 2$, then $\det E(1) = 1 - |\gamma_{01}|^2 > 0$. Let

$$E(2) = \begin{bmatrix} 1 & \gamma_{01} & \gamma_{02} & 1 \\ \gamma_{10} & 1 & \gamma_{01} & \gamma_{10} \\ \gamma_{20} & \gamma_{10} & c_{11} & c_{12} \\ 1 & \gamma_{01} & c_{21} & c_{22} \end{bmatrix},$$

where $\gamma_{20} = \overline{\gamma_{02}}, c_{21} = \overline{c_{12}}$. Then rank $E(2) =$ rank $E(1)$ if and only if

$$\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} = W^* E(1) W,$$

where

$$W = E(1)^{-1} \begin{bmatrix} \gamma_{02} & 1 \\ \gamma_{01} & \gamma_{10} \end{bmatrix}.$$

Let

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = E(1)^{-1} \begin{bmatrix} \gamma_{02} \\ \gamma_{01} \end{bmatrix} = \frac{1}{1 - |\gamma_{01}|^2} \begin{bmatrix} \gamma_{02} - \gamma_{01}^2 \\ \gamma_{01} - \gamma_{10}\gamma_{02} \end{bmatrix},$$
$$\begin{bmatrix} a \\ b \end{bmatrix} = E(1)^{-1} \begin{bmatrix} 1 \\ \gamma_{10} \end{bmatrix} = \frac{1}{1 - |\gamma_{01}|^2} \begin{bmatrix} 1 - |\gamma_{01}|^2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Then $E(2)$ will be of the form of a moment matrix if and only if $c_{11} = c_{22}$. That is,

$$\alpha\gamma_{20} + \beta\gamma_{10} = 1.$$

It is equivalent to

$$1 - |\gamma_{02}|^2 + 2\mathrm{Re}(\gamma_{01}^2\gamma_{20} - |\gamma_{01}|^2) = 0,$$

i.e.,

$$|\gamma_{02}|^2 - 2\mathrm{Re}(\gamma_{01}^2\gamma_{20}) = 1 - 2|\gamma_{01}|^2.$$

Let $\gamma_{01} = c + di, \gamma_{02} = r + si$. Then

$$(r - c^2 + d^2)^2 + (s - 2cd)^2 = (1 - c^2 - d^2)^2.$$

For each $\gamma_{02} = r + si$, the corresponding 2-atomic representing measure is supported in $\mathbb{T}$, since $\bar{Z}Z = 1$.

Furthermore, the 2-atomic representing measure

$$\mu := \rho_0 \delta_{z_0} + \rho_0 \delta_{z_1},$$

where z_0, z_1 are the roots of $z^2 - (\alpha + \beta z) = 0$ and $\bar{z}z = 1$, and

$$\rho_0 = \frac{z_1 - \gamma_{01}}{z_1 - z_0}, \qquad \rho_1 = \frac{\gamma_{01} - z_0}{z_1 - z_0}.$$

The proof is complete. □

Example 3.2. (Unit Circle) Let

$$E(1) = \begin{bmatrix} 1 & \frac{1+i}{2} \\ \frac{1-i}{2} & 1 \end{bmatrix}.$$

Then we have

$$r^2 + (s - \frac{1}{2})^2 = \frac{1}{4}.$$

$\alpha = -i, \beta = 1 + i$. So, $z^2 - (1+i)z + i = 0$. We obtain the two atoms $z_0 = i, z_1 = 1$, both on the unit circle $\mathbb{T}$. And $\rho_0 = \rho_1 = \frac{1}{2}$. Thus we obtain a 2-atomic representing measure $\mu = \frac{1}{2}\delta_i + \frac{1}{2}\delta_1$.

4. On the unit disc $\mathbb{D}$

In this section, we consider the quadratic moment problem on the unit disc $\mathbb{D} := \{z \in \mathbb{C} : |z| \leq 1\}$ as in [10, Theorem 1.8].

Theorem 4.1. (Unit Disc) *Let $r = rank\ E(1)$. The following statements are equivalent for* γ.

(i) *There exists a representing measure supported in $\mathbb{D}$;*
(ii) *There exists an r-atomic representing measure supported in $\mathbb{D}$;*
(iii) $E(1) \geq 0$ *and* $\gamma_{11} \leq \gamma_{00}$.

Proof. (i) ⇒ (iii): From Theorem 1.1 or Theorem 2.1, we have $E(1) \geq 0$, and

$$\gamma_{11} = \int_{\mathbb{D}} \bar{z}z d\mu \leq \int_{\mathbb{D}} 1 d\mu = \gamma_{00}.$$

(ii) ⇒(i): Trivial.
(iii) ⇒ (ii): Assume $E(1) \geq 0$, and $\gamma_{11} \leq \gamma_{00} = 1$.
If $r = 1$, then $\gamma_{11} = |\gamma_{01}|^2 \leq \gamma_{00} = 1$. So $\mu = \delta_{\gamma_{01}}$.
If $r = 2$, then $\det E(1) = \gamma_{11} - |\gamma_{01}|^2 > 0$.
To find a 2-atomic representing measure $\mu = \rho_0 \delta_{z_0} + \rho_1 \delta_{z_1}$, we shall solve the following equations

$$\rho_0 + \rho_1 = 1,$$
$$\rho_0 z_0 + \rho_1 z_1 = \gamma_{01},$$
$$\rho_0 |z_0|^2 + \rho_1 |z_1|^2 = \gamma_{11}.$$

Let $\rho_0 = \rho$. Then from second equation, we have

$$z_1 = \frac{\gamma_{01} - \rho z_0}{1-\rho}.$$

Take it into third equation, we have

$$(1-\rho)|z_0|^2 - \overline{z_0}\gamma_{01} - z_0\gamma_{10} + \rho|z_0|^2 + |\gamma_{01}|^2 = \frac{1-\rho}{\rho}(\gamma_{11} - |\gamma_{01}|^2),$$

i.e.,

$$|z_0 - \gamma_{01}|^2 = \frac{1-\rho}{\rho}\det E(1).$$

Since $\det E(1) > 0$, we may put

$$t := \sqrt{\frac{1-\rho}{\rho}\det E(1)}, \qquad \Rightarrow \quad \rho = \frac{\det E(1)}{t^2 + \det E(1)}.$$

Let $z_0 = x + yi, \gamma_{01} = a + bi$. Then we have, $(x-a)^2 + (y-b)^2 = t^2$. Thus, $x = a + t\cos\theta, y = b + t\sin\theta$. Hence,

$$\begin{aligned} z_0 &= x + yi = a + t\cos\theta + bi + ti\sin\theta \\ &= \gamma_{01} + te^{i\theta}, \\ z_1 &= \frac{\gamma_{01} - \rho z_0}{1-\rho} = \frac{\gamma_{01} - \rho\gamma_{01} - \rho t e^{i\theta}}{1-\rho} \\ &= \gamma_{01} - \frac{\det E(1)}{t}e^{i\theta}. \end{aligned}$$

We want to make $|z_0| \leq 1$ and $|z_1| \leq 1$.
Since

$$\begin{aligned} |z_0|^2 &= |\gamma_{01}|^2 + 2t\mathrm{Re}(\gamma_{10}e^{i\theta}) + t^2, \\ |z_1|^2 &= |\gamma_{01}|^2 - \frac{2}{t}\det E(1)\mathrm{Re}(\gamma_{10}e^{i\theta}) + \frac{\det E(1)^2}{t^2}. \end{aligned}$$

Hence $|z_0| \leq 1$ and $|z_1| \leq 1$ if and only if $(t,\theta) \in R(t,\theta)$, where

$$R(t,\theta) := \{(t,\theta) \in (R_+, [0,2\pi]) | \quad \begin{array}{c} t^2 + 2t\mathrm{Re}(\gamma_{10}e^{i\theta}) + |\gamma_{01}|^2 \leq 1 \text{ and} \\ |\gamma_{01}|^2 - \frac{2}{t}\det E(1)\mathrm{Re}(\gamma_{10}e^{i\theta}) + \frac{\det E(1)^2}{t^2} \leq 1\}. \end{array}$$

We can show that the set $R(t,\theta)$ is not empty as in the proof of [10, Theorem 1.8]. Thus we can obtain a 2-atomic representing measure. The proof is complete. □

Example 4.2. (Unit Disc) Let $\gamma_{01} = 0$. Then

$$R(t,\theta) := \{(t,\theta) \in (R_+, [0,2\pi]) | \det E(1) \leq t \leq 1\}.$$

Let

$$E(1) = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}.$$

Then

$$R(t,\theta) := \{(t,\theta) \in (R_+, [0,2\pi]) | \frac{1}{2} \le t \le 1\}.$$

Take $t = \frac{2}{3}$. Then $\rho = \frac{9}{17}$, and

$$z_0 = \frac{2}{3}(\cos\theta + i\sin\theta),$$
$$z_1 = -\frac{3}{4}(\cos\theta + i\sin\theta), \quad \forall\theta \in [0,2\pi].$$

Thus, we obtain a 2-atomic representing measure

$$\mu = \frac{9}{17}\delta_{z_0} + \frac{8}{17}\delta_{z_1}.$$

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