CHARACTERIZATION OF BEST APPROXIMATIONS IN METRIC LINEAR SPACES

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ABSTRACT. Let (X, d) be a real metric linear space, with translation-invariant metric d and G a linear subspace of X. In this paper we use functionals in the Lipschitz dual of X to characterize those elements of G which are best approximations to elements of X.

We also give simultaneous characterization of elements of best approximation and also consider elements of ϵ-approximation.

1 Introduction and Notation.

Let (X, d) be a real metric linear space, with translation-invariant metric d and G a linear subspace of X. For a given element $x \in X \setminus G$, a **best approximation** to x from G is any element g_0 in G satisfying

$$d(x, g_0) = d(x, G) ,$$

where $d(x, G) := \inf\{d(x, g) : g \in G\}$ – the distance from x to G. The (possibly empty) set of all best approximations to x from G is denoted by $P_G(x)$. Thus,

$$P_G(x) = \{g \in G \ : d(x, g) = d(x, G)\}.$$

The mapping $P_G : X \to 2^G$ which associates with each x in X its set of best approximations in G is called the **metric projection**, or nearest point mapping, onto G.

The set G is called

(1) **proximinal** (or an existence set) if $P_G(x)$ is nonempty for each x in X;

(2) **semi-Chebyshev** (or a uniqueness set) if $P_G(x)$ contains at most one point for every x in X;

(3) **Chebyshev** if G is both proximinal and semi-Chebyshev, i.e., each point in X has exactly one best approximation in G.

One of the major problems in Approximation Theory is that of characterizing elements of best approximation. That is, given an $x \in X \setminus G$, how does one characterize elements of the set $P_G(x)$?

In the setting of normed linear spaces $(X, \|\cdot\|)$, such a characterization can be found in [2] or [7] in the case where G is a subspace of X, and in [1] in the case where G is a convex set. The development of a fairly complete and unified theory in normed linear spaces has been made possible by the existence of non-trivial dual spaces.

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In [6], Pantelidis investigated, *inter alia*, the question of characterization of best approximations in the setting of metric linear spaces. A characterization of best approximations in (not necessarily *linear*) metric spaces was given by Mustăţa [3].

Let (X,d) be a real metric linear space. A mapping $f: X \to \mathbb{R}$ is

(a) **subadditive** if $f(x+y) \leq f(x) + f(y)$ for all $x, y \in X$;

(b) G**-periodic** if $f(x+g) = f(x)$ for all $x \in X$ and all $g \in G$.

Let $x \in X$ and $r > 0$. By $\overset{\circ}{B}(x,r)$ and $B(x,r)$ we mean the sets

$$\overset{\circ}{B}(x,r) := \{y \in X : d(x,y) < r\}, \quad \text{and} \quad B(x,r) := \{y \in X : d(x,y) \leq r\},$$

respectively.

Denote by

$$X_0^{\#} = \{\ f: X \to \mathbb{R} : \|f\|_d < \infty;\ f(0) = 0,\ f \text{ subadditive } \},$$

where

$$\|f\|_d := \sup_{x \in X \setminus \{0\}} \frac{|f(x)|}{d(x,0)}.$$

It is easy to show that $\|\cdot\|_d$ defines a norm on $X_0^{\#}$. In fact, $(X_0^{\#}, \|\cdot\|_d)$ is a Banach algebra. The space $X_0^{\#}$ is called the **Lipschitz dual** of the space X. If X is a normed linear space, then $X^* \subset X_0^{\#}$.

Let G be a subspace of a metric linear space (X,d). Denote by

$$\begin{aligned} G^{\perp} &:= \{f \in X_0^{\#} : f(g) = 0 \text{ for all } g \in G\}, \quad \text{and for } x \in X, \\ d_{G^{\perp}}(x,0) &:= \sup_{f \in G^{\perp} \setminus \{0\}} \frac{|f(x)|}{\|f\|_d}. \end{aligned}$$

It is straightforward to show that for each $x \in X$, $d_{G^{\perp}}(x,0) \leq d(x,0)$. Note also that $G^{\perp}$ is a linear subspace of $X_0^{\#}$.

Let us first highlight the following important fact:

Lemma 1.1. *Let G be a subspace of X and $f: X \to \mathbb{R}$ be a subadditive function such that $f(0) = 0$. Then f is G-periodic if and only if $f(g) = 0$ for all $g \in G$.*

Proof. Assume that f is G-periodic. Then, we have

$$f(g) = f(0+g) = f(0) = 0 \ \text{ for all } \ g \in G.$$

Conversely, assume that $f(g) = 0$ for all $g \in G$. Then for all $x \in X$ and all $g \in G$, we have, by subadditivity of f, that

$$f(x) = f(x+g-g) \leq f(x+g) + f(-g) = f(x+g) \leq f(x) + f(g) = f(x);$$

whence $f(x+g) = f(x)$ for all $x \in X$ and all $g \in G$. ∎

2 Characterization of Elements of Best Approximation. In this section we give a theorem that characterizes elements of best approximation from a linear subspace G of a metric linear space (X,d). It sharpens that given by Pantelidis [6].

Pantelidis [6] gave the following characterization theorem of elements of best approximation in metric linear spaces.

Theorem 2.1 [6]. *Let G be a nonempty linear subspace of a metric linear space X, $x \in X \setminus G$, and $g_0 \in G$. Then $g_0 \in P_G(x)$ if and only if there exists an element $f \in X_0^{\#}$ such that*

(i) $|f(x) - f(y)| \le d(x,y)$ *for all* $x, y \in X$;

(ii) $f(x+g) = f(x),\ x \in X,\ g \in G$ *or* $f|_G = 0$;

(iii) $f(x - g_0) = f(x) = d(x, g_0)$.

It is easy to deduce from (i) that $\|f\|_d \le 1$.

We show that the $f \in X_0^{\#}$ that works in Pantelidis' theorem can be chosen fron an even smaller set, namely, the set of all $f \in X_0^{\#}$ of norm 1. This then gives a direct analogue of a similar characterization in normed linear spaces [7].

Theorem 2.2 (Characterization of Best Approximations). *Let G be a nonempty linear subspace of a translation-invariant metric linear space X, $x \in X \setminus G$, and $g_0 \in G$. Then $g_0 \in P_G(x)$ if and only if there exists an element $f \in X_0^{\#}$ such that*

(i) $\|f\|_d = 1$;

(ii) $f(g) = 0$ *for all* $g \in G$; *and*

(iii) $f(x - g_0) = f(x) = d(x, g_0)$.

Proof. "$\Rightarrow$": Assume that $g_0 \in P_G(x)$. For all $y \in X$, define

$$f(y) = d(y, G).$$

We first show that $f \in X_0^{\#}$. It is clear that $f(g) = 0$ for all $g \in G$.

Let $z \in X \setminus \{0\}$. Then $|f(z)| = f(z) = d(z,G) \le d(z,0)$, whence $\dfrac{|f(z)|}{d(z,0)} \le 1$, and consequently, $\sup\limits_{z \in X\setminus\{0\}} \dfrac{|f(z)|}{d(z,0)} \le 1 < \infty$.

Next, we show that f is subadditive. Let $y, z \in X$. Then, by repeatedly using the fact that d is translation-invariant, we have

$$\begin{aligned} f(y+z) &= \inf_{g\in G} d(y+z, g) = \inf_{g,g'\in G} d(y-g, g'-z) \\ &\le \inf_{g,g'\in G} [d(y-g,0) + d(0, g'-z)] = \inf_{g\in G} d(y,g) + \inf_{g'\in G} d(z,g') \\ &= f(y) + f(z). \end{aligned}$$

We have shown that $\|f\|_d \le 1$. We need to show that $\|f\|_d \ge 1$. To that end, let $\epsilon > 0$ be given. Then there is an element $g_\epsilon \in G$ such that

$$d(x,G) + \epsilon > d(x, g_\epsilon).$$

Since f is subadditive and $f(g) = 0$ for all $g \in G$, it follows from Lemma 1.1 that f is G-periodic. Hence

$$|f(x - g_\epsilon)| = |f(x)| = d(x, G) > d(x, g_\epsilon) - \epsilon = d(x - g_\epsilon, 0) - \epsilon.$$

Therefore

$$\|f\|_d \geq \frac{|f(x - g_\epsilon)|}{d(x - g_\epsilon, 0)} > 1 - \frac{\epsilon}{d(x - g_\epsilon, 0)}.$$

Since ϵ is arbitrary, it follows that $\|f\|_d \geq 1$.

Using Lemma 1.1 again, we have that $f(x - g_0) = f(x) = d(x, G) = d(x, g_0)$, which verifies (iii).

"$\Leftarrow$": Assume that there is an $f \in X_0^{\#}$ which satisfies (i), (ii), and (iii). For each $g \in G$,

$$d(x, g_0) = f(x) = f(x - g) = |f(x - g)| \leq \|f\|_d d(x - g, 0) = d(x - g, 0) = d(x, g).$$

Hence $g_0 \in P_G(x)$. ■

We now give a characterization of elements of best approximation in terms of the "annihilator" $G^\perp$ of the subspace G in $X_0^{\#}$. An analogous result in the setting of metric spaces is due to Mustăţa [3].

Proposition 2.3. *Let G be a nonempty linear subspace of a translation-invariant metric linear space X, $x \in X \setminus G$, and $g_0 \in G$. Then $g_0 \in P_G(x)$ if and only if $d_{G^\perp}(x - g_0, 0) = d(x, g_0)$.*

Proof. Assume that $g_0 \in P_G(x)$. Since $d_{G^\perp}(x - g_0, 0) \leq d(x - g_0, 0) = d(x, g_0)$, it remains to show that $d_{G^\perp}(x - g_0, 0) \geq d(x, g_0)$. By Theorem 2.2, there is an element $f \in X_0^{\#}$ such that $\|f\|_d = 1$, $f(g) = 0$ for all $g \in G$ and $f(x - g_0) = f(x) = d(x, g_0)$. It now follows that

$$d_{G^\perp}(x - g_0, 0) \geq \frac{|f(x - g_0)|}{\|f\|_d} = d(x, g_0).$$

Conversely, assume that $d_{G^\perp}(x - g_0, 0) = d(x, g_0)$. Then, for each $g \in G$,

$$\begin{aligned} d(x, g_0) &= \sup_{f \in G^\perp \setminus \{0\}} \frac{|f(x - g_0)|}{\|f\|_d} = \sup_{f \in G^\perp \setminus \{0\}} \frac{|f(x - g)|}{\|f\|_d} \\ &= d_{G^\perp}(x - g, 0) \leq d(x - g, 0) = d(x, g). \end{aligned}$$

Hence, $g_0 \in P_G(x)$. ■

3 Simultaneous Characterization of Best Approximations.

In this section we consider the problem of simultaneous characterization of a set of elements of best approximation in metric linear spaces. The corresponding theorem in normed space setting can be found in [7] and in metric space setting in [5].

Theorem 3.1 (Simultaneous Characterization of Best Approxiamtions). *Let G be a nonempty linear subspace of a translation-invariant metric linear space X, $x \in X \setminus G$, and $M \subset G$. Then $M \subset P_G(x)$ if and only if there exists an element $f \in X_0^{\#}$ such that*

(i) $\|f\|_d = 1$;

(ii) $f(g) = 0$ *for all* $g \in G$; *and*

(iii) $f(x-m) = f(x) = d(x,m)$ *for all* $m \in M$.

Proof. The proof is an immediate consequence of Theorem 2.2. ■

Following is a simultaneous characterization of best approximations in terms of the annihilator $G^{\perp}$ of the subspace G in $X_0^{\#}$. An analogous result in the setting of metric spaces is due to Narang [5].

Proposition 3.2. *Let* G *be a nonempty linear subspace of* X, $x \in X \setminus G$, *and* $M \subset G$. *Then* $M \subset P_G(x)$ *if and only if* $d_{G^{\perp}}(x-m,0) = d(x,m)$ *for all* $m \in M$.
Proof. The proof is similar to that of Proposition 2.3. ■

4 Characterization of Semi-Chebyshev Subspaces.

In this section we characterize semi-Chebyshev subspaces of a metric linear space X using elements of the Lipschitz dual $X_0^{\#}$. An analogous result in the normed space setting can be found in [7].

Theorem 4.1. *Let* G *be a nonempty linear subspace of a translation-invariant metric linear space* (X,d). *The following statements are equivalent:*

(1) G *is a semi-Chebyshev subspace of* X;

(2) There do not exist $f \in X_0^{\#}$, $x_1, x_2 \in X$ *with* $x_1 - x_2 \in G \setminus \{0\}$ *such that*

- *(i)* $\|f\|_d = 1$;
- *(ii)* $f(g) = 0$ *for all* $g \in G$ *and*
- *(iii)* $f(x_1) = d(x_1,0)$ *and* $f(x_2) = d(x_2,0)$;

(3) There do not exist $f \in X_0^{\#}$, $x \in X$, $g_0 \in G \setminus \{0\}$ *with properties (i), (ii) and*

(iii)' $f(x) = d(x,0) = d(x,g_0)$.

Proof. "(1) $\Rightarrow$ (2)": If (2) fails, then there is an $f \in X_0^{\#}$, points x_1, x_2 in X with $x_1 - x_2 \in G \setminus \{0\}$ and satisfying conditions (i) - (iii) of (2). Let $g_0 = x_1 - x_2$. Then, since f is G-periodic and d is translation-invariant,

$$f(x_1) = f(x_1 - g_0) = f(x_2) = d(x_2,0) = d(x_1 - g_0, 0) = d(x_1, g_0).$$

Hence $g_0 \in P_G(x_1)$. Also, $f(x_1) = f(x_1 - 0) = d(x_1, 0)$ implies that $0 \in P_G(x_1)$. Since $x_1 \neq x_2$, 0 and g_0 are two distinct best approximations to x_1 in G. Hence G is not semi-Chebyshev.
"(2) $\Rightarrow$ (3)": If (3) fails, then there are elements $f \in X_0^{\#}$, $x \in X$, $g_0 \in G \setminus \{0\}$ with properties (i), (ii) and (iii)'. Let $x = x_1$ and $x_2 = x - g_0$. Then $g_0 \in G \setminus \{0\}$, $f(x_1) = d(x_1, 0) = d(x_1, g_0)$, and

$$f(x_2) = f(x_1 - g_0) = f(x_1) = d(x_1, g_0) = d(x_1 - g_0, 0) = d(x_2, 0).$$

Hence (2) fails.
"(3) $\Rightarrow$ (1)": Assume that G is not semi-Chebyshev. Then there are elements $y \in X \setminus G$, g_1, $g_2 \in P_G(y)$ with $g_1 \neq g_2$. Let $x = y - g_1$ and $g_0 = g_2 - g_1$. Then $x \in X \setminus G$ and $g_0 \in G \setminus \{0\}$. Now

$$d(x,G) = d(y - g_1, G) = d(y,G) = d(y,g_1) = d(y,g_2).$$

Therefore,

$$d(x, g_0) = d(y - g_1, g_2 - g_1) = d(y, g_2) = d(x, G).$$

That is, $g_0 \in P_G(x)$, and

$$d(x, G) = d(y, g_1) = d(y - g_1, 0) = d(x, 0),$$

whence $0 \in P_G(x)$. By Theorem 3.1, there is an $f \in X_0^{\#}$ such that $\|f\|_d = 1$, $f(g) = 0$ for all $g \in G$ and $f(x) = d(x, 0) = d(x, g_0)$. Hence (3) fails. ■

5 Elements of ϵ-approximation. Let (X, d) be a metric linear space, G a subspace of X and $x \in X$. For $\epsilon \geq 0$, denote by

$$P_G^{\epsilon}(x) := \{g \in G : d(x, g) \leq d(x, G) + \epsilon\}.$$

Each element of $P_G^{\epsilon}(x)$ is called an **ϵ-approximation to x from G.** Elements of $P_G^{\epsilon}(x)$ are also referred to as **good approximations**.

If $\epsilon = 0$, then $P_G^{\epsilon}(x) = P_G(x)$. It is clear that for each $\epsilon > 0$ and each $x \in X$ the set $P_G^{\epsilon}(x)$ is nonempty and

$$P_G^{\epsilon}(x) = G \cap B(x, d(x, G) + \epsilon).$$

Let $g_0 \in G$. Then $g_0 \in P_G^{\epsilon}(x)$ if and only if $G \cap \overset{\circ}{B}(x, d(x, g_0) - \epsilon) = \emptyset$.

The problem of ϵ-approximation consists in characterizing the elements of $P_G^{\epsilon}(x)$ for each $x \in X$. Following the above observation, this is equivalent to characterizing those elements g_0 in G for which $G \cap \overset{\circ}{B}(x, d(x, g_0) - \epsilon) = \emptyset$.

Theorem 5.1 (Characterization of elements of ϵ-approximation). *Let G be a nonempty linear subspace of a translation-invariant metric linear space X, $x \in X \setminus G$, $g_0 \in G$ and $\epsilon > 0$. Then $g_0 \in P_G^{\epsilon}(x)$ if and only if there exists an element $f \in X_0^{\#}$ such that*

(i) $\|f\|_d = 1$;

(ii) $f(g) = 0$ *for all* $g \in G$; *and*

(iii) $f(x - g_0) \geq d(x, g_0) - \epsilon$.

Proof. "$\Rightarrow$": Assume that $g_0 \in P_G^{\epsilon}(x)$. We show that the function $f : X \to \mathbb{R}$ defined by

$$f(y) = d(y, G) \quad \text{for all} \quad y \in X$$

satisfies conditions (i), (ii) and (iii). It follows from the proof of Theorem 2.2 that f satisfies conditions (i) and (ii). Since $g_0 \in P_G^{\epsilon}(x)$ and f is G-periodic, it follows that

$$f(x - g_0) = f(x) = d(x, G) \geq d(x, g_0) - \epsilon,$$

which verifies (iii).

"$\Leftarrow$": Assume that there is an element $f \in X_0^{\#}$ which satisfies conditions (i), (ii) and (iii). For all $g \in G$, we have

$$\begin{aligned} d(x, g_0) \leq f(x - g_0) + \epsilon &= f(x - g) + \epsilon \leq |f(x - g)| + \epsilon \\ &\leq \|f\|_d d(x - g, 0) + \epsilon = d(x, g) + \epsilon. \end{aligned}$$

Taking the infimum over all $g \in G$, we get that $d(x, g_0) \le d(x, G) + \epsilon$, whence $g_0 \in P_G^\epsilon(x)$. ■

We now give an alternative characterization of ϵ-approximation in terms of the annihilator $G^\perp$ of the subspace G.

Proposition 5.2 *Let G be a nonempty linear subspace of a translation-invariant metric linear space X, $x \in X \setminus G$, and $g_0 \in G$. Then $g_0 \in P_G^\epsilon(x)$ if and only if $d_{G^\perp}(x - g_0, 0) \ge d(x, g_0) - \epsilon$.*

Proof. Assume that $g_0 \in P_G^\epsilon(x)$. Then by Theorem 5.1, there is an element $f \in X_0^\#$ such that $\|f\|_d = 1$, $f(g) = 0$ for all $g \in G$ and $f(x - g_0) \ge d(x, g_0) - \epsilon$. Thus,

$$d_{G^\perp}(x - g_0, 0) \ge \frac{|f(x - g_0)|}{\|f\|_d} \ge f(x - g_0) \ge d(x, g_0) - \epsilon.$$

Conversely, assume that $d_{G^\perp}(x - g_0, 0) \ge d(x, g_0) - \epsilon$. Then, for each $g \in G$,

$$\begin{aligned} d(x, g_0) &\le \sup_{f \in G^\perp \setminus \{0\}} \frac{|f(x - g_0)|}{\|f\|_d} + \epsilon = \sup_{f \in G^\perp \setminus \{0\}} \frac{|f(x - g)|}{\|f\|_d} + \epsilon \\ &= d_{G^\perp}(x - g, 0) + \epsilon \le d(x - g, 0) + \epsilon = d(x, g) + \epsilon. \end{aligned}$$

Taking the infimum over all $g \in G$, we have that $d(x, g_0) \le d(x, G) + \epsilon$ and, consequently, $g_0 \in P_G^\epsilon(x)$. ■

The following simultaneous characterization of ϵ-approximations holds.

Theorem 5.3. *Let G be a nonempty linear subspace of a translation-invariant metric linear space X, $x \in X \setminus G$, $M \subset G$ and $\epsilon > 0$. Then $M \subset P_G^\epsilon(x)$ if and only if there exists an element $f \in X_0^\#$ such that*

(i) $\|f\|_d = 1$;

(ii) $f(g) = 0$ *for all* $g \in G$; *and*

(iii) $f(x - m) \ge d(x, m) - \epsilon$ *for all* $m \in M$.

Proof. This is an immediate consequence of Theorem 5.1. ■

The following simultaneous characterization of ϵ-approximations in terms of the annihilator $G^\perp$ of the subspace G holds.

Proposition 5.4. *Let G be a nonempty linear subspace of X, $x \in X \setminus G$, and $M \subset G$. Then $M \subset P_G^\epsilon(x)$ if and only if $d_{G^\perp}(x - m, 0) \ge d(x, m) - \epsilon$ for all $m \in M$.*

References

[1] Deutsch F.R. and P.H. Maserick, Applications of the Hahn-Banach Theorem in approximation theory, *SIAM Review* **6** (1967), 516-530.

[2] Holland A.S.B. and B.N Sahney, *The General Problem of Approximation and Spline Functions*, Krieger, Huntington, NY (1979).

[3] Mustăţa C., On the best approximation in metric spaces, *Mathematica - Revue d'Analyse Numerique et de la Théorie de l'Approximation, L'Analyse Numerique et la Théorie de l'Approximation* **4** (1975), 45 - 50.

[4] Mustăţa C., A Characterization of semi-Chebyshev sets in metric Space, *Mathematica - Revue d'Analyse Numerique et de la Théorie de l'Approximation, L'Analyse Numerique et la Théorie de l'Approximation* **7** (1978), 169 - 170.
[5] Narang T. D., On some approximation problems in metric spaces, *Tamkang Journal of Mathematics* **22** (1991), 99 - 103.
[6] Pantelidis G., Approximationstheorie für metrische lineare Räume, *Math. Ann.* **184** (1969), 30 - 48.
[7] Singer I., *Best Approximation in Normed Linear Spaces by Elements of Linear Subspaces*, Springer-Verlag · Berlin · Heidelberg · New York, 1970.

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