THE SPECIAL COPRODUCT OF OCKHAM ALGEBRAS*

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Abstract. In this paper we introduce an algebraic concept of the coproduct of Ockham algebras called the special coproduct. We show that if $L_i \in DMS(i = 1, 2, \dots, n)$ then the special coproduct of $L_i(i = 1, 2, \dots, n)$ exists if and only if $L_1, \dots, L_n$ have isomorphic skeletons.

An Ockham algebra is an algebra $\langle L; \vee, \wedge, f, 0, 1\rangle$ of type $\langle 2, 2, 1, 0, 0\rangle$ such that $\langle L; \vee, \wedge, 0, 1\rangle$ is a bounded distributive lattice and f is a unary operation defined on L satisfying, for all $x, y \in L$,

$$f(x \wedge y) = f(x) \vee f(y), f(x \vee y) = f(x) \wedge f(y), f(0) = 1, f(1) = 0.$$

In such an algebra $\langle L; f\rangle$ the subset $S(L) = \{f(x)|x \in L\}$ is a subalgebra which we call the *skeleton* of L. The class of all Ockham algebras is a variety, denoted by O. Cleariy, if $L_i \in O(i = 1, \dots, n)$, the direct product $\prod_{i=1}^{n} L_i$, where the operation $\sim$ is defined by $(x_1, \dots, x_n)^{\sim} = (x_1^{\sim}, \dots, x_n^{\sim})$, is also an Ockham algebra.

The study of Ockham algebras has been initiated by J.Berman [2] who gave particular attention to certain subvariety $K_{p,q}$ of Ockham algebra $\langle L; f\rangle$ in which $f^q = f^{2p+q}$. The subvariety of $K_{p,q}$ defined by the inequality $x \geq f^2(x)$ is denoted by *DMS*, and its members are called dual *MS*-algebras .

We recall that a mapping $h : X \to Y$, where X, Y are lattices, is a *homomorphism* if, for any $a, b \in X, h(a \wedge b) = h(a) \wedge h(b)$ and $h(a \vee b) = h(a) \vee h(b)$. Such an h is said to be an *isomorphism* if it is one-to-one. A mapping h is called a *kernel* if $h^2 = h \leq id$. Let $\langle L; \sim\rangle, \langle M; \sim\rangle$ be Ockham algebras.We say a lattice homomorphism $h : L \to M$ is an *(Ockham) homomorphism* if $(h(a))^{\sim} = h(a^{\sim})$. Such an (Ockham) homomorphism h is an (Ockham) isomorphism if it is one-to-one,denoted by $\simeq$.

Here we introduce a particular algebraic concept of the coproduct of Ockham algebras which is called the *special coproduct*.We show that the special coproduct of Ockham algebras is a subalgebra of the direct product Ockham algebras. In particular, if $L_i \in DMS(i = 1, 2, \dots, n)$ then the special coproduct of $L_1, L_2, \dots, L_n$ has isomorphic skeletons.

For later convenience we denote the category of bounded distributive lattices with 0 and 1 by $D_{0,1}$.

Definition. *Let $L_1, \dots, L_n$ be bounded distributive lattices with* 0 *and* 1 *, and let the maps $f_{ij} : L_i \to L_j$ be lattice homomorphisms such that*

1. $(\forall i) f_{ii} = id_{L_i}$
2. $(\forall i, j, k) f_{ij} \geq f_{kj} \circ f_{ik}$

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By the special coproduct of $L_1, \dots, L_n$ relative to the family of homomorphisms f_{ij} we mean the subset $\sum_{i=1}^{n} L_i$ of $\prod_{i=1}^{n} L_i$ given by

$$\sum_{i=1}^{n} L_i = \{(x_1, x_2, \dots, x_n) \in \prod_{i=1}^{n} L_i : x_i = \bigvee_{j=1}^{n} f_{ji}(x_j) \text{ , for any } i \}$$

We can see from the definition that

$$(x_1, \dots, x_n) \in \sum_{i=1}^{n} L_i \Leftrightarrow (\forall i) x_i = \bigvee_{j=1}^{n} f_{ji}(x_j) \Leftrightarrow (\forall i)(\forall j) f_{ij}(x_i) \leq x_j.$$

It is then easy to see that $\sum_{i=1}^{n} L_i$ is a sublattice of $\prod_{i=1}^{n} L_i$ containing $(1, \dots, 1)$ and $(0, \dots, 0)$. We first show the following fact.

Theorem 1. *Let $L_1, \dots, L_n \in D_{0,1}$. Then L is the special coproduct of $L_1, \dots, L_n$ relative to homomorphisms $f_{ij} : L_i \to L_j$ if and only if there exist kernel homomorphisms $h_1, \dots, h_n$ on L such that $\bigvee_{i=1}^{n} h_i = id_L$,where $(\bigvee_{i=1}^{n} h_i)(x) = \bigvee_{i=1}^{n} h_i(x) (\forall x \in L)$.*

Proof. $\Longrightarrow$: Suppose that L is the special coproduct of $L_1, \dots, L_n$ relative to homomorphisms $f_{ij} : L_i \to L_j$. Then, for any i, j, k, $f_{ij}(x_i) \geq f_{kj}(f_{ik}(x_i))$ and $(f_{j1}(x_j), f_{j2}(x_j), \dots, f_{jn}(x_j)) \in \sum_{i=1}^{n} L_i$(for any j). Define the mapping $h_j : \sum_{i=1}^{n} L_i \to \sum_{i=1}^{n} L_i$ by $h_j(x_1, \dots, x_n) = (f_{j1}(x_j), \dots, f_{jn}(x_j))$. It is clear that each h_j is a homomorphism,and the fact that $f_{ij}(x_i) \leq x_j$ (for any i, j). We have

$$\begin{aligned} h_j^2(x_1, \dots, x_n) &= h_j(f_{j1}(x_j), \dots, f_{jn}(x_j)) \\ &= (f_{j1}(f_{jj}(x_j)), \dots, f_{jn}(f_{jj}(x_j))) \\ &= (f_{j1}(x_j), \dots, f_{jn}(x_j)) \\ &= h_j(x_1, \dots, x_n) \\ &\leq (x_1, \dots, x_n) \end{aligned}$$

thus $h_j^2 = h_j \leq id$, so h_j is a kernel. Finally,

$$(\bigvee_{j=1}^{n} h_j)(x) = \bigvee_{j=1}^{n} h_j(x) = (\bigvee_{j=1}^{n} f_{j1}(x_j), \dots, \bigvee_{j=1}^{n} f_{jn}(x_j)) = (x_1, \dots, x_n)$$

, and so $\bigvee_{j=1}^{n} h_j = id_L$.

$\Longleftarrow$: Suppose now that there are kernel homomorphisms $h_1, \dots, h_n$ on L such that $\bigvee_{j=1}^{n} h_j = id_L$. Write $L_i = Imh_i$, the image of h_i, and define $f_{ij} : L_i \to L_j$ by $f_{ij}(h_i(x)) = h_j(h_i(x))$, namely, f_{ij} is induced by the restriction of h_j to Imh_i. Since $h_i^2 = h_i \leq id_{L_i}$ by the hypothesis,we have $f_{ii} = id_{L_i}$ and $f_{kj} \circ f_{ik}(h_i(x)) = f_{kj}(h_k(h_i(x))) = h_j(h_k(h_i(x))) \leq h_j(h_i(x)) = f_{ij}(h_i(x)), i.e., f_{ij} \geq f_{kj} \circ f_{ik}$. It follows that $\sum_{i=1}^{n} L_i$ exists. Consider the homomorphism $h : L \to \sum_{i=1}^{n} L_i$ defined by $h(x) = (h_1(x), \dots, h_n(x))$. Observe that

$$h(x) = h(y) \Rightarrow h_i(x) = h_i(y)(\text{ for any } i) \Rightarrow x = \bigvee_{i=1}^{n} h_i(x) = \bigvee_{i=1}^{n} h_i(y) = y,$$

so h is injective. Now, for $(h_1(x_1),\dots,h_n(x_n))\in\sum_{i=1}^{n}L_i$, we have $h_j(x_j)=\bigvee_{i=1}^{n}f_{ij}(h_i(x_i))$. Let $z=\bigvee_{i=1}^{n}h_i(x_i)$ .Then $h_j(z)=h_j(\bigvee_{i=1}^{n}h_i(x_i))=\bigvee_{i=1}^{n}h_j(h_i(x_i))=\bigvee_{i=1}^{n}f_{ij}(h_i(x_i))=h_j(x_j)$, whence

$$h(z)=(h_1(z),\dots,h_n(z))=(h_1(x_1),\dots,h_n(x_n)).$$

It follows that h is surjective and so $L\simeq\sum_{i=1}^{n}L_i$.

Corollary 1. *$L\in D_{0,1}$ is the special coproduct of $L_1,L_2\in D_{0,1}$ if and only if L induces a pair of kernel homomorphisms h_1,h_2 such that $h_1\vee h_2=1$.*

Theorem 2. *If $L_1,\dots,L_n\in O$ then the special coproduct of $L_1,\dots,L_n$ relative to (Ockham) homomorphisms $f_{ij}:L_i\to L_j$ is a subalgebra of $\prod_{i=1}^{n}L_i$.*

Proof. It suffices to show that $(x_1,\dots,x_n)\in\sum_{i=1}^{n}L_i$ implies $(x_1^\sim,\dots,x_n^\sim)\in\sum_{i=1}^{n}L_i$. Let $(x_1,\dots,x_n)\in\sum_{i=1}^{n}L_i$. Then

$(\forall i)(\forall j)f_{ij}(x_i)\le x_j$ and $(\forall i)(\forall j)f_{ij}(x_i^\sim)\ge x_j^\sim$.

We thus have

$$(\forall i)(\forall j)f_{ji}(x_j^\sim)=f_{ji}(f_{jj}(x_j^\sim))\ge f_{ji}(f_{ij}(f_{ji}(x_j^\sim)))\ge f_{ji}(f_{ij}(x_i^\sim))\ge f_{ji}(x_j^\sim).$$

So $(\forall i)(\forall j)f_{ji}(x_j^\sim)=f_{ji}(f_{ij}(x_i^\sim))\le f_{ii}(x_i^\sim)=x_i^\sim$ and $(\forall i)(\forall j)f_{ij}(x_i^\sim)=x_j^\sim$. It follows that $(\forall j)\bigvee_{i=1}^{n}f_{ij}(x_i^\sim)=x_j^\sim$, whence $(x_1^\sim,\dots,x_n^\sim)\in\sum_{i=1}^{n}L_i$.

Corollary 2. *If $L_1,\dots,L_n\in K_{p,q}$ then the special coproduct of $L_1,\dots,L_n$ relative to (Ockham) homomorphisms $f_{ij}:L_i\to L_j$ belongs to $K_{p,q}$.*

Observe that Theorem 1 carries over to Ockham algebras. In fact, it suffices to replace all the bounded disreibutive lattices by Ockham algebras, and all the homomorphisms in $D_{0,1}$ by Ockham homomorphisms. That the lattice homomorphisms $h_j:\sum_{i=1}^{n}L_i\to\sum_{i=1}^{n}L_i$ given by $h_j(x_1,\dots,x_n)=(f_{j1}(x_j),\dots,f_{jn}(x_j))$ are Ockham homomorphisms follows from the fact $(\forall i)(\forall j)f_{ij}(x_i^\sim)=x_j^\sim$, for then

$$\begin{aligned}(*)\quad (h_j(x_1,\dots,x_n))^\sim &= (f_{j1}(x_j^\sim),\dots,f_{jn}(x_j^\sim))\\ &= (x_1^\sim,\dots,x_n^\sim)\\ &= h_j(x_1^\sim,\dots,x_n^\sim).\end{aligned}$$

We now give a descripition of some properies of the special coproduct of Ockham algebras.

Theorem 3. *Let $L_1,\dots,L_n\in O$ and let $\sum_{i=1}^{n}L_i$ be the special coproduct of $L_1,\dots,L_n$ relative to (Ockham) homomorphisms $f_{ij}:L_i\to L_j$. Then there exist kernel homomorphisms $h_k:\prod_{i=1}^{n}L_i\to\sum_{i=1}^{n}L_i$ such that $\bigvee_{k=1}^{n}h_k=id$ and $L_k\simeq Imh_k(k=1,\dots,n)$. Moreover, $S(Imh_i)\simeq S(Imh_j)(\forall i,j)$.*

Proof. Let $L_j^*=\{(f_{j1}(x_j),\dots,f_{jn}(x_j))|x_j\in L_j\}(j=1,\dots,n)$. Write $y_i=f_{ji}(x_j)$. Then $y_i=f_{ji}(x_j)\ge f_{ki}(f_{jk}(x_j))=f_{ki}(y_k)(\forall j,k)$, and then each L_j^* is a subalgebra of

$\sum_{i=1}^{n} L_i$ .Since $f_{jj}(x_j) = x_j$, it is easy to see that $L_j \simeq L_j^*$ by $x_j \to (f_{j1}(x_j), \dots, f_{jn}(x_j))$. Define the mapping $h_j \prod_{i=1}^{n} L_i \to \sum_{i=1}^{n} L_i$ by $h_j(x_1, \dots, x_n) = (f_{j1}(x_j), \dots, f_{jn}(x_j))$. Then $Imh_j = L_j^* \simeq L_j$. Arguing as in the proof of Theorem 1 we have $h_j^2 = h_j \leq id$ and $\bigvee_{j=1}^{n} h_j = id$. Finally, we can see from (*) above that $S(Imh_j) \simeq S(\sum_{i=1}^{n} L_i)$.

Theorem 4. *Let $L_1, \dots, L_n$ be DMS-algebras having isomorphic skeletons. Then the special coproduct of $L_1, \dots, L_n$ exists, and moreover, it can be simultaneously (Ockham) isomorphically embedded by each L_i.*

Proof. Assume that $\alpha_i : S(L_i) \to S(L_1)$ is an (Ockham) isomorphism. Let $f_{ii} = id_{L_i}(\forall i)$, and for $i \neq j$ define $f_{ij} : L_i \to L_j$ by $f_{ij}(x_i) = \alpha_j^{-1}(\alpha_i(x_i^{\sim\sim}))(\forall x_i \in L_i)$. Then these f_{ij} are (Ockham) homomorphisms. Observe that

$$\begin{aligned}(i \neq j, k \neq i, k \neq j) f_{kj}(f_{ik}(x_i)) &= f_{kj}(\alpha_k^{-1}(\alpha_i(x_i^{\sim\sim}))) \\ &= \alpha_j^{-1}(\alpha_k(\alpha_k^{-1}(\alpha_i(x_i^{\sim\sim\sim\sim})))) \\ &= \alpha_j^{-1}(\alpha_i(x_i^{\sim\sim})) \\ &= f_{ij}(x_i).\end{aligned}$$

and

$$f_{ki}(f_{ik}(x_i)) = f_{ki}(\alpha_k^{-1}(\alpha_i(x_i^{\sim\sim}))) = \alpha_i^{-1}(\alpha_k(\alpha_k^{-1}(\alpha_i(x_i^{\sim\sim})))) = x_i^{\sim\sim} \leq x_i$$

It follows that these f_{ij} satisfy (1) and (2) in the definition above, and hence the special coproduct of $L_1, \dots, L_n$ relative to f_{ij} exists.

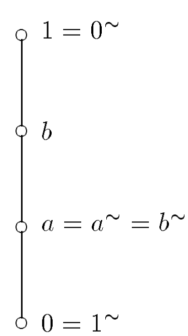
Define now $g_i : L_i \to \sum_{i=1}^{n} L_i$ by $g_i(x_i) = (f_{i1}(x_i), \dots, f_{in}(x_i))$ .We can see from the proof of Theorem 3 that each g_i is a homomorphism and is injective. Consequently, under g_i, each L_i can be (Ockham) isomorphically embeded into $\sum_{i=1}^{n} L_i$. By Theorem 3 and 4 we have immediately the following interesting fact.

Corollary 3. *Let $L_1, \dots, L_n$ be DMS-algebras.Then the special coproduct of $L_1, \dots, L_n$ exists if and only if $L_1, \dots, L_n$ have isomorphic skeletons .*

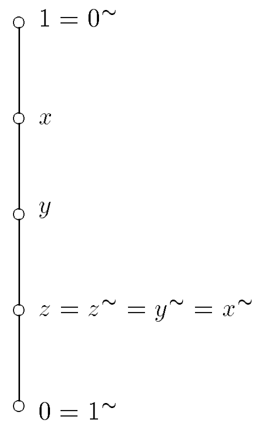
Example 1. Let L be a *DMS*-algebra. Define $L_1 = L_2 = L$ and let $f_{11} = f_{12} = f_{22} = id, f_{21}(x) = x^{\sim\sim}(\forall x \in L)$. Then the conditions (1) and (2) in the definition above are satisfied,and so we can form the *DMS*-algebra

$$L + L = \{(x, y) \in L \times L | y^{\sim\sim} \leq x \leq y\}$$

which then induces a pair of kernel homomorphisms h_1, h_2 such that $h_1 \vee h_2 = id$. More specifically, taking L to be the algebra S_9 described by



we have $S_9 + S_9 = \{(0,0),(a,a),(a,b),(b,b),(1,1)\}$, and so $S_9 + S_9$ is isomorphic to the following *DMS*-algebra



and the homomorphisms h_1 and h_2 are given by

$$h_1(x_1, x_2) = (x_1, x_1) \text{ and } h_2(x_1, x_2) = (x_2^{\sim\sim}, x_2) \ .$$

So $Imh_1 = \{(0,0),(a,a),(b,b),(1,1)\}$ and $Imh_2 = \{(0,0),(a,a),(a,b),(1,1)\}$ .The corresponding partitions are then

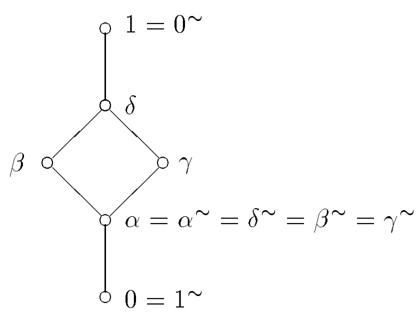
$$h_1 \sim \{\{0\},\{y,z\},\{x\},\{1\}\} \text{ and } h_2 \sim \{\{0\},\{x,y\},\{z\},\{1\}\},$$

thus proving that $h_1 \vee h_2 = id$.

Example 2. Let L be a *DMS*-algebra. Define $L_1 = L_2 = L$ and let $f_{11} = f_{22} = id, f_{12}(x) = f_{21}(x) = x^{\sim\sim}(\forall x \in L)$. Then the conditions (1) and (2) in the definition above are satisfied, and so the special coproduct relative to $f_{ij}(i,j = 1,2)$ is as follows:

$$L + L = \{(x,y) \in L \times L | x^{\sim\sim} = y^{\sim\sim}\}$$

More specifically, taking $L = S_9$ as in Example 1,we have
$S_9 + S_9 = \{(0,0),(a,a),(a,b),(b,a),(b,b),(1,1)\}$,
which is isomorphic to the *DMS*-algebra



The associate kernel homomorphisms have the partitions

$$h_1 \sim \{\{0\},\{\alpha,\beta\},\{\delta,\gamma\},\{1\}\} \text{ and } h_2 \sim \{\{0\},\{\alpha,\gamma\},\{\beta,\delta\},\{1\}\} \ .$$

thus proving that $h_1 \vee h_2 = id$.

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