FIXED POINT SETS OF NORMAL SELFMAPS

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ABSTRACT. The main result of this note is the following common generalization of a theorem of J.-P. Vigué on selfmaps of bounded convex domains and a classical Hurwitz's theorem: Let X be a complex submanifold of a complex manifold Y and let f_n be a sequence in $\mathcal{H}(X,Y)$ converging to $f \in \mathcal{H}(X,Y)$ with $X \cap \limsup \mathrm{Fix}(f_n) = \emptyset$, where $\mathcal{H}(X,Y)$ represents the space of holomorphic maps from X to Y with the compact-open topology and $\mathrm{Fix}(f)$ denotes the set of fixed points of f. Then either $\mathrm{Fix}(f) = \emptyset$ or $\dim \mathrm{Fix}(f) \geq 1$.

In addition, properties of fixed point sets of selfmaps of bounded convex domains discovered by Vigué - for example, that such sets are holomorphic retracts - are extended to other normal maps.

Introduction.

J.-P. Vigué has discovered new and interesting properties of the set of fixed points of holomorphic selfmaps of bounded convex domains in $\mathbf{C}^n$, the n-dimensional complex plane, producing Theorems A, B, C and D below. The notation $\mathcal{H}(X,Y)$ represents the space of holomorphic maps from a complex space X to a complex space Y with the compact-open topology and $\mathrm{Fix}(f)$ denotes the set of fixed points of $f \in \mathcal{H}(X,X)$.

Theorem A [12, 8]. *Let X be a hyperbolic manifold and let $f \in \mathcal{H}(X,X)$. Then* $\mathrm{Fix}(f)$ *is a submanifold (not necessarily connected) of X. Moreover, if $p \in \mathrm{Fix}(f)$, then*

$$T_p(\mathrm{Fix}(f)) = \{\psi \in T_p(X) : df_p(\psi) = \psi\}$$

where T_p denotes the tangent space at p and df_p is the differential map.

Theorem B [10]. *Let X be a bounded convex domain in $\mathbf{C}^n$. If $f \in \mathcal{H}(X,X)$ and the set of fixed points,* $\mathrm{Fix}(f)$, *is not empty, then* $\mathrm{Fix}(f)$ *is a holomorphic retract.*

Theorem C [11]. *Let X be a bounded convex domain in $\mathbf{C}^n$. If f_n is a sequence in $\mathcal{H}(X,X)$ converging to $f \in \mathcal{H}(X,X)$ such that for some compact $K \subset X$,* $\mathrm{Fix}(f_n) \cap K \neq \emptyset$, *then* $\limsup \dim \mathrm{Fix}(f_n) \leq \dim \mathrm{Fix}(f)$.

Theorem D [11]. *Let X be a bounded convex domain in $\mathbf{C}^n$. Let f_n be a sequence in $\mathcal{H}(X,X)$ converging to $f \in \mathcal{H}(X,X)$ and suppose that the corresponding sequence of fixed points of f_n converges to the boundary of X. Then either the limit map f has no fixed point or* $\mathrm{Fix}(f)$ *is a complex manifold of dimension at least 1.*

Extending W. K. Hayman's notion of a uniformly normal family of functions [5], the authors [6, 9] defined uniformly normal families of holomorphic maps between complex spaces. Singleton uniformly normal families encompass the normal maps previously studied by various authors. Other familiar examples of uniformly normal families are the important families of holomorphic maps into either taut, tautly imbedded, hyperbolic or hyperbolically imbedded spaces. Classical theorems such as the big Picard theorem, Schottky's theorem,

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Lappan's five-point theorem, Lohwater-Pommerenke's theorem and theorems on normal maps by Hahn and Järvi have been extended to uniformly normal families by the authors (see [6, 9]).

In this note a common generalization of Theorem D and Hurwitz's Theorem [3] is established and extensions of Theorems A, B and C to other normal maps are obtained. These are listed below. Recall, if A_n is a sequence of subsets of a topological space X, $\limsup A_n = \{x : x = \lim a_{n_k}, a_{n_k} \in A_{n_k}\}$.

• Let X be a complex submanifold of a complex manifold Y and let f_n be a sequence in $\mathcal{H}(X,Y)$ converging to $f \in \mathcal{H}(X,Y)$ with $X \cap \limsup \mathrm{Fix}(f_n) = \emptyset$. Then either $\mathrm{Fix}(f) = \emptyset$ or $\dim \mathrm{Fix}(f) \geq 1$.

• Let X be a complex subspace of a complex space Y and let $f \in \mathcal{H}(X,X)$ be a normal map in $\mathcal{H}(X,Y)$. Then $\mathrm{Fix}(f)$ is a closed complex subspace (not necessarily connected) of X and is singular only where X is singular. Moreover, if $p \in \mathrm{Fix}(f)$ is regular, then

$$T_p(\mathrm{Fix}(f)) = \{\psi \in T_p(X) : df_p(\psi) = \psi\}$$

where T_p denotes the tangent space at p and df_p is the differential map.

• Let X be a complex space and let f_n be a sequence of normal maps in $\mathcal{H}(X,X)$ converging to a normal map $f \in \mathcal{H}(X,X)$. If for each n all of the connected components of $\mathrm{Fix}(f_n)$ have the same dimension and $\limsup \mathrm{Fix}(f_n) \neq \emptyset$, then $\limsup \dim \mathrm{Fix}(f_n) \leq \dim \mathrm{Fix}(f)$.

• Let f be a holomorphic selfmap of a hyperbolic convex domain X in $\mathbf{C}^n$ with $\mathrm{Fix}(f) \neq \emptyset$. Then each connected component of $\mathrm{Fix}(f)$ is a holomorphic retract. In addition, if X is taut, $\mathrm{Fix}(f)$ is connected and hence is a holomorphic retract.

• Let X be a complex subspace of a complex space Y and let $f \in \mathcal{H}(X,Y)$ be a normal map such that $f(X) \subset X$ with a nonsingular fixed point p where the eigenvalues of df_p are in $\{z \in \mathbf{C} : |z| < 1\} \cup \{1\}$. Then the connected component of $\mathrm{Fix}(f)$ containing p is a holomorphic retract.

In Section 1 some notations and properties of uniformly normal families are presented. In Section 2 the main results about fixed point sets are provided.

1. **Preliminaries**.

Let X, Y and Z be topological spaces. The one-point compactification of the space Y will be denoted by Y^* and $\mathcal{C}(X,Y)$ will denote the space of continuous maps from X to Y with the compact-open topology. If $\mathcal{F} \subset \mathcal{C}(X,Y)$ and $\mathcal{G} \subset \mathcal{C}(Y,Z)$, $\mathcal{G} \circ \mathcal{F} = \{g \circ f : f \in \mathcal{F}, g \in \mathcal{G}\}$. The notation $\overline{A}$ (A') will denote the closure (the derived set) of a subset A of a topological space X and $\Delta = \{z \in \mathbf{C} : |z| < 1\}$.

Definition 1. Let X, Y be complex spaces. The family $\mathcal{F} \subset \mathcal{H}(X,Y)$ is said to be *uniformly normal in* $\mathcal{H}(X,Y)$ (or simply uniformly normal) if $\mathcal{F} \circ \mathcal{H}(\Delta, X)$ is relatively compact in $\mathcal{C}(\Delta, Y^*)$. A map f is simply called *normal* if the singleton set $\{f\}$ is uniformly normal.

A complex space X is *taut* if and only if the family $\mathcal{H}(\Delta, X) \cup \{\infty\}$ is compact in $\mathcal{C}(\Delta, X^*)$ and hence $\mathcal{H}(X,X)$ is uniformly normal if X is taut. M. Abate [2] showed that a complex space X is hyperbolic if and only if the family $\mathcal{H}(\Delta, X)$ is relatively compact in $\mathcal{C}(\Delta, X^*)$, i.e., $\mathcal{H}(\Delta, X)$ is uniformly normal. A complex subspace X of a complex space Y is *tautly imbedded* in Y if and only if the family $\mathcal{H}(\Delta, X)$ is ‘ relatively compact in $\mathcal{H}(\Delta, Y)$ and hence $\mathcal{H}(X,X)$ is a uniformly normal family in $\mathcal{H}(X,Y)$ if X is tautly imbedded in Y. The authors [6] showed that $\mathcal{H}(X,X)$ is a uniformly normal family in $\mathcal{H}(X,Y)$ if and only if X is hyperbolically imbedded in Y. In particular a complex subspace X of a complex space

Y is tautly imbedded in Y if and only if X is relatively compact in Y and hyperbolically imbedded in Y.

Proposition 2 [7]. *Let X, Y be complex spaces. If $\mathcal{F} \subset \mathcal{H}(X,Y)$ is uniformly normal, then $\mathcal{F}$ is relatively compact in $\mathcal{C}(X,Y^*)$.*

Let X be a complex subspace of a complex space Y and let $f \in \mathcal{H}(X,X)$. The notation Γ_f will be used for the sequence f^n of iterates of f where $f^1 = f$, $f^2 = f \circ f$, $\cdots$, and Γ'_f will denote the set of subsequential limits of Γ_f in $\mathcal{C}(X,Y^*)$.

Proposition 3 [7]. *Let X be a complex subspace of a complex space Y and let $f \in \mathcal{H}(X,X)$ be a normal map in $\mathcal{H}(X,Y)$. Then Γ_f is uniformly normal in $\mathcal{H}(X,Y)$ and in particular is relatively compact in $\mathcal{C}(X,Y^*)$.*

The following proposition for a normal selfmap is a local version of a theorem for holomorphic selfmaps of a taut space by Abate [1] and may be proved similarly. Insuring that the compositions of the maps involved are valid is what is essentially needed. A holomorphic retraction of a complex space X is a holomorphic map $\rho \in \mathcal{H}(X,X)$ such that $\rho^2 = \rho$. It was shown in [1] that the image $\rho(X)$ of a holomorphic retraction $\rho \in \mathcal{H}(X,X)$, called a holomorphic retract, is a closed complex subspace of a complex space X and if $p \in \rho(X)$ is a nonsingular point of X, then $\rho(X)$ is also nonsingular at p.

Proposition 4 [7]. *Let X be a complex subspace of a complex space Y. Let $f \in \mathcal{H}(X,X)$ be a normal map in $\mathcal{H}(X,Y)$. Suppose there is a nonempty open subset $U \subset X$ such that $\alpha(U) \subset X$ for $\alpha \in \Gamma'_f$. Then*

(1) There exists a unique map $\rho \in \Gamma'_f$ such that $\rho^2 = \rho$ on U and $\alpha \in \Gamma'_f$ have the form $\alpha = \gamma \circ \rho$ on U where γ is an automorphism of the complex subspace $U_0 = \rho(U)$.

(2) $h(U) = U_0$ for all $h \in \Gamma'_f$.

(3) The restriction of f to U_0, $f|_{U_0}$, is an automorphism of U_0.

The following proposition is also proved in [7].

Proposition 5. *Let X be a complex subspace of a complex space Y let $f \in \mathcal{H}(X,X)$ be a normal map in $\mathcal{H}(X,Y)$. If the set of eigenvalues of df_p are in $\Delta \cup \{1\}$, there is a map $\rho \in \mathcal{C}(X,Y^*)$ such that on the connected component, U, of $\rho^{-1}(X)$ containing p the sequence of iterates f^n converges to ρ and $\rho|_U \in \mathcal{H}(U,U)$ is a holomorphic retraction. Furthermore, if $\Gamma'_f \subset \mathcal{H}(X,Y)$, then f^n converges to a map ρ in $\mathcal{H}(X,Y)$ and $\rho \circ \rho = \rho$ on $\rho^{-1}(X)$.*

2. **Main Results**.

The conclusion of Vigué's Theorem D is the same as that of Hurwitz's Theorem [3] which may be stated as follows: If $X \subset \mathbf{C}$ and f_n is a sequence in $\mathcal{H}(X,\mathbf{C})$ converging to $f \in \mathcal{H}(X,\mathbf{C})$ such that $\mathrm{Fix}(f_n) = \emptyset$, then either $\mathrm{Fix}(f) = \emptyset$ or f is the identity. Our first result is a common generalization of Hurwitz's Theorem and Theorem D. If X is a complex subspace of a complex space Y and $f \in \mathcal{H}(X,Y)$, then $\mathrm{Fix}(f)$ is an analytic subset of Y and its dimension is defined by $\dim \mathrm{Fix}(f) = \max\{\dim_z \mathrm{Fix}(f) : z \in \mathrm{regFix}(f)\}$ where $\mathrm{regFix}(f)$ is the set of regular points of $\mathrm{Fix}(f)$ (see [4]).

Theorem 1. *Let X be a complex submanifold of a complex manifold Y and let f_n be a sequence in $\mathcal{H}(X,Y)$ converging to $f \in \mathcal{H}(X,Y)$ such that the corresponding sequence of fixed points of f_n converges to the boundary of X. Then either $\mathrm{Fix}(f) = \emptyset$ or $\dim \mathrm{Fix}(f) \geq 1$.*

Proof. Suppose $\mathrm{Fix}(f) \neq \emptyset$ and $\dim \mathrm{Fix}(f) = 0$. Let $p \in \mathrm{Fix}(f)$ and for $r > 0$ let $B_r = \{z \in \mathbf{C}^n : \|z\| < r\}$. Real numbers a, b, may be assumed with $0 < b < a$ such that

(i) B_a is a relatively compact local coordinate neighborhood of p with the center at p,

(ii) $B_a \cap \mathrm{Fix}(f) = \{p\}$, $f_n(\overline{B}_b) \subset B_a$, $f(\overline{B}_b) \subset B_a$, and
(iii) $\|f_n - f\| < \delta/2$ on ∂B_b where $\delta = \inf\{\|(f - id)(q)\| : q \in \partial B_b\}$.

Then

$$\|(f_n - id) - (f - id)\| < \|f - id\| \quad \text{on} \quad \partial B_b$$

and hence by an extension of Rouché's theorem [4] the number of zeros counted with multiplicities in B_b of the maps $f_n - id$ and $f - id$ are equal. It follows that

$$\mathrm{Fix}(f_n) \cap B_b \neq \emptyset$$

which is a contradiction. □

The following theorem extends a result of Wavre [8] for a taut complex space.

Theorem 2. *Let X be a complex space with no complex subspaces with positive dimension and let $f \in \mathcal{H}(X, X)$ be a normal map with a relatively compact image. Then f has a unique fixed point $x_o \in X$ and the sequence of iterates of f converges to x_o.*

Proof. Since the sequence of iterates Γ_f is relatively compact in $\mathcal{H}(X, X)$, by Proposition 4 there is a holomorphic retraction $\rho \in \Gamma'_f$. The set $\rho(X)$ is compact connected complex subspace of X and so is a singleton x_o. It follows that $\Gamma'_f = \{\rho\}$. □

Vigué [12, 8] proved the following result when X is a hyperbolic manifold (Theorem A).

Theorem 3. *Let X be a complex subspace of a complex space Y and let $f \in \mathcal{H}(X, X)$ be a normal map in $\mathcal{H}(X, Y)$. Then $\mathrm{Fix}(f)$ is a closed complex subspace (not necessarily connected) of X and is singular only where X is singular. Moreover, if $p \in \mathrm{Fix}(f)$, then*

$$T_p(\mathrm{Fix}(f)) = \{\psi \in T_p(X) : df_p(\psi) = \psi\}$$

where T_p denotes the tangent space at p and df_p, the differential map.

Proof. Let $p \in \mathrm{Fix}(f)$. Since the sequence of iterates Γ_f is relatively compact in $\mathcal{C}(X, Y^*)$, there are relatively compact neighborhoods W, V of p such that V is biholomorphic to a bounded domain in C^n, $\overline{W} \subset V$ and $\alpha(\overline{W}) \subset V$ for $\alpha \in \Gamma'_f$. By Proposition 4, there is a map $\rho \in \Gamma'_f$ such that $\rho \circ \rho = \rho$ on W. Let U be the connected component of $\rho^{-1}(W) \cap W$ containing p. Then U is a hyperbolic space and $\rho \in \mathcal{H}(U, U)$ is a holomorphic retraction. Since $\mathrm{Fix}(f) \cap U \subset \rho(U)$ and $\rho(U)$ is singular only at singular points of U, the proof is reduced to the case where X is a hyperbolic manifold. It is then completed by appeal to Vigué's Theorem A. □

In Theorem C, selfmaps of bounded convex domains in C^n are shown to satisfy the inequality $\limsup \dim \mathrm{Fix}(f_n) \leq \dim\ \mathrm{Fix}(f)$, which is local in nature. It might be suspected from this observation that Theorem C can be generalized to more general selfmaps of more general domains. Theorem 4 is one such generalization. In Theorem 4, it is assumed that, for each map under consideration, all connected components of the fixed point set have the same dimension. This assumption is motivated by the discovery of Vigué that on some domains, all connected components of the fixed point set of a selfmap have the same dimension [12].

Theorem 4. *Let X be a complex space and let f_n be a sequence of normal maps in $\mathcal{H}(X, X)$ converging to a normal map $f \in \mathcal{H}(X, X)$. If for each n all of the connected components of $\mathrm{Fix}(f_n)$ have the same dimension and $\limsup \mathrm{Fix}(f_n) \neq \emptyset$, then $\limsup \dim \mathrm{Fix}(f_n) \leq \dim\ \mathrm{Fix}(f)$.*

Proof. Suppose $p \in \limsup \mathrm{Fix}(f_n)$, that $p_n \in \mathrm{Fix}(f_n)$ and $p_n \to p \in \mathrm{Fix}(f)$. It may be assumed that the points p_n and p are regular. Since by Theorem 3, $\dim \mathrm{Fix}(f_n)$

coinsides with the dimension of the eigenspace of $(df_n)_{p_n}$ corresponding to the eigenvalue 1 and $(df_n)_{p_n} \to (df)_p$, it follows that

$$\dim_{p_n} \mathrm{Fix}(f_n) \;\leq\; \dim_p \mathrm{Fix}(f).$$

□

The following theorem extends Theorem B by Vigué, a result for bounded convex domains.

Theorem 5. *Let X be a convex hyperbolic domain in $\mathbf{C}^n$ and let $f \in \mathcal{H}(X,X)$ with* $\mathrm{Fix}(f) \neq \emptyset$*. Then each connected component of* $\mathrm{Fix}(f)$ *is a holomorphic retract and in particular a complex submanifold of X. If , in addition, X is taut then* $\mathrm{Fix}(f)$ *is connected and hence is a holomorphic retract.*

The following lemma on a property of hyperbolic convex domains will be useful.

Lemma. *Let X be a convex hyperbolic domain in $\mathbf{C}^n$. Then the hyperbolic balls of X (i.e., the balls with respect to the hyperbolic distance k_X) are convex sets.*

Proof. Let $y, z \in B_r(p)$ with $k_X(p,z) \leq k_X(p,y)$ where $B_r(p) = \{x \in X : k_X(p,x) \leq r\}$ for $p \in X,\ r > 0$ and let $\epsilon > 0$. Since X is convex,

$$k_X(x,x') = \lim k_\Delta(0,a_n) \quad \text{for } x,x' \in X$$

where $a_n \in \Delta,\ \ g_n(0) = x,\ \ g_n(a_n) = x',\ \ g_n \in \mathcal{H}(\Delta, X)$.

It follows that there exist maps $f, g \in \mathcal{H}(\Delta, X)$ and reals $a,\ b$ such that

(i) $0 < b \leq a < 1$,

(ii) $f(0) = g(0) = p,\ \ f(a) = y,\ \ g(b) = z$, and

(iii) $k_\Delta(0,a) < k_X(p,y) + \epsilon,\ \ $ and $k_\Delta(0,b) < k_X(p,z) + \epsilon$.

Define $h_t \in \mathcal{H}(\Delta, X)$, $0 \leq t \leq 1$, by

$$h_t(w) = tf(w) + (1-t)g(bw/a), \quad w \in \Delta.$$

Then

$$k_X(p, ty + (1-t)z) = k_X(p, h_t(a)) \leq k_\Delta(0,a) < k_X(p,y) + \epsilon.$$

Hence B_r is convex. □

Proof of Theorem 5. Let $\mathrm{Fix}_c(f)$ denote a nonempty connected component of $\mathrm{Fix}(f)$. For each $p \in \mathrm{Fix}_c(f)$ choose a relatively compact open neighborhood W_p of p such that

(i) W_p is a hyperbolic ball centered at p and

(ii) $W_p \cap \mathrm{Fix}(f) \subset \mathrm{Fix}_c(f)$.

Then $f(\overline{W}_p) \subset \overline{W}_p$ and $\alpha(\overline{W}_p) \subset \overline{W}_p$ for $\alpha \in \Gamma_f$. Let $W = \cup_{p \in \mathrm{Fix}_c(f)} W_p$. Define, as in Vigué [10], a sequence of maps ϕ_k in $\mathcal{H}(X,X)$ by

$$\phi_k = \frac{1}{k}(id + f + f^2 + \cdots + f^{k-1})$$

where id denotes the identity map. Then a subsequence ϕ_{k_j} converges to a limit $\phi \in \mathcal{C}(X, X^*)$ with $\phi(\overline{W}) \subset \overline{W}$. Each eigenvalue of $d\phi_p$ at $p \in \mathrm{Fix}_c(f)$ is either 1 or 0 and , by proposition 5, the sequence of iterates ϕ^n in $\mathcal{H}(W,X)$ converges to $\rho \in \mathcal{H}(W,X)$ which is a holomorphic retraction on $V = \rho^{-1}(W) \cap W$. The first conclusion follows from

$$\mathrm{Fix}_c(f) = \mathrm{Fix}_c(f) \cap W = \rho(W).$$

The last conclusion follows from the observation that if X is taut convex, ρ is a holomorphic retraction on X and $\mathrm{Fix}(f) = \rho(X)$. □

From Theorem 5, $\mathrm{Fix}(f)$ is connected for $f \in \mathcal{H}(X,X)$ when X is a taut convex domain in $\mathbf{C}^n$. In the light of this observation, Corollary 1 derives from Theorem 4.

Corollary 1. *Let X be a taut convex domain in $\mathbf{C}^n$, and let f_n be a sequence in $\mathcal{H}(X,X)$ converging to $f \in \mathcal{H}(X,X)$. If $X \cap \limsup \mathrm{Fix}(f_n) \neq \emptyset$, then $\limsup \dim \mathrm{Fix}(f_n) \leq \dim \mathrm{Fix}(f)$.*

Theorem 6 and Corollary 2 show that fixed point sets of other normal maps satisfy similar properties to a property enjoyed by fixed point sets of selfmaps on bounded domains in C^n,

Theorem 6. *Let X be a complex subspace of a complex space Y and let $f \in \mathcal{H}(X,X)$ be a normal map in $\mathcal{H}(X,Y)$. Suppose that f has a nonsingular fixed point p where the set of eigenvalues of the differential map df_p are in $\Delta \cup \{1\}$. Then the connected component of $\mathrm{Fix}(f)$ containing p is a holomorphic retract. If, in addition, $\Gamma'_f \subset \mathcal{H}(X,Y)$, then each nonempty connected component of $\mathrm{Fix}(f)$ is a holomorphic retract and if $\Gamma'_f \subset \mathcal{H}(X,X)$ then $\mathrm{Fix}(f)$ is a holomorphic retract.*

Proof. Let $\mathrm{Fix}_c(f)$ denote the connected component of $\mathrm{Fix}(f)$ containing p. By Proposition 5 the sequence of iterates Γ_f converges to a holomorphic retraction ρ on U, the connected component of $\rho^{-1}(X)$ containing p and $\rho(U) = \mathrm{Fix}_c(f)$. If $\Gamma'_f \subset \mathcal{H}(X,Y)$, then $\{f^n\}$ converges to a map $\rho \in \mathcal{H}(X,Y)$ and the map ρ satisfies $\rho \circ \rho = \rho$ on $\rho^{-1}(X)$ which is not empty and $\rho(\rho^{-1}(X)) = \mathrm{Fix}_c(f)$. Finally if $\Gamma'_f \subset \mathcal{H}(X,X)$ then ρ is a holomorphic retraction on X and $\rho(X) = \mathrm{Fix}(f)$. □

Corollary 2 follows from the fact that $\Gamma'_f \subset \mathcal{H}(X,Y)$ when X is tautly imbedded and $\Gamma'_f \subset \mathcal{H}(X,X)$ when X is taut.

Corollary 2. *Let X be a complex subspace tautly imbedded in a complex space Y. Suppose that f is a holomorphic selfmap of X with a nonsingular fixed point p where the set of eigenvalues of the differential map df_p are in $\Delta \cup \{1\}$. Then each connected component of $\mathrm{Fix}(f)$ is a holomorphic retract. If X is assumed to be taut instead of tautly imbedded, then $\mathrm{Fix}(f)$ is a holomorphic retract.*

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