UNIQUELY DETERMINEDNESS OF THE APPROXIMATELY ORDER DERIVATIVE

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Abstract. In the previous paper [1], we defined the approximately order derivative. But it is so to be regreted that we did not show the uniqueness of the derivative, that is, the derivative turned out to be the set. In this paper, we show that the derivative consists of one point.

1 Introduction Let X be the N-dimensional Euclidean space, Y a complete vector lattice, μ the Lebesgue measure on X, μ^* the Lebesgue outer measure on X and $\mathcal{A}$ the family of all Lebesgue measurable sets in X. Let $\mathcal{E}_X$ be the family of all Archimedean units in X, that is, $\mathcal{E}_X = \{(x_1, \cdots, x_N) | x_i > 0 \text{ for all } i\}$. Let $\mathcal{L}(X, Y)$ be the family of all linear operators translated bounded sets in X to bounded sets in Y. Then for $l_1, l_2 \in \mathcal{L}(X, Y)$ we define that $l_1 \leq l_2$ if $l_1(x) \leq l_2(x)$ for any $x \in X$ with $x \geq 0$. With regard to this order, $\mathcal{L}(X, Y)$ is a complete vector lattice.

We say that x is a right density point in $E \subset X$ if for any real number $\varepsilon > 0$ there exists some $e \in \mathcal{E}_X$ such that $\mu^*(E^C \cap [x, x+h]) < \varepsilon\mu([x, x+h])$ for any $h \in \mathcal{E}_X$ with $0 < h \leq e$; and, that x is a left density point in $E \subset X$ if for any real number $\varepsilon > 0$ there exists some $e \in \mathcal{E}_X$ such that $\mu^*(E^C \cap [x-h, x]) < \varepsilon\mu([x-h, x])$ for any $h \in \mathcal{E}_X$ with $0 < h \leq e$; moreover, that x is a density point if it is a right density point and a left density point. We say that x is a right dispersion point in $E \subset X$ if for any real number $\varepsilon > 0$ there exists some $e \in \mathcal{E}_X$ such that $\mu^*(E \cap [x, x+h]) < \varepsilon\mu([x, x+h])$ for any $h \in \mathcal{E}_X$ with $0 < h \leq e$; and, that x is a left dispersion point in $E \subset X$ if for any real number $\varepsilon > 0$ there exists some $e \in \mathcal{E}_X$ such that $\mu^*(E \cap [x-h, x]) < \varepsilon\mu([x-h, x])$ for any $h \in \mathcal{E}_X$ with $0 < h \leq e$; moreover, that x is a dispersion point if it is a right dispersion point and a left dispersion point. When we consider the density and the dispersion, in [1] we assume that $E \in \mathcal{A}$, but in this paper we assume only that E is a subset of X. Under this situation we can prove the all statements in [1] similarly.

There is no point that it is a right density point and a right dispersion point, or a left density point and a left dispersion point, simultaneously. However any points are not always either a right density point or a right dispersion point, a left density point or a left dispersion point.

2 Main result In this section for $l \in \mathcal{L}(X, Y)$, $F : E \longrightarrow Y$ and $x \in E$ we use the symbols defined in [1], $E^+_{sup}(l; F, x)$, $L^+_{sup}(F, x)$, $o\text{-}\overline{AD}^+F(x)$, $o\text{-}AD^+F(x)$, $o\text{-}ADF(x)$, etc.

Lemma 2.1. *Let X, Y be vector lattices and $l \in \mathcal{L}(X, Y)$. Then if $\{x_n\}$ in X converges to 0, $\{l(x_n)\}$ converges to 0.*

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Proof. We assume that $\{x_n\}$ converges to 0, that is, there exists some $x_0 \in X$ with $x_0 > 0$ such that, for any natural number m there exists some natural number N such that $|x_n| \leq \frac{1}{m}x_0$ for any natural number $n > N$.

There exists a monotone increasing sequence $\{r_n\}$ of real numbers divergent to the infinity such that $\{r_n x_n\}$ converges to 0. In fact there exists a monotone increasing sequence $\{N(m)\}$ of natuaral numbers such that $|x_n| \leq \frac{1}{m^2}x_0$ for any $n > N(m)$. Let

$$r_n = \begin{cases} 1 & \text{if } n \leq N(1), \\ m & \text{if } N(m) < n \leq N(m+1)\,(m = 1, 2, \cdots). \end{cases}$$

Then since

$$|r_n x_n| = \begin{cases} |x_n| & \text{if } n \leq N(1), \\ m|x_n| \leq \dfrac{1}{m}x_0 & \text{if } N(m) < n \leq N(m+1)\,(m = 1, 2, \cdots), \end{cases}$$

we get that $\{r_n x_n\}$ converges to 0 and $\{r_n\}$ diverges to the infinity.

$\{r_n x_n\}$ is bounded because $\{r_n x_n\}$ converges to 0. Since $l \in \mathcal{L}(X, Y)$, $\{r_n l(x_n)\}$ is also bounded, that is, there exists some $y_0 \in Y$ with $y_0 > 0$ such that $r_n|l(x_n)| \leq y_0$. If for m we select N satisfied $r_{N+1} \geq m$, for any $n > N$ we get $|l(x_n)| \leq \frac{1}{r_n}y_0 \leq \frac{1}{m}y_0$, that is, $\{l(x_n)\}$ converges to 0. □

Lemma 2.2. *Let $l \in \mathcal{L}(X, Y)$ and $E \in \mathcal{A}$.*

(1) *For a right density point $x_0 \in E$*

$$o\text{-}\overline{AD}^{+} l(x_0) = o\text{-}\underline{AD}^{+} l(x_0) = \{l\}.$$

(2) *For a left density point $x_0 \in E$*

$$o\text{-}\overline{AD}^{-} l(x_0) = o\text{-}\underline{AD}^{-} l(x_0) = \{l\}.$$

Proof. We get that $l \in o\text{-}\overline{AD}^{+} l(x_0)$ because l is satisfied the following conditions for $F = l$.

(a-SUP1$_R$) For any $l' \in \mathcal{L}(X, Y)$ with $l' > 0$, there exists some $l'' \in L^{+}_{sup}(F, x_0)$ such that $l \leq l'' < l + l'$.

(a-SUP2$_R$) For any $l'' \in L^{+}_{sup}(F, x_0)$, $l'' \not< l$.

Similarly we get that $l \in o\text{-}\underline{AD}^{+} l(x_0)$, $l \in o\text{-}\overline{AD}^{-} l(x_0)$ and $l \in o\text{-}\underline{AD}^{-} l(x_0)$.

If $l'' > l$, $E^{+}_{sup}(l''; l, x_0) = \{x | x \in E, x \geq x_0, l(x - x_0) = l''(x - x_0)\}$. Then the dimension of $E^{+}_{sup}(l''; l, x_0)$ is less than N and x_0 is a right dispersion point of $E^{+}_{sup}(l''; l, x_0)$.

If $l'' = l$, $E^{+}_{sup}(l''; l, x_0) = \{x | x \in E, x \geq x_0\}$. Then x_0 is not a right dispersion point of $E^{+}_{sup}(l''; l, x_0)$.

Next note that every $x \in X$ is represented by

$$\begin{aligned} x &= f(r, \theta_1, \cdots, \theta_{N-1}) \\ &= r(\cos\theta_1 \cdots \cos\theta_{N-1}, \cos\theta_1 \cdots \sin\theta_{N-1}, \cdots, \sin\theta_1) \end{aligned}$$

for $r \geq 0, 0 \leq \theta_1 < 2\pi, 0 \leq \theta_i < \pi\,(i = 2, \cdots, N-1)$.

If $l'' \ngeq l$, there exists some $f(r_0, \theta_{1,0}, \cdots, \theta_{N-1,0})$ with $r_0 > 0$ and $0 \leq \theta_{i,0} \leq \dfrac{\pi}{2} (i = 1, \cdots, N-1)$ such that

$$l''(f(r_0, \theta_{1,0}, \cdots, \theta_{N-1,0})) \ngeq l(f(r_0, \theta_{1,0}, \cdots, \theta_{N-1,0})).$$

There exists some α_i with $0 < \alpha_i + \theta_{i,0} < \dfrac{\pi}{2}, \alpha_i \neq 0$ such that for any θ_i with $|\theta_i - \theta_{i,0}| \leq |\alpha_i|, 0 \leq \theta_i \leq \dfrac{\pi}{2}$ for $i = 1, \cdots, N-1$

$$l''(f(r_0, \theta_1, \cdots, \theta_{N-1})) \ngeq l(f(r_0, \theta_1, \cdots, \theta_{N-1})).$$

If not, let $\alpha_{i,0} \neq 0$ for $i = 1, \cdots, N-1$, and for a sequence $\{\alpha_{i,k}\}$ with $0 < \alpha_{i,k} + \theta_{i,0} < \dfrac{\pi}{2}, 0 < |\alpha_{i,k}| \leq \dfrac{1}{2}|\alpha_{i,k-1}|$ for $i = 1, \cdots, N-1$ there exists $\theta_{i,k}$ with $|\theta_{i,k} - \theta_{i,0}| \leq |\alpha_{i,k}|, 0 \leq \theta_{i,k} \leq \dfrac{\pi}{2}$ such that

$$l''(f(r_0, \theta_{1,k}, \cdots, \theta_{N-1,k})) \geq l(f(r_0, \theta_{1,k}, \cdots, \theta_{N-1,k})).$$

The above sequence $\{f(r_0, \theta_{1,k}, \cdots, \theta_{N-1,k})\}$ converges to $f(r_0, \theta_{1,0}, \cdots, \theta_{N-1,0})$, but it is contradictory from Lemma 2.1. Thus there exists some α_i with $0 < \alpha_i + \theta_{i,0} < \dfrac{\pi}{2}, \alpha_i \neq 0$ such that for any θ_i with $|\theta_i - \theta_{i,0}| \leq |\alpha_i|, 0 \leq \theta_i \leq \dfrac{\pi}{2}$ for $i = 1, \cdots, N-1$

$$l''(f(r_0, \theta_1, \cdots, \theta_{N-1})) \ngeq l(f(r_0, \theta_1, \cdots, \theta_{N-1})).$$

Since l'' and l are linear, the above inequality is held for arbitrary $r_0 > 0$. Let

$$W = \{f(r, \theta_1, \cdots, \theta_{N-1}) | r > 0, |\theta_i - \theta_{i,0}| \leq |\alpha_i|, 0 \leq \theta_i \leq \frac{\pi}{2} \, (i = 1, \cdots, N-1)\}.$$

Then $E \cap (x_0 + W) \subset E^+_{sup}(l''; l, x_0)$. Let $El(h) = El(h_1, \cdots, h_N)$ be the intersection of an ellipsoid, which radii are $h_1, \cdots, h_N$, and $\{(x_1, \cdots, x_N) | x_i > 0 \text{ for any } i\}$. Then $x_0 + El(h) \subset [x_0, x_0 + h]$. From the above

$$\mu^*(E^+_{sup}(l''; l, x_0) \cap [x_0, x_0 + h]) \quad \geq \quad \mu(E \cap (x_0 + W) \cap (x_0 + El(h))).$$

Since x_0 is a right density point in E, that is,

$$\begin{aligned} \mu(E^C \cap (x_0 + W) \cap (x_0 + El(h))) &\leq \mu(E^C \cap [x_0, x_0 + h]) \\ &< \varepsilon\mu([x_0, x_0 + h]), \end{aligned}$$

if

$$\mu(E \cap (x_0 + W) \cap (x_0 + El(h))) < \varepsilon\mu([x_0, x_0 + h]),$$

then

$$\mu((x_0 + W) \cap (x_0 + El(h))) < 2\varepsilon\mu([x_0, x_0 + h]).$$

On the other hand,

$$\begin{aligned} \mu((x_0 + W) \cap (x_0 + El(h))) &\geq \frac{|\alpha_1 \cdots \alpha_{N-1}|}{2\pi^{N-1}} \times \frac{\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2} + 1)} \times h_1 \cdots h_N \\ &= \frac{|\alpha_1 \cdots \alpha_{N-1}|}{2\pi^{\frac{N}{2}-1}\Gamma(\frac{N}{2} + 1)} h_1 \cdots h_N \\ &= \frac{|\alpha_1 \cdots \alpha_{N-1}|}{2\pi^{\frac{N}{2}-1}\Gamma(\frac{N}{2} + 1)} \mu([x_0, x_0 + h]), \end{aligned}$$

where Γ is Γ-function. It is contradictory and x_0 is not a right dispersion point of $E^{+}_{sup}(l''; l, x_0)$.

Therefore $l'' \in L^{+}_{sup}(l, x_0)$ if and only if $l'' > l$. Let $l_1 \in o\text{-}\overline{AD}^{+} l(x_0)$. For any $l' > 0$ there exists $l'' \in L^{+}_{sup}(l, x_0)$ such that $l_1 \leq l'' < l_1 + l'$. Since l' is arbitrary, we get that $l \leq l_1$. Since from Theorem 3.1 in [1] any different points in $o\text{-}\overline{AD}^{+} l(x_0)$ is incomparable, we get that $l_1 = l$.

Similarly we can prove the rest. □

Theorem 2.1. *Let $E \in \mathcal{A}$, $F : E \longrightarrow Y$.*

(1) *Let x be a right density point in E. If F is approximately right order differentiable at x, then $o\text{-}AD^{+}F(x)$ consists of one point.*

(2) *Let x be a left density point in E. If F is approximately left order differentiable at x, then $o\text{-}AD^{-}F(x)$ consists of one point.*

Proof. We put $F_1 = F, F_2 = -F$ for Theorem 4.3(1) in [1]. Then from Lemma 2.2

$$o\text{-}AD^{+}F(x) - o\text{-}AD^{+}F(x) = o\text{-}AD^{+}0(x) = \{0\}.$$

Therefore $o\text{-}AD^{+}F(x)$ consists of one point.

Similarly we can prove that $o\text{-}AD^{-}F(x)$ consists of one point. □

References

[1] T. Kawasaki, Approximately order derivatives in vector lattices, Math. Japonica, **49** (1999), 229-239.

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