ON MINIMAL OPEN REFINEMENTS

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Abstract. It is proved first in this paper that all weakly $\kappa\overline{\theta}$-refinable spaces which were defined recently in [4] are irreducible for any infinite cardinal κ, i.e. any open cover of such spaces has a minimal open refinement. The special case $\kappa = \aleph_0$ has been proved before by J.C.Smith. A generalization of weakly $\kappa\overline{\theta}$-refinable and weakly $\overline{\delta\theta}$-refinable spaces is defined as weakly $\overline{\delta_\kappa\theta}$-refinable and it is proved for $\kappa = \aleph_\alpha$ that any $\aleph_{\alpha+1}$-compact, weakly $\overline{\delta_\kappa\theta}$-refinable space has the Lindelöf number $\leq \aleph_\alpha$. Thus it is shown as a corollary that a regular, perfect, $\aleph_1$-compact T_1 space is hereditarily paracompact if it is weakly $\delta\theta$-refinable.

0. Introduction and Definitions

Let X be a topological space without any separation axiom. If $\mathcal{G}$ is a family of nonempty open subsets of X which is not necessarily a cover, then the set $\{x \in X : n < \mathrm{ord}(x, \mathcal{G})\}$ may be empty but is always open and thus $\{x \in X : \mathrm{ord}(x, \mathcal{G}) \leq n\}$ is closed for each $n \in \mathbb{N}$. As is well known $\mathrm{ord}(x, \mathcal{G})$ denotes the cardinality of the subfamily $\{G \in \mathcal{G} : x \in G\}$. For any family $\mathcal{A}$ of subsets of X we briefly write $\bigcup \mathcal{A}$ instead of $\bigcup\{A : A \in \mathcal{A}\}$. We also write $\mathcal{A}(B)$ in this paper exclusively for the family $\{A \cap B : A \in \mathcal{A}\}$ for any subset B of X. A sequence $\{\mathcal{G}_n\}_{n=1}^{\infty}$ of open covers of X is called a **θ-cover** iff for each $x \in X$ there exists an $n_x \in \mathbb{N}$ such that $\mathrm{ord}(x, \mathcal{G}_{n_x}) < \omega_0$. The space X is called **θ-refinable** iff each open cover of X has a θ-cover refinement. The property of θ-refinability is one of the most natural generalizations of metacompactness and subparacompactness, the two most widely known generalized covering properties after paracompactness. This concept has been defined in 1965 by Worrell and Wicke in [9]. They proved some interesting characterizations in that paper including I) A topological space is paracompact and T_2 iff it is collectionwise normal, θ-refinable and T_1; II) A topological space is developable iff it is essentially T_1 (i.e. the closures of any two singletons is either equal or disjoint), θ-refinable and has a base of countable order. Spaces that are θ-refinable are also known as **submetacompact**.

A sequence $\{\mathcal{G}_n\}_{n=1}^{\infty}$ of open families (which are not necessarily covers) of X on the other hand is called a **weak θ-cover** (resp. **weak $\delta\theta$-cover**) iff 1) the family $\mathcal{G} = \bigcup_{n=1}^{\infty} \mathcal{G}_n$ is an open cover of X and 2) for each $x \in X$ there exists an $n_x \in \mathbb{N}$ such that $0 < \mathrm{ord}(x, \mathcal{G}_{n_x}) < \omega_0$ (resp. $0 < \mathrm{ord}(x, \mathcal{G}_{n_x}) \leq \omega_0$). This sequence is called a **weak $\overline{\theta}$-cover** (resp. **weak $\overline{\delta\theta}$-cover**) iff the following extra condition holds : 3) The countable open cover $\{\bigcup \mathcal{G}_n\}_{n=1}^{\infty}$ is point-finite. **Weakly θ-refinable** , **weakly $\delta\theta$-refinable** , **weakly $\overline{\theta}$-refinable** and **weakly $\overline{\delta\theta}$-refinable** spaces can be defined similarly. Worrell and Wicke have proved in 1976 that III) Any weak $\delta\theta$-cover of a countably compact space has a finite subcover, see [8]. Thus they have obtained the following strong generalization of the well known theorem of Arens and Dugundji : IV) A space is compact iff it is countably compact and weakly $\delta\theta$-refinable. Weakly θ-refinable (resp. weakly $\overline{\theta}$-refinable and weakly $\overline{\delta\theta}$-refinable) spaces were defined by Bennett and Lutzer (resp. by J.C. Smith) in the paper [1] (resp. [6]). As

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is well known an open cover $\mathcal{U}$ of a topological space X is called **minimal** iff $X \neq \bigcup \mathcal{U}'$ holds for any proper subfamily $\mathcal{U}'$ of $\mathcal{U}$. Bennett and Lutzer have proved in [1] that V) Quasi-developable spaces are weakly θ-refinable and Smith proved on the other hand in [6] that VI) Any open cover of a weakly $\overline{\theta}$-refinable space has a minimal open refinement, a property which is called **irreducibility** by J.Boone in his paper [2].

Let κ be an infinite cardinal number. An open cover $\mathcal{G}$ of X is called a **weak $\kappa\overline{\theta}$-cover** if $\mathcal{G}$ can be written as $\mathcal{G} = \bigcup_{\alpha<\kappa} \mathcal{G}_\alpha$ so that the following two conditions hold: i) $\forall x \in X,\ \exists \alpha_x < \kappa,\ 0 < \mathrm{ord}(x, \mathcal{G}_{\alpha_x}) < \omega_0$ and ii) $\{\bigcup \mathcal{G}_\alpha\}_{\alpha<\kappa}$ is point-finite cover in X. This type of covers and **weakly $\kappa\overline{\theta}$-refinable** spaces are constructed recently in [4]. S.Fast and J.C.Smith have defined on the other hand in 1995 a weakly $\overline{\theta}$-refinable normal T_2 space which is not $B(D,\omega_0)$-refinable, where a topological space X is called **$B(D,\lambda)$-refinable** for any infinite ordinal number λ iff every open cover of X has a refinement $\mathcal{K} = \bigcup_{\alpha<\lambda} \mathcal{K}_\alpha$ such that i) each $\bigcup_{\beta<\alpha}(\bigcup \mathcal{K}_\beta)$ is closed for any $\alpha < \lambda$ and ii) each $\mathcal{K}_\alpha$ is relatively a closed-discrete family in the open subspace $X_\alpha = X - \bigcup_{\beta<\alpha}(\bigcup \mathcal{K}_\beta)$ $(\alpha < \lambda)$, see [5]. D denotes the relative discreteness of each $\mathcal{K}_\alpha$ in X_α in the symbol $B(D,\lambda)$. This concept has been defined in 1980 by J.C.Smith [7], after the joint paper of J.Chaber and H.Junnila [3]. He proved among several results in [7] that every $B(D,\omega_0)$-refinable space is weakly $\overline{\theta}$-refinable. If one writes closed-locally finite instead of closed-discrete in condition ii) above then one gets the weaker concept **$B(LF,\lambda)$-refinability**. Several related papers on irreducibility can also be found in the reference list of [7].

The aim of this paper is expressed in the abstract.

1. Results on irreducibility

We prove in this section that every open cover of any weak $\kappa\overline{\theta}$-refinable space has a minimal open refinement whatever the infinite cardinal number κ is, i.e. these spaces are irreducible for any infinite κ. We need first some preparatory propositions and lemmas. A family $\mathcal{A}$ of subsets of X is called **κ-discrete** if it can be written as the union $\bigcup_{\alpha<\kappa} \mathcal{A}_\alpha$ of κ many discrete families of X. An **F_κ-subset** of X is a subset which can be written as the union κ number of closed subsets of X. The dual concept of G_κ-subsets can be defined similarly. X is called **κ-perfect** iff each open set in X is an F_κ-set.

We start with the following basic results:

Lemma 1 (J.C.Smith, [6]) : *If $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$ is a minimal open cover of X then there exists a discrete family $\mathcal{K} = \{K_\alpha : \alpha \in \Lambda\}$ of non-empty closed subsets of X such that $K_\alpha \subseteq U_\alpha - \bigcup_{\beta\neq\alpha} U_\beta$ for each $\alpha \in \Lambda$.*

Proof: Straightforward.

Proposition 1 : *Let $F = \bigcup_{\alpha<\kappa} K_\alpha$ be an F_κ-subset of X such that each union $\bigcup_{\beta<\alpha} K_\beta$ is closed for each $\alpha < \kappa$. If there exists an open minimal cover $\mathcal{U}_\alpha$ (in X) of $K_\alpha - \bigcup_{\beta<\alpha}(\bigcup \mathcal{U}_\beta)$ for each $\alpha < \kappa$ then $\mathcal{U} = \bigcup_{\alpha<\kappa} \mathcal{U}_\alpha$ has a minimal open refinement in X covering F.*

Proof: This is nothing but Lemma 3.3 in [7].

Corollary 1 (J.C.Smith, [6]) : *Let $F = \bigcup_{n=1}^{\infty} K_n$ be an F_σ subset of X and let $\{\mathcal{U}_n\}_{n=1}^{\infty}$ be a sequence of open families of X such that each $\mathcal{U}_n$ is a minimal cover of $K_n - \bigcup_{k<n}(\bigcup \mathcal{U}_k)$. Then $\mathcal{U} = \bigcup_{n=1}^{\infty} \mathcal{U}_n$ has an open refinement in X covers F minimally.*

Corollary 2 : *Let $F_1 \subseteq F_2$ and F_i $(i = 1,2)$ be closed and let $\mathcal{U}_1$ (resp. $\mathcal{U}_2$) be an open family in X covering F_1 (resp. $F_2 - \bigcup \mathcal{U}_1$) minimally. Then $\mathcal{U}_1 \cup \mathcal{U}_2$ has an open refinement in X covers F_2 minimally.*

Proposition 2 : *Let $\mathcal{G}$ be a collection of open sets in X. Suppose there exists a κ-discrete collection $\mathcal{K} = \bigcup_{\alpha<\kappa} \mathcal{K}_\alpha$ of nonempty closed subsets of X such that $\mathcal{K}$ refines $\mathcal{G}$ and the union $\bigcup_{\beta<\alpha}(\bigcup \mathcal{K}_\beta)$ is closed for each $\alpha < \kappa$. Then there exists an open refinement $\mathcal{U}$ of $\mathcal{G}$ which covers the set $\bigcup \mathcal{K}$ minimally.*

Proof: Let $\mathcal{K}_\alpha = \{K_{\alpha,\beta} : \beta \in \Lambda_\alpha\}$. Choose a $G_{\alpha,\beta} \in \mathcal{G}$ with $K_{\alpha,\beta} \subseteq G_{\alpha,\beta}$ for each $\beta \in \Lambda_\alpha$, $\alpha < \kappa$. Let $U_{0,\beta} = G_{0,\beta} - \bigcup\{K_{0,\delta} : \delta \in \Lambda_0 \text{ with } \delta \neq \beta\}$ for each $\beta \in \Lambda_0$ and let $\mathcal{U}_0 = \{U_{0,\beta} : \beta \in \Lambda_0\}$. Now, let $\alpha < \kappa$ and assume that $\{\mathcal{U}_\beta : \beta < \alpha\}$ have been already constructed. Furthermore let $\Lambda^*_\alpha = \{\beta \in \Lambda_\alpha : K_{\alpha,\beta} - \bigcup_{\gamma<\alpha}(\bigcup \mathcal{U}_\gamma) \neq \emptyset\}$. For each $\beta \in \Lambda^*_\alpha$, let us define now

$$U_{\alpha,\beta} = G_{\alpha,\beta} - ((\bigcup_{\gamma<\alpha}(\bigcup \mathcal{K}_\gamma)) \cup (\bigcup\{K_{\alpha,\delta} : \delta \in \Lambda_\alpha - \{\beta\}\})).$$

Let $\mathcal{U}_\alpha = \{U_{\alpha,\beta} : \beta \in \Lambda^*_\alpha\}$. By induction, $\{\mathcal{U}_\alpha : \alpha < \kappa\}$ is now constructed. Note that it satisfies $\bigcup_{\gamma<\alpha}(\bigcup \mathcal{K}_\gamma) \subset \bigcup_{\gamma<\alpha}(\bigcup \mathcal{U}_\gamma)$ for each $\alpha < \kappa$. Here we set $\mathcal{U} = \bigcup_{\alpha<\kappa} \mathcal{U}_\alpha$. We show that $\mathcal{U}$ is a refinement of $\mathcal{G}$ which covers $\bigcup \mathcal{K}$ minimally. Clearly, $\mathcal{U}$ refines $\mathcal{G}$. Pick any $x \in \bigcup \mathcal{K}$. Find $\alpha_0 \in \kappa$ with $x \in \bigcup \mathcal{K}_{\alpha_0} - \left(\bigcup_{\gamma<\alpha_0}(\bigcup \mathcal{K}_\gamma)\right)$. Assume $x \notin \bigcup_{\gamma<\alpha_0}(\bigcup \mathcal{U}_\gamma)$. Take $\beta_0 \in \Lambda_{\alpha_0}$ with $x \in K_{\alpha_0,\beta_0}$. Since $x \in K_{\alpha_0,\beta_0} - \bigcup_{\gamma<\alpha_0}(\bigcup \mathcal{U}_\gamma)$, we have $\beta_0 \in \Lambda^*_{\alpha_0}$. Hence it follows that $x \in U_{\alpha_0,\beta_0} \in \mathcal{U}_{\alpha_0}$. This implies that $\bigcup \mathcal{K} \subset \bigcup \mathcal{U}$. On the other hand, take any $U_{\alpha,\beta} \in \mathcal{U}_\alpha \subset \mathcal{U}$. It follows from $\beta \in \Lambda^*_\alpha$ that $K_{\alpha,\beta} - \bigcup_{\gamma<\alpha}(\bigcup \mathcal{U}_\gamma) \neq \emptyset$. It is easily verified that each member $V \in \mathcal{U}$ with $V \neq U_{\alpha,\beta}$ does not meet this nonempty set. Therefore, $\mathcal{U}$ is a minimal cover of $\bigcup \mathcal{K}$.

Corollary 3 (J.C.Smith, [6]) : *Let $\mathcal{K}_n = \{K(\alpha, n) : \alpha \in \Lambda_n\}$ be a discrete family of closed subsets and let $\mathcal{U}_n = \{U(\alpha, n) : \alpha \in \Lambda_n\}$ be an open family such that $K(\alpha, n) \subseteq U(\alpha, n)$ for each $n \in \mathbb{N}$ and $\alpha \in \Lambda_n$. Then $\mathcal{U} = \bigcup_{n=1}^\infty \mathcal{U}_n$ has an open refinement in X covers the set $\bigcup_{n=1}^\infty(\bigcup \mathcal{K}_n)$ minimally.*

Corollary 4 : *Let K be a nonempty closed subset of X and let the family $\mathcal{G}$ of open subsets of X be a cover for K. If there exists a κ-discrete collection $\mathcal{K} = \bigcup_{\alpha<\kappa} \mathcal{K}_\alpha$ of nonempty closed subsets of X such that $\mathcal{K}$ refines $\mathcal{G}$, covers K and if $\bigcup_{\beta<\alpha}(\bigcup \mathcal{K}_\beta(K))$ is closed for each $\alpha < \kappa$, then there exists an open refinement of $\mathcal{G}$ covers K minimally.*

Proof: Notice that $K = \bigcup_{\alpha<\kappa}(\bigcup \mathcal{K}_\alpha(K)) = \bigcup \mathcal{K}(K)$.

Remark 1 : Let $\mathcal{G} = \bigcup_{\alpha<\kappa} \mathcal{G}_\alpha$ be a weak $\kappa\overline{\theta}$-cover of X where $\mathcal{G}_\alpha = \{G_{\alpha,\beta} : \beta \in \Lambda_\alpha\}$. Now we are going to define the following $K_{n,m}$ sets of X which are closed. They have an important role in the proof of Proposition 3. Let

$$K_{n,m} = \{x \in X : \text{i) } \mathrm{ord}(x, \mathcal{G}^*) < n \text{ or ii) } \mathrm{ord}(x, \mathcal{G}^*) = n \text{ and } \exists \alpha < \kappa, 0 < \mathrm{ord}(x, \mathcal{G}_\alpha) \leq m\}$$

where $\mathcal{G}^*$ denotes the open cover $\{\bigcup \mathcal{G}_\alpha\}_{\alpha<\kappa}$ and n, m are positive integers. By hypothesis $\mathcal{G}^*$ is point-finite i.e. $0 < \mathrm{ord}(x, \mathcal{G}^*) < \omega_0$ for each $x \in X$. Notice that $K_{n,m}$ is **closed** since if $x \notin K_{n,m}$, then we either have i) $n < \mathrm{ord}(x, \mathcal{G}^*)$ or ii) $\mathrm{ord}(x, \mathcal{G}^*) = n$ and $m < \mathrm{ord}(x, \mathcal{G}_{\alpha_k})$ for each indice $\alpha_k < \kappa$ where $x \in \bigcup \mathcal{G}_{\alpha_k}$ $(k = 1, 2, \ldots, n)$ holds; and x has evidently an open nbhd missing $K_{n,m}$ in each cases. Notice also that

$$K_{n,m} \subseteq K_{n,m'} \qquad \text{if} \quad m \leq m'$$

and one can also easily notice that

$$K_{n,\omega} = \bigcup_{m=1}^{\infty} K_{n,m} = \{x \in X : \mathrm{ord}(x, \mathcal{G}^*) \leq n\}$$

is closed and contained in $K_{n+1,1}$. These notations are used in the next two propositions. We utilize a method of J.C.Smith.

Proposition 3 : *Let $\mathcal{G}_\alpha = \{G_{\alpha,\beta} : \beta \in \Lambda_\alpha\}$ for each $\alpha < \kappa$ and let $\mathcal{G} = \bigcup_{\alpha<\kappa} \mathcal{G}_\alpha$ be a weak $\kappa\overline{\theta}$-cover of X. If $\mathcal{G}$ has a refinement $\mathcal{U}_{n,m}$ covering $K_{n,m}$ minimally, then $\mathcal{G}$ also have open refinements $\mathcal{U}_{n,m+1}$ and $\mathcal{V}_{n,m+1}$ covering $K_{n,m+1} - \bigcup \mathcal{U}_{n,m}$ and $K_{n,m+1}$ minimally.*

Proof: During the proof the positive integers n and m are fixed. We suppose that $K_{n,m+1} - \bigcup \mathcal{U}_{n,m}$ is nonempty since otherwise the statement is clear. Let us define

$$F(\alpha, \Lambda) = \bigcap_{\beta\in\Lambda} G_{\alpha,\beta} \cap A_{\alpha,m+1} \cap (K_{n,m+1} - \bigcup \mathcal{U}_{n,m})$$

for each $\Lambda \in [\Lambda_\alpha]^{m+1}$ where $A_{\alpha,m+1} = \{x \in X : 0 < \operatorname{ord}(x, \mathcal{G}_\alpha) \leq m+1\}$ and let $\mathcal{F}_\alpha = \{F(\alpha, \Lambda) : \Lambda \in [\Lambda_\alpha]^{m+1}\}$. We claim that $\mathcal{F} = \bigcup_{\alpha<\kappa} \mathcal{F}_\alpha$ is a κ-discrete collection of closed sets and covers $K_{n,m+1} - \bigcup \mathcal{U}_{n,m}$. The covering assertion can be proved easily since $K_{n,m+1} - \bigcup \mathcal{U}_{n,m} \subseteq K_{n,m+1} - K_{n,m} \subseteq \{x \in X : \operatorname{ord}(x, \mathcal{G}^*) = n \text{ and } \exists\alpha < \kappa, \operatorname{ord}(x, \mathcal{G}_\alpha) = m+1\}$. Now let us prove that the family $\mathcal{F}_\alpha$ is discrete. Take any point $x \in X$. If $\operatorname{ord}(x, \mathcal{G}^*) < n$ then $x \in K_{n,m} \subseteq \bigcup \mathcal{U}_{n,m}$, and if $n < \operatorname{ord}(x, \mathcal{G}^*)$ then $x \in X - K_{n,\omega}$. Therefore x has an open basic nbhd disjoint from $K_{n,m+1} - \bigcup \mathcal{U}_{n,m}$ in both cases. Now let us suppose that $\operatorname{ord}(x, \mathcal{G}^*) = n$. If $x \notin \bigcup \mathcal{G}_\alpha$ then there are suitable indexes $\alpha_1, \alpha_2, \ldots, \alpha_n$ all different from α such that $x \in \bigcap_{1\leq k\leq n} (\bigcup \mathcal{G}_{\alpha_k})$ and this intersection set is open and disjoint from $\bigcap_{\beta\in\Lambda} G_{\alpha,\beta} \cap K_{n,m+1}$ for each $\Lambda \in [\Lambda_\alpha]^{m+1}$ since we evidently have $\bigcap_{\beta\in\Lambda} G_{\alpha,\beta} \subseteq \bigcup \mathcal{G}_\alpha$. If $x \in \bigcup \mathcal{G}_\alpha$ we should just examine the case $m+1 \leq \operatorname{ord}(x, \mathcal{G}_\alpha)$ since otherwise we would have $x \in K_{n,m} \subseteq \bigcup \mathcal{U}_{n,m}$. If now $m+1 < \operatorname{ord}(x, \mathcal{G}_\alpha)$ holds then the open set $\{y \in X : m+1 < \operatorname{ord}(y, \mathcal{G}_\alpha)\}$ contains x and disjoint from $\bigcup \mathcal{F}_\alpha$. Finally if $\operatorname{ord}(x, \mathcal{G}_\alpha) = m+1$ holds then there is a $\Lambda_x \in [\Lambda_\alpha]^{m+1}$ such that $x \in \bigcap_{\beta\in\Lambda_x} G_{\alpha,\beta}$ and this open set does not meet the set $F(\alpha, \Lambda)$ for any $\Lambda \neq \Lambda_x$, $\Lambda \in [\Lambda_\alpha]^{m+1}$. One can easily and similarly observe that all members $F(\alpha, \Lambda)$ of $\mathcal{F}_\alpha$ are actually closed. We now claim that each union

$$\bigcup_{\beta<\alpha} (\bigcup \mathcal{F}_\beta(K_{n,m+1} - \bigcup \mathcal{U}_{n,m}))$$

is closed for any $\alpha < \kappa$. For simplicity let us write $K = K_{n,m+1} - \bigcup \mathcal{U}_{n,m}$ and $E_\alpha = \bigcup_{\beta<\alpha} (\mathcal{F}_\beta(K))$. Notice first that $E_\alpha \subseteq K$. Take any point $x \in X - E_\alpha$. If $x \in X - K$ then clearly this open set does not intersect E_α. Suppose now that $x \in K - E_\alpha$ and let us write explicitly all indices $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that $x \in \bigcup \mathcal{G}_{\alpha_i}$ for each $1 \leq i \leq n$. Suppose some of them, say, $\alpha_1, \alpha_2, \ldots, \alpha_k$ are less than α and all others are equal or greater than α. Our first important observation is that x belongs to open

$$W_x = \bigcap_{1\leq i\leq k} \{y \in X : m+1 < \operatorname{ord}(y, \mathcal{G}_{\alpha_i})\}$$

which is certainly disjoint from $\bigcap_{1\leq i\leq k} (\bigcup \mathcal{F}_{\alpha_i}(K))$. In fact if $\operatorname{ord}(x, \mathcal{G}_{\alpha_1}) \leq m+1$ holds for instance, then we would have first $\operatorname{ord}(x, \mathcal{G}_{\alpha_1}) = m+1$ (by remembering $\operatorname{ord}(x, \mathcal{G}_{\alpha_1}) \leq m$ yields the contradiction $x \in K_{n,m} \subseteq \bigcup \mathcal{U}_{n,m}$) and then we would have the contradiction

$$x \in F(\alpha_1, \Lambda_1) \cap K \subseteq \bigcup \mathcal{F}_{\alpha_1}(K) \subseteq \bigcup_{\beta<\alpha} (\bigcup \mathcal{F}_\beta(K)) = E_\alpha$$

for some appropriate subindex set $\Lambda_1 \in [\Lambda_{\alpha_1}]^{m+1}$. Since we also have $\alpha < \alpha_{k+1}, \ldots, \alpha < \alpha_n$ and $x \in \bigcap_{k<i\leq n} (\bigcup \mathcal{G}_{\alpha_i})$, one can easily prove that the open set

$$W_x \cap \bigcap_{k<i\leq n} (\bigcup \mathcal{G}_{\alpha_i})$$

does not intersect E_α. Thus our claim has been proved now. Then $\mathcal{G}$ has an open refinement in X covering $K = K_{n,m+1} - \bigcup \mathcal{U}_{n,m}$ minimally by Corollary 4. The proposition follows from Corollary 2.

Proposition 4 (Continued) : *If $\mathcal{G}$ has an open refinement $\mathcal{U}_{n,\omega}$ covering $K_{n,\omega} = \bigcup_{m=1}^{\infty} K_{n,m}$ minimally, then $\mathcal{G}$ also have open refinements covering respectively $K_{n+1,1} - \bigcup \mathcal{U}_{n,\omega}$ and $K_{n+1,1}$ minimally.*

Proof: Since

$$K_{n+1,1} - \bigcup \mathcal{U}_{n,\omega} \subseteq \{x \in X : \mathrm{ord}(x, \mathcal{G}^*) = n+1 \text{ and } \exists \alpha < \kappa, \mathrm{ord}(x, \mathcal{G}_\alpha) = 1\}$$

it is not difficult to prove that the family $\mathcal{F}_\alpha = \{F(\alpha, \Lambda) : \Lambda \in [\Lambda_\alpha]^1\}$ is closed and discrete in X for each $\alpha < \kappa$ whereas

$$F(\alpha, \Lambda) = \bigcap_{\beta \in \Lambda} G_{\alpha,\beta} \cap A_{\alpha,1} \cap (K_{n+1,1} - \bigcup \mathcal{U}_{n,\omega})$$

for each $\Lambda \in [\Lambda_\alpha]^1$ and $A_{\alpha,1} = \{x \in X : \mathrm{ord}(x, \mathcal{G}_\alpha) = 1\}$. Besides $\mathcal{F} = \bigcup_{\alpha<\kappa} \mathcal{F}_\alpha$ is a κ-discrete family covering $K_{n+1,1} - \bigcup \mathcal{U}_{n,\omega}$ and $\bigcup_{\beta<\alpha}(\bigcup \mathcal{F}_\beta(K_{n+1,1} - \bigcup \mathcal{U}_{n,\omega}))$ is closed for any $\alpha < \kappa$. Thus the proof can be achieved just as in the above proposition.

Theorem 1 : *Weakly $\kappa\overline{\theta}$-refinable spaces are irreducible.*

Proof: Let a weak $\kappa\overline{\theta}$-cover $\mathcal{G} = \bigcup_{\alpha<\kappa} \mathcal{G}_\alpha$ of a weakly $\kappa\overline{\theta}$-refinable space X be given. Then the set $K_{1,1}$ i.e. the set of all points $x \in X$ satisfying the condition

$$\mathrm{ord}(x, \mathcal{G}^*) = 1 \text{ and } \exists \alpha_x < \kappa,\ \mathrm{ord}(x, \mathcal{G}_{\alpha_x}) = 1,$$

is closed as we have already noticed. Now let $F(\alpha, \Lambda) = \bigcap_{\beta \in \Lambda} G_{\alpha,\beta} \cap A_{\alpha,1} \cap K_{1,1}$ as in the proof of Proposition 3 where $\Lambda \in [\Lambda_\alpha]^1$ and $A_{\alpha,1} = \{x \in X : 0 < \mathrm{ord}(x, \mathcal{G}_\alpha) = 1\}$. Then it is not difficult to observe that all unions $\bigcup_{\beta<\alpha}(\bigcup \mathcal{F}_\alpha(K_{1,1}))$ are closed for each $\alpha < \kappa$, where $\mathcal{F}_\alpha = \{F(\alpha, \Lambda) : \Lambda \in [\Lambda_\alpha]^1\}$. Thus $K_{1,1}$ can be covered minimally by an open refinement $\mathcal{U}_{1,1}$ of $\mathcal{G}$. The closed subsets $K_{n,m}$ and $K_{n,\omega}$ ($n, m \in \mathbb{N}$) defined in Remark 2 satisfy the inclusions $K_{1,1} \subseteq K_{1,2} \subseteq \ldots \subseteq K_{1,\omega} \subseteq K_{2,1} \subseteq K_{2,2} \subseteq \ldots \subseteq K_{2,\omega} \subseteq K_{3,1} \subseteq \ldots$ and besides X is the union of these sets. It is proved in Proposition 3 and Proposition 4 that there exists a sequence of open families $\mathcal{U}_{1,1}, \mathcal{U}_{1,2}, \ldots, \mathcal{U}_{1,\omega}, \mathcal{U}_{2,1}, \ldots$ such that each is a refinement of $\mathcal{G}$ and moreover $\mathcal{U}_{n,m+1}$ (resp. $\mathcal{U}_{n+1,1}$) covers $K_{n,m+1} - \bigcup \mathcal{U}_{n,m}$ (resp. $K_{n+1,1} - \bigcup \mathcal{U}_{n,\omega}$) minimally. Thus this theorem follows after Corollary 1, i.e. $\mathcal{G}$ has an open refinement covering X.

Corollary 5 (J.C.Smith, [6]) : *Weakly $\overline{\theta}$-refinable spaces are irreducible.*

Proof: Weakly $\overline{\theta}$-refinable spaces are nothing but weakly $\aleph_0\overline{\theta}$-refinable spaces.

Remark 2 : Thus we have shown the following interesting result above: As long as the open cover $\mathcal{G}^* = \{\bigcup \mathcal{G}_\alpha\}_{\alpha<\kappa}$ is point-finite and the basic condition $(\forall x \in X, \exists \alpha_x < \kappa, \mathrm{ord}(x, \mathcal{G}_{\alpha_x}) < \omega_0)$ is satisfied, it is completely irrelevant what the infinite cardinal number κ actually is in the process of getting the conclusion of the existence of a minimal open refinement of the open cover $\mathcal{G} = \bigcup_{\alpha<\kappa} \mathcal{G}_\alpha$.

2. Results on Lindelöf number

The primary aim of this section is to prove the Theorem 2 and Corollary 10.

Remark 3 : If A is a subset in any topological space X and the open family $\mathcal{G}$ is a cover for A then it is well known that there exists a maximal subset M_A of A by utilizing Zorn's Lemma such that $M_A \subseteq A \subseteq \mathrm{st}(M_A, \mathcal{G})$ and $\mathrm{st}(x, \mathcal{G}) \cap M_A = \{x\}$ for each $x \in M_A$. This set is evidently a discrete subset in X and is called a **maximal distinguished subset** of A with respect to the open family $\mathcal{G}$. The following lemma is important and useful in this section. All topological spaces in this section are T_1.

Lemma 2 (Continued) : *M_A is a closed-discrete subset in X if there exists a closed set K satisfying $M_A \subseteq K \subseteq \bigcup \mathcal{G}$.*

Proof: Let us take any point $x \in X - M_A$. If $x \in X - K$ then x evidently has an open basic nbhd disjoint from M_A. If on the other hand $x \in K - M_A$ then by choosing a member $G \in \mathcal{G}$ satisfying $x \in G$ one can easily define an open basic nbhd U_x of x contained in G and disjoint with M_A since $G \cap M_A$ is finite and therefore closed in the T_1 space X (we even have $|G \cap M_A| \leq 1$) and $x \notin G \cap M_A$. Thus $X - M_A$ is in fact open.

One should remember here that a topological space is called **$\aleph_\alpha$-compact** iff every closed-discrete subset A of X has cardinality less than $\aleph_\alpha$. As is well known A is closed-discrete iff the derived set A^d, i.e. the set of the all limit points of A in X, is empty, and X is $\aleph_\alpha$-compact iff every net which is directed by an index set with cardinality $\aleph_\alpha$ has at least one adherent point. The cardinal number $\aleph_\alpha$ will be written briefly as κ in Theorem 2 and thereafter. Thus κ^+ denotes as usual its immediate successor $\aleph_{\alpha+1}$; ω_α denotes the least ordinal number having the cardinality $\aleph_\alpha$. The **Lindelöf number** $L(X)$ of a topological space X on the other hand, as is well known, is the least infinite cardinal number having the property that every open cover of X has a subcover with cardinality $\leq L(X)$. Thus X is a Lindelöf space iff $L(X) = \aleph_0$. It is not difficult to observe that X is $\aleph_\alpha$-compact if $L(X) \leq \aleph_\alpha$. Let us give now the definition of the basic concepts of this section.

Definition: An open cover $\mathcal{G} = \bigcup_{\alpha<\kappa} \mathcal{G}_\alpha$ is called a **weak $\overline{\delta_\kappa\theta}$-cover** iff i) each $\mathcal{G}_\alpha$ is an open family, ii) for each $x \in X$ there exists an $\alpha_x < \kappa$ such that $\underline{\mathrm{ord}}(x, \mathcal{G}_{\alpha_x}) \leq \omega_0$, iii) $\mathcal{G}^* = \{\bigcup \mathcal{G}_\alpha\}_{\alpha<\kappa}$ is point-finite. A topological space X called **weakly $\overline{\delta_\kappa\theta}$-refinable** iff every open cover of X has a weak $\overline{\delta_\kappa\theta}$-cover refinement. These concepts generalize evidently both weakly $\kappa\overline{\theta}$-refinable spaces and weakly $\overline{\delta\theta}$-refinable spaces. On the other hand X is called **weakly $\delta_\kappa\theta$-refinable** iff each open cover of X has a weak $\delta_\kappa\theta$-cover refinement where an open cover $\mathcal{G} = \bigcup_{\alpha<\kappa} \mathcal{G}_\alpha$ is called a **weak $\delta_\kappa\theta$-cover** iff conditions i) and ii) hold.

Theorem 2 : *An $\aleph_{\alpha+1}$-compact weakly $\overline{\delta_\kappa\theta}$-refinable space X has the Lindelöf number $L(X) \leq \aleph_\alpha$.*

Proof : Let $\mathcal{G} = \bigcup_{\alpha<\kappa} \mathcal{G}_\alpha$ be a weak $\overline{\delta_\kappa\theta}$-cover of the $\aleph_{\alpha+1}$-compact space X. We first prove the following claim: If the closed set $F_n = \{x \in X : \mathrm{ord}(x, \mathcal{G}^*) \leq n\}$ $(n \in \mathbb{N})$ is covered by a subfamily $\mathcal{U}_n$ of $\mathcal{G}$ satisfying $|\mathcal{U}_n| \leq \aleph_\alpha$ then F_{n+1} is also covered by a similar subfamily. Here $\mathcal{G}^*$ denotes, as in the above definition, the open cover $\{\bigcup \mathcal{G}_\alpha\}_{\alpha<\kappa}$. Let us define now

$$K(\Lambda) = (F_{n+1} - \bigcup \mathcal{U}_n) \cap \bigcap_{\alpha\in\Lambda} (\bigcup \mathcal{G}_\alpha)$$

for each $\Lambda \in [\omega_\alpha]^{n+1}$. It is not difficult to observe that each $K(\Lambda)$ is closed in X for any $\Lambda \in [\omega_\alpha]^{n+1}$; in fact any point $x \in X - K(\Lambda)$ has an open basic nbhd disjoint from $K(\Lambda)$ if $x \in X - (F_{n+1} - \bigcup \mathcal{U}_n)$ or $x \in (F_{n+1} - \bigcup \mathcal{U}_n) - K(\Lambda)$ since we evidently have $F_{n+1} - \bigcup \mathcal{U}_n \subseteq \{x \in X : \mathrm{ord}(x, \mathcal{G}^*) = n+1\}$. It is not difficult to observe that $\mathcal{K}_{n+1} = \{K(\Lambda) : \Lambda \in [\omega_\alpha]^{n+1}\}$ is actually a discrete family in X. Now let for each $\Lambda = \{\alpha_i : 1 \leq i \leq n+1\}$ and $\alpha_i \in \Lambda$,

$M(\Lambda, \alpha_i)$ be a maximal distinguished subset of $\{x \in K(\Lambda) : 0 < \text{ord}(x, \mathcal{G}_{\alpha_i}) \leq \omega_0\}$ with respect to $\mathcal{G}_{\alpha_i}$. Each $M(\Lambda, \alpha_i)$ set is closed-discrete in X after Lemma 3 since $M(\Lambda, \alpha_i) \subseteq K(\Lambda) \subseteq \bigcup \mathcal{G}_{\alpha_i}$. Furthermore we have

$$\bigcup_{i=1}^{n+1} M(\Lambda, \alpha_i) \subseteq K(\Lambda) \subseteq \bigcup_{i=1}^{n+1} \text{st}\big(M(\Lambda, \alpha_i), \mathcal{G}_{\alpha_i}\big).$$

Thus each $K(\Lambda) \in \mathcal{K}_{n+1}$ can be covered by a subfamily of $\mathcal{G}$ having cardinality $\leq \aleph_\alpha$. Since

$$F_{n+1} \subseteq \bigcup \mathcal{U}_n \cup \bigcup \{K(\Lambda) : \Lambda \in [\omega_\alpha]^{n+1}\}$$

our claim is established now. Consequently the theorem follows easily since F_1 can evidently be covered by a subfamily $\mathcal{U}_1 \subseteq \mathcal{G}$ with $|\mathcal{U}_1| \leq \aleph_\alpha$ and we furthermore have $X = \bigcup_{n=1}^{\infty} F_n$.

As an immediate consequence of Theorem 2, we have the following generalization of J.C.Smith [6, Theorem 3.6]:

Corollary 6 : *Any $\aleph_1$-compact weakly $\overline{\delta_\kappa}\theta$-refinable T_1 space is Lindelöf.*

Corollary 7 : *Every countably compact weakly $\overline{\delta_\kappa}\theta$-refinable T_1 space is compact.*

Proposition 5 : *An $\aleph_{\alpha+1}$-compact weakly $\delta_\kappa\theta$-refinable space X has the Lindelöf number $L(X) \leq \aleph_\alpha$ if every closed set in X is a G_κ-set.*

Proof : Let $\{\mathcal{G}_\alpha\}_{\alpha<\kappa}$ be a weak $\delta_\kappa\theta$-cover for X and let each $\bigcup \mathcal{G}_\alpha$ be written as the union $\bigcup_{\beta<\kappa} K_{\alpha,\beta}$ where each $K_{\alpha,\beta}$ is closed for each $\alpha < \kappa$ and $\beta < \kappa$. Let $F_{\alpha,\beta}(\gamma)$ be the maximal distinguished subset of

$$E_{\alpha,\beta}(\gamma) = \{x \in K_{\alpha,\beta} : 0 < \text{ord}(x, \mathcal{G}_\gamma) \leq \aleph_\alpha\} \qquad (\gamma < \kappa)$$

with respect to $\mathcal{G}_\gamma$ which is closed-discrete in X as we have shown in Lemma 3. Then we have $K_{\alpha,\beta} = \bigcup_{\gamma<\kappa} E_{\alpha,\beta}(\gamma)$ and $E_{\alpha,\beta}(\gamma) \subseteq \text{st}(F_{\alpha,\beta}(\gamma), \mathcal{G}_\gamma)$. It is not difficult to see now that the cardinality of this star set is not greater than $\aleph_\alpha$. Thus each $K_{\alpha,\beta}$ satisfies $|K_{\alpha,\beta}| \leq \aleph_\alpha$. Theorem follows now.

Corollary 8 (J.C.Smith, [6]) : *Every weak $\delta\theta$-cover of a perfect $\aleph_1$-compact T_1 space has a countable subcover.*

Proposition 6 : *Every subspace in any κ-perfect, $\aleph_{\alpha+1}$-compact, weakly $\delta_\kappa\theta$-refinable T_1 space has Lindelöf number $\leq \aleph_\alpha$.*

Proof : Let X be a T_1 space possessing all the properties of the hypothesis. It is certainly sufficient to prove $L(X_0) \leq \aleph_\alpha$ for any non-empty open subspace X_0 of X. Let X_0 be the union $\bigcup_{\alpha<\kappa} K_\alpha$ of the closed subsets K_α of X and let $A \subseteq X_0$ be a subset whose derived set in the subspace X_0 is empty. Then we have $\bigcup_{\alpha<\kappa}(A^d \cap K_\alpha) = A^d \cap X_0 = \emptyset$, and thus $|K_\alpha \cap A| \leq \kappa$ for each $\alpha < \kappa$, since $(K_\alpha \cap A)^d \subseteq K_\alpha \cap A^d$. Thus $A = \bigcup_{\alpha<\kappa}(A \cap K_\alpha)$ has cardinality $\leq \kappa$. It is also straightforward to observe that X_0 is a weakly $\delta_\kappa\theta$-refinable subspace. Therefore this proposition follows from Proposition 4.

Corollary 9 (J.C.Smith, [6]) : *Any perfect, $\aleph_1$-compact, weakly $\delta\theta$-refinable T_1 space is hereditarily Lindelöf.*

Corollary 10 : *Any perfect $\aleph_1$-compact weakly $\delta\theta$-refinable regular T_1 space is hereditarily paracompact and thus hereditarily irreducible.*

Proof : It is well known that every regular Lindelöf T_1 space is paracompact.

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