CONVERSES OF FURUTA TYPE INEQUALITIES

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Received April 24, 2002; revised June 3, 2002

Abstract. Let A and B be positive operators on a Hilbert space. We consider what kind of conditions induces the order between A and B. Such an attempt was done in recent works due to Yang and Ito. Based on their results, we prove that if $A^t \natural_{\frac{\gamma-t}{p-t}} B^p \geq B^\gamma$ for $0 < p < t$ and $p < \gamma$, then $A^\beta \geq B^\beta$ for $\beta = \min\{\gamma, t\}$, where the binary operation $\natural$ is used as a generalized formula of the geometric mean. Moreover we give an extension of Ito's theorem.

1. Introduction. Throughout this note, A and B are positive operators on a Hilbert space. An operator T is positive (resp. strictly positive, i.e., positive invertible), we use the notation $T \geq 0$ (resp. $T > 0$). The α-power mean of A and B given by

$$A \sharp_\alpha B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\alpha A^{\frac{1}{2}} \quad \textit{for} \quad 0 \leq \alpha \leq 1$$

is the essential tool in this note which is introduced by Kubo-Ando [15]. Similarly we use the notation $A \natural_s B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^s A^{\frac{1}{2}}$ for $s \in \mathbf{R}$ and $\natural_s = \sharp_s$ when $s \in [0, 1]$.

We cite the mean theoretic expression of the Furuta inequality which is sometimes useful (cf.[2],[11]).

Furuta inequality:(cf.[6],[7]) *If $A \geq B \geq 0$, then*

$$\text{(F)} \qquad A^u \sharp_{\frac{1-u}{p-u}} B^p \leq A \quad \textit{and} \quad B \leq B^u \sharp_{\frac{1-u}{p-u}} A^p$$

holds for $u \leq 0$ and $p \geq 1$.

It is natural to consider whether a similar inequality to (F) holds when the exponent of A transfers to the non-negative part. Firstly, Yoshino [17] pointed out that (F) type inequality holds. Afterward the domain was spreaded and attained to the following theorem.

Complementary theorem of the Furuta inequality:(cf. [4],[8],[9],[12])
If $A \geq B \geq 0$ with $A > 0$, then the following inequalities hold:

$$\text{(CF1)} \qquad A^t \natural_{\frac{1-t}{p-t}} B^p \leq B \quad \textit{for} \quad 0 \leq t < p \quad \textit{and} \quad \frac{1}{2} \leq p \leq 1.$$

$$\text{(CF2)} \qquad A^t \natural_{\frac{2p-t}{p-t}} B^p \leq B^{2p} \quad \textit{for} \quad 0 \leq t < p \leq \frac{1}{2}.$$

2000 *Mathematics Subject Classification.* 47A63 and 47A64.
Key words and phrases. Positive operators, Operator mean, Löwner-Heinz inequality, Furuta inequality, Chaotic order.

In this note, we consider the coverses of the above theorem in some sense. Such a problem is initiated by Yang [16] and Ito [10] has given an including form of Yang's one as follows:

Theorem A. *Let* $A,\ B > 0$ *and* $0 < p < t$.

$$\text{(A)} \qquad If \quad A^t \natural_{\frac{\gamma-t}{p-t}} B^p \geq B^\gamma \quad for \quad p < \gamma \leq 2p, \quad then \quad A^\delta \geq B^\delta \quad for \quad \delta = \min\{\gamma,\ t\}.$$

Our view point is to divide into two cases $t \geq \gamma$ and $t \leq \gamma$. Then we can obtain more precise result, by which the following theorem due to Ito [10] is extended.

Theorem B. *Let* $A,\ B,\ C > 0$ *and* $0 < p < t$. *If* $A^t \natural_{\frac{\gamma-t}{p-t}} B^p \geq B^\gamma$ *for* $p < \gamma \leq 2p$ *and*
$B \gg C$ (i.e., $\log B \geq \log C$), *then for* α *such that* $0 \leq \alpha \leq \min\{\gamma, t\}$,

$$\text{(B)} \qquad C^{-r} \sharp_{\frac{\alpha+r}{t+r}} A^t \geq C^{-r} \natural_{\frac{\alpha+r}{p+r}} B^p$$

holds for $r \geq 0$.

2. Theorems by Yang and Ito. First of all, we review Yang's results. Yang [16] had shown the following:

Let $A,\ B > 0$.

$$\text{(Y1)} \quad If \quad A^t \natural_{\frac{1-t}{p-t}} B^p \leq B \quad for \quad 1 < p < 2p-1 < t, \quad then \quad A^\alpha \geq B^\alpha \quad for \quad 0 \leq \alpha \leq 2p-1.$$

$$\text{(Y2)} \quad If \quad A^t \natural_{\frac{2p-t}{p-t}} B^p \geq B^{2p} \quad for \quad 1 < p < 2p < t, \quad then \quad A^\alpha \geq B^\alpha \quad for \quad 0 \leq \alpha \leq 2p.$$

We here note that (A) interpolates (Y1) and (Y2), i.e., the case $\gamma = 2p$ is (Y2) clearly. On the other hand, the case $\gamma = 2p-1$ is (Y1). Actually, the assumption of (Y1) can be rephrased as $A^t \natural_{\frac{2p-1-t}{p-t}} B^p \geq B^{2p-1}$. Since $B^p \natural_{\frac{1-p}{p-t}} A^t = A^t \natural_{\frac{2p-1-t}{p-t}} B^p \geq B^{2p-1}$, the assumption in (A) is equivalent to $(B^{-\frac{p}{2}} A^t B^{-\frac{p}{2}})^{\frac{1-p}{p-t}} \geq B^{p-1}$ and so is to $A^t \natural_{\frac{1-t}{p-t}} B^p = B^p \natural_{\frac{p-1}{p-t}} A^t \leq B$.

From our view point, we give an interpretation on Yang and Ito's proposal. For $A,\ B > 0$, we denote $A \gg B$ if $\log A \geq \log B$ and call it the chaotic order, which is weaker than the usual order because $\log x$ is an operator monotone function. On the chaotic order, we have the next Theorem C which is essential in our following discussion.

Theorem C. *The chaotic order* $A \gg B$ *for* $A, B > 0$ *if and only if*

$$\text{(C)} \qquad A^u \sharp_{\frac{\delta-u}{p-u}} B^p \leq B^\delta \quad and \quad A^\delta \leq B^u \sharp_{\frac{\delta-u}{p-u}} A^p \quad for \quad u \leq 0 \quad and \quad 0 \leq \delta \leq p.$$

The above theorem is a generalization of our result in [3], we call it chaotic Furuta inequality.

Chaotic Furuta inequality(cf. [1],[3],[5],[13],[14]): If $A \gg B$ for $A, B > 0$, then

$$\text{(FC)} \qquad A^u \sharp_{\frac{-u}{p-u}} B^p \leq I \leq B^u \sharp_{\frac{-u}{p-u}} A^p$$

holds for $u < 0$ and $p > 0$.

Theorem 1. *Let* $A^t \natural_{\frac{\gamma-t}{p-t}} B^p \ge B^\gamma$ *for* $A,\ B > 0,\ 0 < p < t$ *and* $p < \gamma$.

(1) $$If\ \ \gamma \le t, \ \ then\ \ A^\gamma \ge A^t \sharp_{\frac{\gamma-t}{p-t}} B^p\ (\ge B^\gamma).$$

(2) $$If\ \ \gamma \ge t, \ \ then\ \ A^t \ge B^t.$$

Proof. (1) Since $A \natural_s B = B \natural_{1-s} A$, the condition $A^t \sharp_{\frac{\gamma-t}{p-t}} B^p = B^p \sharp_{\frac{\gamma-p}{t-p}} A^t \ge B^\gamma$ is equivalent to $(B^{-\frac{p}{2}} A^t B^{-\frac{p}{2}})^{\frac{\gamma-p}{t-p}} \ge B^{\gamma-p}$. Since $(B^{-\frac{p}{2}} A^t B^{-\frac{p}{2}})^{\frac{1}{t-p}} \gg B$, Theorem C implies that

$$B^{-p} \sharp_{\frac{(\gamma-p)+p}{(t-p)+p}} B^{-\frac{p}{2}} A^t B^{-\frac{p}{2}} \ge (B^{-\frac{p}{2}} A^t B^{-\frac{p}{2}})^{\frac{\gamma-p}{t-p}}.$$

So $I \sharp_{\frac{\gamma}{t}} A^t \ge B^{\frac{p}{2}} (B^{-\frac{p}{2}} A^t B^{-\frac{p}{2}})^{\frac{\gamma-p}{t-p}} B^{\frac{p}{2}}$ holds, that is,

$$A^\gamma \ge B^p \sharp_{\frac{\gamma-p}{t-p}} A^t = A^t \sharp_{\frac{\gamma-t}{p-t}} B^p.$$

Next we prove (2). $A^t \natural_{\frac{\gamma-t}{p-t}} B^p = B^p \natural_{\frac{p-\gamma}{p-t}} A^t \ge B^\gamma$ is equivalent to

$$B^{\frac{p}{2}} (B^{-\frac{p}{2}} A^t B^{-\frac{p}{2}})^{\frac{p-\gamma}{p-t}} B^{\frac{p}{2}} \ge B^\gamma, \quad \text{so} \quad (B^{-\frac{p}{2}} A^t B^{-\frac{p}{2}})^{\frac{p-\gamma}{p-t}} \ge B^{\gamma-p}.$$

We have $B^{-\frac{p}{2}} A^t B^{-\frac{p}{2}} \ge B^{t-p}$ by Löwner-Heinz inequality since $\frac{p-\gamma}{p-t} \ge 1$, so that $A^t \ge B^t$.

3. An extension of Ito's theorm. Ito [10] has shown Theorem B cited in §1 which also includes Yang's one [16] as the cases $\gamma = 2p - 1$ and $\gamma = 2p$. Theorem B has the following extension.

Theorem 2. *Let* $A,\ B,\ C > 0$ *with* $B \gg C$ *and* $0 < p < t$.
(1) *If* $A^\gamma \ge B^\gamma$ *for* $p < \gamma \le \min\{t,\ 2p\}$, *then for given* $r \ge 0$,

$$C^{-r} \sharp_{\frac{\alpha+r}{t+r}} A^t \ge C^{-r} \natural_{\frac{\alpha+r}{p+r}} B^p$$

holds for $-r \le \alpha \le \gamma$.
(2) *If* $A^t \ge B^t$, *then for given* $r \ge 0$,

$$C^{-r} \sharp_{\frac{\alpha+r}{t+r}} A^t \ge C^{-r} \natural_{\frac{\alpha+r}{p+r}} B^p$$

holds for $-r \le \alpha \le \min\{t,\ 2p\}$.

Proof. First of all, we remark that $A \natural_s (A \natural_t B) = A \natural_{st} B$ for $s,\ t \in \mathbf{R}$ and $A,\ B \ge 0$. The case (1) is obtained by applying Theorem C to both $A \gg C$ and $B \gg C$ and the monotone property of operator means for $A^\gamma \ge B^\gamma$ as follows: In the case $-r \le \alpha \le p$,

$$\begin{aligned} C^{-r} \sharp_{\frac{\alpha+r}{t+r}} A^t \ &= \ C^{-r} \sharp_{\frac{\alpha+r}{\gamma+r}} (C^{-r} \sharp_{\frac{\gamma+r}{t+r}} A^t) \ge C^{-r} \sharp_{\frac{\alpha+r}{\gamma+r}} A^\gamma \\ &\ge \ C^{-r} \sharp_{\frac{\alpha+r}{\gamma+r}} B^\gamma = C^{-r} \sharp_{\frac{\alpha+r}{p+r}} (C^{-r} \sharp_{\frac{p+r}{\gamma+r}} B^\gamma) \ge C^{-r} \sharp_{\frac{\alpha+r}{p+r}} B^p. \end{aligned}$$

In the case $p \le \alpha \le \gamma \le t$, since $B \natural_{-s} C = B(B^{-1} \natural_s C^{-1})B$, we have

$$\begin{aligned} & C^{-r} \sharp_{\frac{\alpha+r}{t+r}} A^t = C^{-r} \sharp_{\frac{\alpha+r}{\gamma+r}} (C^{-r} \sharp_{\frac{\gamma+r}{t+r}} A^t) \ge C^{-r} \sharp_{\frac{\alpha+r}{t+r}} A^\gamma \\ \ge \ & C^{-r} \sharp_{\frac{\alpha+r}{\gamma+r}} B^\gamma \ge B^\alpha = B^p (B^{-p} \sharp_{\frac{\alpha-p}{p}} I) B^p \ge B^p (B^{-p} \sharp_{\frac{\alpha-p}{p}} (B^{-p} \sharp_{\frac{p}{p+r}} C^r)) B^p \\ = \ & B^p (B^{-p} \sharp_{\frac{\alpha-p}{p+r}} C^r) B^p = B^p \natural_{\frac{p-\alpha}{p+r}} C^{-r} = C^{-r} \natural_{\frac{\alpha+r}{p+r}} B^p. \end{aligned}$$

The first and Third inequalities follow from (C) because $A \gg C$, the second one follows from the monotone property of means and the final one is given by (FC).

Similarly the case (2) gives (B) as follows:
If $-r \leq \alpha \leq p$, then

$$C^{-r} \natural_{\frac{\alpha+r}{t+r}} A^t = C^{-r} \natural_{\frac{\alpha+r}{p+r}} (C^{-r} \natural_{\frac{p+r}{t+r}} A^t) \geq C^{-r} \natural_{\frac{\alpha+r}{p+r}} A^p \geq C^{-r} \natural_{\frac{\alpha+r}{p+r}} B^p.$$

If $0 < p \leq \alpha \leq \min\{t,\ 2p\}$, then

$$\begin{aligned} & C^{-r} \natural_{\frac{\alpha+r}{p+r}} B^p = B^p \natural_{\frac{p-\alpha}{p+r}} C^{-r} = B^p(B^{-p} \natural_{\frac{\alpha-p}{p+r}} C^r)B^p \\ = \;& B^p(B^{-p} \natural_{\frac{\alpha-p}{p}} (B^{-p} \natural_{\frac{p}{p+r}} C^r))B^p \\ \leq \;& B^p(B^{-p} \natural_{\frac{\alpha-p}{p}} I)B^p = B^\alpha \leq A^t \natural_{\frac{t-\alpha}{t-p}} B^p \leq A^\alpha \leq C^{-r} \natural_{\frac{\alpha+r}{t+r}} A^t. \end{aligned}$$

The first inequality follows from (FC), the second and third ones are mean property and the final one is (C) because $A \gg C$.

Acknowledgement. The author would like to express his thanks to the colleagues of the Saturday semminar in Osaka Kyoiku University for their warm and critical discussions.

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