ON LOCAL TOPOLOGICAL ALGEBRAS

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Abstract. Every unital regular uniform complete locally m-convex algebra is local.

1 Introduction The purpose of this note is to prove the *local theorem* (cf. Definition 2.1) for uniform algebras. This theorem was obtained in the case of unital commutative regular semi-simple Banach algebras (see e.g. [4: p. 191, Theorem 1]), in which case the spectrum is compact and all (open) covers reduce to finite ones. Here finite partitions of unity suffice. The proof is still valid for suitable topological algebras with compact spectrum; cf. A. Mallios [7: p. 307, Lemma 2.1]. The same result is obtained for F-algebras, by R.M. Brooks [3: p. 271, Theorem 2.6]. In that case the spectrum is Lindelöf, and all covers reduce to countable ones, hence countable partitions of unity suffice [3: p. 271, Theorem 2.5]. However, this is not always the case for uniform algebras, since the spectrum is not, in general, Lindelöf. Now, our crucial point here is to prove that *the property of being the algebra local is preserved by projective limits, without employing partitions of unity*, that was actually of importance in the previous cases.

2 Preliminaries Let $(E, \Gamma \equiv (p_\alpha)_{\alpha \in I})$ be a *complete locally m-convex algebra*, where p_α, $\alpha \in I$, is a family of submultiplicative semi-norms (see E.A. Michael [8: p. 20, Theorem 5.1], or even A. Mallios [6: p. 88, Theorem 3.1]). Then, one considers the so-called *Arens-Michael decomposition* of E, such that one has;

$$E = \varprojlim \hat{E}_\alpha, \tag{2.1}$$

within a topological algebra isomorphism (ibid).

Definition 2.1. [6: p. 348, Theorem 5.1]. Let E be a topological algebra with spectrum $\mathcal{M}(E)$. We shall say that E is a *local algebra*, if every continuous $\mathbb{C}$-valued function α on $\mathcal{M}(E)$, which locally belongs to $E^\wedge$ (i.e., for every $f \in \mathcal{M}(E)$, there exists a neighborhood U of f and an element $x \in E$, with $\alpha \mid_U = \hat{x} \mid_U$), is realized by an elements of E (viz. there exists an elements $y \in E$, with $\alpha = \hat{y}$ on $\mathcal{M}(E)$).

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Definition 2.2. [6: p. 274, Definition 5.1]. Let $(E, \Gamma \equiv (p_\alpha)_{\alpha \in I})$ be a (Hausdorff) locally m-convex algebra, such that $p_\alpha(x^2) = p_\alpha(x^2)$, $x \in E$, for every $\alpha \in I$. Then, we say that E is a *uniform algebra*.

3 Main Results

Theorem 1. *Let $(E_\alpha, \pi_{\alpha\beta})_{\alpha \in I})$ be a strictly dense projective system* [6: p. 174] *of local semi-simple topological algebras. Then $E = \varprojlim E_\alpha$ is local.*

Proof. For $\alpha \leq \beta$, let $\pi_{\alpha\beta} : E_\beta \to E_\alpha$ and $\pi_\alpha : E \to E_\alpha$ be the canonical maps. We consider the maps

$$\rho_{\alpha\beta} = {}^t\pi : \mathcal{M}(E_\beta) \to \mathcal{M}(E_\alpha) \ \text{ and } \ \rho_\alpha = {}^t\pi_\alpha : \mathcal{M}(E_\alpha) \to \mathcal{M}(E).$$

Let $\phi : \mathcal{M}(E) \to \mathbb{C}$ be a continuous $\mathbb{C}$-*valued* function on $\mathcal{M}(E)$ which locally belongs to $E^\wedge = \mathcal{G}(E)(\mathcal{G} : E \to \mathcal{C}(\mathcal{M}(E))$, the corresponding *Gel'fand map of* E). Our aim is to show that ϕ globally belongs to $E^\wedge$ (i.e., there exists $x \in E$, such that $\phi = \hat{x}$). For $\alpha \in I$, let $\phi_\alpha = \phi \circ \rho_\alpha : \mathcal{M}(E_\alpha) \to \mathbb{C}$. Then ϕ_α locally belongs to $E_\alpha^\wedge$: Indeed, let $f \in \mathcal{M}(E_\alpha)$, so that $\rho_\alpha(f) \in \mathcal{M}(E)$. Now, since, by hypothesis, ϕ locally belongs to $E^\wedge$, there exists U neighborhood of $\rho_\alpha(f)$, and $x \in E$, such that $\phi \mid_U = \hat{x} \mid_U$. Moreover since ρ_α is continuous, there exists a neighborhood V of $f \in \mathcal{M}(E_\alpha)$, such that $\rho_\alpha(V) \subset U$. Therefore,

$$\phi \mid_{\rho_\alpha(V)} = \hat{x} \mid_{\rho_\alpha(V)},$$

$$\phi \circ \rho_\alpha \mid_V = \hat{x} \circ \rho_\alpha \mid_V .$$

On the other hand, $\hat{x} \circ \rho_\alpha = \widehat{\pi_\alpha(x)} \mid_V$, $\alpha \in I$. Hence,

$$\phi_\alpha \mid_V = \hat{\pi}_\alpha(x) \mid_V, \quad \text{with} \quad \pi_\alpha(x) \in E_\alpha,$$

which gives that ϕ_α *locally belongs to* E_α, for every $\alpha \in I$. Consequently, ϕ_α *globally belongs to* E_α, for, by hypothesis, E_α is local, so that there exists $x_\alpha \in E_\alpha$, such that $\phi_\alpha = \hat{x}_\alpha$. Thus, by considering the element $x = (x_\alpha)_{\alpha \in I} \in \prod_{\alpha \in I} E_\alpha$, we actually obtain:

i) $x \in E = \varprojlim E_\alpha$, and ii) $\phi = \hat{x}$.

Indeed, concerning i), what we have to do is to prove that;

$$\pi_{\alpha\beta}(x_\beta) = x_\alpha, \quad \text{for} \quad \alpha \leq \beta, \quad \text{with} \quad (\alpha, \beta) \in I^2.$$

First, we have $\rho_\alpha = \rho_\beta \circ \rho_{\alpha\beta}$, so that

$$\phi \circ \rho_\alpha = \phi \circ \rho_\beta \circ \rho_{\alpha\beta},$$

hence, $\phi_\alpha = \phi_\beta \circ \rho_{\alpha\beta}$, that is, $\hat{x}_\alpha = \hat{x}_\beta \circ \rho_{\alpha\beta}$, then, $\hat{x}_\alpha = \hat{x}_\beta \circ^t \pi_{\alpha\beta} = \widehat{\pi_{\alpha\beta}(x_\beta)}$. Now, since E_α is, by hypothesis, semi-simple, one has

$$\pi_{\alpha\beta}(x_\beta) = x_\alpha, \quad \text{for} \quad \alpha \leq \beta,$$

that is, $x \in E$, as desired.

ii) On the other hand, we have $\mathcal{M}(E) = \varprojlim \mathcal{M}(E_\alpha)$ [6: p. 175, Lemma 7.1], hence,

$$\mathcal{M}(E) = \bigcup_{i \in I} \rho_\alpha(\mathcal{M}(E_\alpha)).$$

Therefore, it suffices to verify that:

$$\phi = \hat{x}, \quad \text{when restricted to each} \quad \rho_\alpha(\mathcal{M}(E_\alpha)), \ \alpha \in I.$$

Indeed, one has

$$\phi \circ \rho_\alpha = \widehat{\pi_\alpha(x_\alpha)} = \hat{x}_\alpha,$$

as well as, according to the preceding,

$$\phi \circ \rho_\alpha = \phi_\alpha = \hat{x}_\alpha, \quad \alpha \in I.$$

Therefore, one obtains

$$\phi \circ \rho_\alpha = \hat{x} \circ \rho_\alpha,$$

that is,

$$\phi(\rho_\alpha(\mathcal{M}(E_\alpha))) = \hat{x}(\rho_\alpha(\mathcal{M}(E_\alpha))),$$

which actually proves the assertion. ■

As a consequence of the preceding, we come now to our second main result, as stated in the Abstract. That is, we have.

Theorem 2. *Let E be a unital regular uniform complete locally m-convex algebra. Then, E is local.*

Proof. By considering an *Arens Michael decomposition* of E, say $(\hat{E}_\alpha)_{\alpha \in I}$, one obtains, $E = \varprojlim \hat{E}_\alpha$, as a *strictly dense projective system of Banach algebras* (cf., for instance, [6: p. 176; (7.18)]). Farthermore, due to our hypothesis for E, $\hat{E}_\alpha$ is *semi-simple*; indeed, the norm on $\hat{E}_\alpha$ is given by $\| (\rho_\alpha(x) \|_\alpha = p_\alpha(x)$, so that one has $p_\alpha(x^2)$, hence $\| (\rho_\alpha(x))^2 \|_\alpha = (\| \rho_\alpha(x) \|_\alpha)^2$, for every $x \in E$ (see [6: p. 93; (4.6), as well as, p. 274; (5.1)]). This implies that each one of the algebra $\hat{E}_\alpha$, $\alpha \in I$, is semi-simple (ibid., p. 275, Lemma 5.1). On the other hand, *since* E *is regular,* $\hat{E}_\alpha$ *is regular* (loc. cit., p. 339, Corollary 4.1 and p. 150, Corollary 2.1), therefore, the assertion, according to the previous theorem. ■

Note. The final step in the proof of our argument above that *"E regular implies $\hat{E}_\alpha$ regular"* follows, in effect, from the next more general result (A. Mallios): *The completion of a regular topological algebra with locally equicontinuous spectrum is still regular.* See also A. Mallios [6: p. 150, Theorem 2.1, along with p. 146, Lemma 2.2].

Remarks 1.– i) It has been proved in [6: p. 279, Corollary 5.2] that a semi-simple Michael algebra [6: p. 269, Definition 3.4] is a uniform algebra. Hence, *a unital semi-simple regular complete Michael algebra is local.*

ii) On the other hand, according to [6: p. 271, Theorem 4.1] a Warner algebra (ibid. p. 271, Definition 4.1) with continuous Gel'fand map is a uniform algebra. Now, a Fréchet algebra is Mackey (see e.g. [5: p. 218, Proposition 8]). Thus, *a Fréchet algebra is a Warner algebra, having continuous Gel'fand map* (see also [6: p. 183, Corollary 1.1]), *hence, a uniform algebra.* Consequently, one gets the following.

Corollary 1. *A semi-simple regular Fréchet algebra is local.*

Remark 2. The previous result follows also from [3: p. 271, Theorem 2.6]. However, the proof there is based on the existence of partitions of unity, something that we do not apply here.

We give an example of a unital regular uniform complete locally m-convex algebra, which is not Fréchet.

Example.– Let X be a completely regular k-space and $\mathcal{C}_c(X)$ the locally m-convex algebra of complex-valued continuous function on X, endowed with the compact-open topology [7: p. 19, Example 3.1]. So $\mathcal{C}_c(X)$ is a unital commutative complete uniform (lmc) algebra [7: p. 276, Theorem 5.1], then local (see Theorem 2, as above). On the other hand, according to Warner's Theorem [9: p. 267, Theorem 2], choosing X not semi-compact, we have the algebra $\mathcal{C}_c(X)$ not Fréchet, hence, the result of R.M. Brooks [3: p. 271, Theorem 2.6] is not valid here.

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