STRONG CONVERGENCE OF ITERATIVE SEQUENCES FOR ASYMPTOTICALLY NONEXPANSIVE MAPPINGS IN BANACH SPACES

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ABSTRACT. In this paper, we deal with an iteration process for an asymptotically nonexpansive mapping and prove a strong convergence theorem for the mapping in Banach spaces, which is a generalization of the recent result of Shioji and Takahashi [12].

1. Introduction

Let C be a nonempty closed convex subset of a real Banach space E and let T be a mapping of C into itself. Then, we denote by $F(T)$ the set of fixed points of T. A mapping T of C into itself is said to be *nonexpansive* if $\|Tx - Ty\| \leq \|x-y\|$ for every $x, y \in C$ and a mapping T of C into itself is said to be *asymptotically nonexpansive* with Lipschitz constants $\{k_n\}$ if $\varliminf_{n\to\infty} k_n \leq 1$ and $\|T^n x - T^n y\| \leq k_n \|x-y\|$ for every $x, y \in C$ (see [3]).

Let C be a nonempty closed convex subset of a real Hilbert space H and let T be a nonexpansive mapping of C into itself. Let $x \in C$. Halpern [4] and Reich [9] considered the following iteration process:

$$x_0 \in C, \quad x_{n+1} = \alpha_n x + (1-\alpha_n) T x_n \tag{1}$$

for each $n = 0, 1, 2, \ldots$, where $\{\alpha_n\}$ is a sequence in $[0,1]$. Wittmann [15] showed that $\{x_n\}$ defined by (1) converges strongly to the element of $F(T)$ which is nearest to x if $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ and $F(T) \neq \emptyset$. Shioji and Takahashi [10] extended the result of Wittmann [15] to a Banach space.

Let T be an asymptotically nonexpansive mapping of a nonempty bounded closed convex subset C of H and let $x \in C$. Using the concept of mean, Shimizu and Takahashi [13] studied the strong convergence of the following iteration process for an asymptotically nonexpansive mapping:

$$x_0 \in C, \quad x_n = \alpha_n x + (1-\alpha_n) \frac{1}{n+1} \sum_{j=0}^{n} T^j x_n \tag{2}$$

for sufficient large integer n, where $\{\alpha_n\}$ is a sequence in $[0,1]$. Shioji and Takahashi [11] extended the result of [13] to a Banach space. Further, Shioji and Takahashi [12] proved the following theorem by using the results of [11] (see also [14]): Let E be a uniformly convex Banach space whose norm is uniformly Gâteaux differentiable and let C be a nonempty bounded closed convex subset of E. Let T be an asymptotically nonexpansive mapping on C with Lipschitz constants $\{k_n\}$. Let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \leq$

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$\alpha_n \le 1$, $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$ and $\sum_{n=0}^{\infty} \left((1-\alpha_n)\left(\frac{1}{n+1}\sum_{j=0}^{n} k_j\right)^2 - 1\right)_+ < \infty$. Let $x \in C$ and let $\{x_n\}$ be the sequence defined by

$$x_0 \in C, \quad x_{n+1} = \alpha_n x + (1-\alpha_n)\frac{1}{n+1}\sum_{j=0}^{n} T^j x_n \tag{3}$$

for each $n = 0, 1, 2, \ldots$. Then, $\{x_n\}$ converges strongly to Px, where P is the sunny nonexpansive retraction from C onto $F(T)$. Mann [6] introduced the following iteration process for approximating fixed points of a nonexpansive mapping T on a nonempty closed convex subset C in a Hilbert space:

$$x_0 \in C, \quad x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tx_n \tag{4}$$

for each $n = 0, 1, 2, \ldots$, where $\{\alpha_n\}$ is a sequence in $[0, 1]$. Later, Reich [8] studied the sequence defined by (4) in a uniformly convex Banach space whose norm is Fréchet differentiable and obtained a weak convergence theorem (see also [1]).

In this paper, we introduce an iteration process for mappings of C into itself by using the ideas of [1, 6, 12]. We prove a strong convergence theorem for an asymptotically nonexpansive mapping, which is a generalization of the result of Shioji and Takahashi [12].

2. Preliminaries

Throughout this paper, E is a real Banach space and E^* is the dual space of E. We write $x_n \to x$ (or $\lim_{n\to\infty} x_n = x$) to indicate that the sequence $\{x_n\}$ of vectors converges strongly to x. We also denote by $\langle y, x^* \rangle$ the value of $x^* \in E^*$ at $y \in E$. We denote by $\mathbb{N}$ the set of all nonnegative integers. We also denote $\max\{a, 0\}$ by $(a)_+$ for a real number a.

A Banach space E is said to be strictly convex if $\|x + y\|/2 < 1$ for $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. In a strictly convex Banach space, we have that if $\|x\| = \|y\| = \|(1-\lambda)x + \lambda y\|$ for $x, y \in E$ and $\lambda \in (0, 1)$ then $x = y$. For every ε with $0 \le \varepsilon \le 2$, we define the modulus $\delta(\varepsilon)$ of convexity of E by

$$\delta(\varepsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} \;\middle|\; \|x\| \le 1, \|y\| \le 1, \|x - y\| \ge \varepsilon\right\}.$$

A Banach space E is said to be uniformly convex if $\delta(\varepsilon) > 0$ for every $\varepsilon > 0$. If E is uniformly convex, then for r, ε with $r \ge \varepsilon > 0$, there exists $\delta\left(\frac{\varepsilon}{r}\right) > 0$ such that

$$\left\|\frac{x+y}{2}\right\| \le r\left(1 - \delta\left(\frac{\varepsilon}{r}\right)\right)$$

for every $x, y \in E$ with $\|x\| \le r$, $\|y\| \le r$ and $\|x - y\| \ge \varepsilon$. It is well-known that a uniformly convex Banach space is reflexive and strictly convex.

The multi-valued mapping J from E into E^* defined by

$$J(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\} \quad \text{for every} \quad x \in E$$

is called the duality mapping of E. From the Hahn-Banach theorem, we see that $J(x) \neq \emptyset$ for all $x \in E$. A Banach space E is said to be smooth if the limit

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each x and y in S_1, where $S_1 = \{u \in E : \|u\| = 1\}$. The norm of E is said to be uniformly Gâteaux differentiable if for each y in S_1, the limit is attained uniformly for x in S_1. We know that if E is smooth then the duality mapping is single-valued and norm to weak-star continuous and that if the norm of E is uniformly Gâteaux differentiable then the duality mapping is single-valued and norm to weak-star uniformly continuous on each bounded subset of E.

Let C be a nonempty convex subset of E and let K be a nonempty subset of C. A mapping P of C onto K is said to be sunny if $P(Px + t(x - Px)) = Px$ for each $x \in C$ and $t \geq 0$ with $Px + t(x - Px) \in C$. A mapping P of C onto K is said to be a retraction if $Px = x$ for each $x \in K$. We know from [2, 7] that if E is smooth, then a retraction P of C onto K is sunny and nonexpansive if and only if

$$\langle x - Px, J(y - Px)\rangle \leq 0 \quad \text{for all} \quad x \in C \quad \text{and} \quad y \in K.$$

Hence, there is at most one sunny nonexpansive retraction of C onto K. If there is a sunny nonexpansive retraction of C onto K, K is said to be a sunny nonexpansive retract of C. The following proposition related to the existence of sunny nonexpansive retractions was proved in [11].

Proposition 2.1. *Let C be a nonempty closed convex subset of a uniformly convex Banach space whose norm is uniformly Gâteaux differentiable and let T be an asymptotically nonexpansive mapping on C with Lipschitz constants $\{k_n\}$ such that $F(T) \neq \emptyset$. Then, $F(T)$ is a sunny nonexpansive retract of C.*

3. Lemmas

Let C be a nonempty closed convex subset of a Banach space E and let T be a mapping of C into itself. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences of real numbers such that $0 \leq \alpha_n \leq 1, 0 \leq \beta_n \leq 1$, and let $x \in C$. Now consider the following iteration process:

$$\begin{cases} x_0 \in C \\ x_{n+1} = \alpha_n x + (1 - \alpha_n)\dfrac{1}{n+1}\displaystyle\sum_{j=0}^{n} T^j y_n, \\ y_n = \beta_n x_n + (1 - \beta_n)\dfrac{1}{n+1}\displaystyle\sum_{j=0}^{n} T^j x_n \end{cases} \tag{5}$$

for each $n \in \mathbb{N}$. Especially, if $\beta_n = 1$ for each $n \in \mathbb{N}$, then the sequence $\{x_n\}$ is written by (3). We prove a strong convergence theorem for an asymptotically nonexpansive mapping T on C with Lipschitz constants $\{k_n\}$, which is a generalization of the result of Shioji and Takahashi [12]. Without loss of generality, we may assume $k_n \geq 1$ for each $n \in \mathbb{N}$. Since $k_n \geq 1$ for each $n \in \mathbb{N}$, we obtain the following lemmas.

Lemma 3.1. *Let C be a nonempty closed convex subset of a Banach space E and let T be an asymptotically nonexpansive mapping on C with Lipschitz constants $\{k_n\}$ such that $F(T) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences of real numbers such that $0 \leq \alpha_n \leq 1$, $0 \leq \beta_n \leq 1$ and*

$$\sum_{n=0}^{\infty}(1 - \alpha_n)(M_n - 1) < \infty, \tag{6}$$

where $M_n = \left(\frac{1}{n+1}\sum_{j=0}^{n} k_j\right)\left(\beta_n + (1 - \beta_n)\left(\frac{1}{n+1}\sum_{j=0}^{n} k_j\right)\right)$. Let $x \in C$, and let $\{x_n\}$ and $\{y_n\}$ be the sequences defined by (5). Then, $\{x_n\}$ and $\{y_n\}$ are bounded. Further, $\{T^j x_n\}$ and $\{T^j y_n\}$ are bounded for each $j \in \mathbb{N}$.

Proof. Let $K_0 = \sup_n k_n$. We obtain

$$1 \leq \beta_n + (1 - \beta_n)\left(\frac{1}{n+1}\sum_{j=0}^{n} k_j\right) \leq K_0$$

for each $n \in \mathbb{N}$. Set $M_n = \left(\frac{1}{n+1}\sum_{j=0}^{n} k_j\right)\left(\beta_n + (1-\beta_n)\left(\frac{1}{n+1}\sum_{j=0}^{n} k_j\right)\right)$. Then, we obtain $1 \leq M_n \leq K_0^2$ for each $n \in \mathbb{N}$. Let $z \in F(\mathcal{S})$. Then, it follows from (5) that

$$
\begin{aligned}
\|y_n - z\| &= \left\|\beta_n(x_n - z) + (1-\beta_n)\frac{1}{n+1}\sum_{j=0}^{n}(T^j x_n - z)\right\| \\
&\leq \beta_n\|x_n - z\| + (1-\beta_n)\frac{1}{n+1}\sum_{j=0}^{n}\|T^j x_n - z\| \\
&\leq \left(\beta_n + (1-\beta_n)\left(\frac{1}{n+1}\sum_{j=0}^{n} k_j\right)\right)\|x_n - z\| && (7) \\
&\leq K_0\|x_n - z\| && (8)
\end{aligned}
$$

for each $n \in \mathbb{N}$. By (5), we also obtain

$$
\begin{aligned}
\|x_{n+1} - z\| &= \left\|\alpha_n(x - z) + (1-\alpha_n)\frac{1}{n+1}\sum_{j=0}^{n}(T^j y_n - z)\right\| \\
&\leq \alpha_n\|x - z\| + (1-\alpha_n)\frac{1}{n+1}\sum_{j=0}^{n}\|T^j y_n - z\| \\
&\leq \alpha_n\|x - z\| + (1-\alpha_n)\left(\frac{1}{n+1}\sum_{j=0}^{n} k_j\right)\|y_n - z\| && (9) \\
&\leq \|x - z\| + K_0\|y_n - z\| && (10)
\end{aligned}
$$

for each $n \in \mathbb{N}$. Since $F(T) \neq \emptyset$, from (8) and (10), we see that $\{x_n\}$ is bounded if and only if $\{y_n\}$ is bounded.

By (7) and (9), for each $n \in \mathbb{N}$, we have

$$
\begin{aligned}
&\|x_{n+1} - z\| \\
&\quad \leq \alpha_n\|x - z\| + (1-\alpha_n)\left(\frac{1}{n+1}\sum_{j=0}^{n} k_j\right)\left(\beta_n + (1-\beta_n)\left(\frac{1}{n+1}\sum_{j=0}^{n} k_j\right)\right)\|x_n - z\| \\
&\quad = \alpha_n\|x - z\| + (1-\alpha_n)M_n\|x_n - z\|. && (11)
\end{aligned}
$$

Set $h_n = ((1-\alpha_n)M_n - 1)_+$. Since $h_n = ((1-\alpha_n)M_n - 1)_+ \leq (1-\alpha_n)(M_n - 1)$ for each $n \in \mathbb{N}$, we obtain

$$
\sum_{n=0}^{\infty} h_n < \infty \tag{12}
$$

by (6). By (11), for each $n \in \mathbb{N}$, we have

$$
\begin{aligned}
&\|x_{n+1} - z\| \\
&\quad \le (1-(1-\alpha_n))\|x - z\| + (1-\alpha_n)M_n\|x_n - z\| \\
&\quad \le \{1 + (1-\alpha_n)(M_n - 1) - (1-\alpha_n)M_n(1-\alpha_{n-1})\}\|x - z\| \\
&\qquad + (1-\alpha_n)M_n(1-\alpha_{n-1})M_{n-1}\|x_{n-1} - z\| \\
&\quad \le \Big\{1 + (1-\alpha_n)(M_n - 1) + (1-\alpha_n)M_n(1-\alpha_{n-1})(M_{n-1} - 1) \\
&\qquad\qquad - (1-\alpha_n)M_n(1-\alpha_{n-1})M_{n-1}(1-\alpha_{n-2}\Big\}\|x - z\| \\
&\qquad + (1-\alpha_n)M_n(1-\alpha_{n-1})M_{n-1}(1-\alpha_{n-2})M_{n-2}\|x_{n-2} - z\| \\
&\qquad\qquad \vdots \\
&\quad \le \left\{1 + (1-\alpha_n)(M_n - 1) + \sum_{i=1}^{n-1}\Big\{(1-\alpha_i)(M_i - 1)\prod_{j=i+1}^{n}[(1-\alpha_j)M_j]\Big\}\right. \\
&\qquad\qquad \left. - (1-\alpha_0)\prod_{j=1}^{n}[(1-\alpha_j)M_j]\right\}\|x - z\| + \prod_{j=0}^{n}[(1-\alpha_j)M_j]\|x_0 - z\| \qquad (13) \\
&\quad \le \left\{1 + (1-\alpha_n)(M_n - 1) + \sum_{i=1}^{n-1}\Big\{(1-\alpha_i)(M_i - 1)\prod_{j=i+1}^{n}(1+h_j)\Big\}\right\}\|x - z\| \\
&\qquad + \prod_{j=0}^{n}(1+h_j)\|x_0 - z\| \\
&\quad \le \prod_{j=0}^{n}(1+h_j)\left\{\Big[1 + \sum_{i=1}^{n}(1-\alpha_i)(M_i - 1)\Big]\|x - z\| + \|x_0 - z\|\right\} \\
&\quad \le \exp\left(\sum_{j=0}^{\infty} h_j\right)\left\{\Big[1 + \sum_{i=1}^{\infty}(1-\alpha_i)(M_i - 1)\Big]\|x - z\| + \|x_0 - z\|\right\}.
\end{aligned}
$$

Hence by (6) and (12), we obtain that $\{\|x_n - z\|\}$ is bounded. Therefore, $\{x_n\}$ and $\{y_n\}$ are bounded.

Let $L_0 = \sup_n\{\|x_n - z\|\}$. Then, it follows from (8) that

$$\|T^j x_n - z\| \le k_j\|x_n - z\| \le K_0 L_0$$

and

$$\|T^j y_n - z\| \le k_j\|y_n - z\| \le K_0 \cdot K_0 L_0 = K_0^2 L_0$$

for each $j, n \in \mathbb{N}$. Hence, $\{T^j x_n\}$ and $\{T^j y_n\}$ are also bounded for each $j \in \mathbb{N}$. $\square$

Lemma 3.2 and Proposition 3.3 were proved by Shioji and Takahashi [11].

Lemma 3.2. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E and let T be an asymptotically nonexpansive mapping on C with Lipschitz constants $\{k_n\}$ such that $F(T) \neq \emptyset$. Then, for each $r > 0$,*

$$\overline{\lim_{m\to\infty}}\ \overline{\lim_{n\to\infty}}\ \sup_{y \in C\cap B_r}\left\|\frac{1}{n+1}\sum_{j=0}^{n} T^j y - T^m\left(\frac{1}{n+1}\sum_{j=0}^{n} T^j y\right)\right\| = 0,$$

where $B_r = \{z \in E : \|z\| \le r\}$.

Proposition 3.3. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E whose norm is uniformly Gâteaux differentiable. Let T be an asymptotically nonexpansive mapping on C with Lipschitz constants $\{k_n\}$ such that $F(T) \neq \emptyset$ and let P be the sunny nonexpansive retraction from C onto $F(T)$. Let $\{d_n\}$ be a sequence of real numbers such that $0 < d_n \leq 1, \lim_{n\to\infty} d_n = 0$ and*

$$\varlimsup_{n\to\infty} \frac{\frac{1}{n+1}\sum_{j=0}^{n} k_j - 1}{d_n} < 1.$$

Let $x \in C$ and let z_n be the unique point of C which satisfies

$$z_n = d_n x + (1 - d_n)\frac{1}{n+1}\sum_{j=0}^{n} T^j z_n \tag{14}$$

for $n \geq m_0$, where m_0 is a sufficiently large integer. Then, $\{z_n\}$ converges strongly to Px.

Remark 3.4. The inequality

$$\varlimsup_{n\to\infty} \frac{\frac{1}{n+1}\sum_{j=0}^{n} k_j - 1}{d_n} < 1$$

yields

$$(1 - d_n) \cdot \frac{1}{n+1}\sum_{j=0}^{n} k_j < 1$$

for all sufficiently large integer n. So for such n, there exists a unique point z_n of C satisfying $z_n = d_n x + (1 - d_n)\frac{1}{n+1}\sum_{j=0}^{n} T^j z_n$, since the mapping T_n from C into itself defined by $T_n u = d_n x + (1 - d_n)\frac{1}{n+1}\sum_{j=0}^{n} T^j u$ is a contraction, that is,

$$\|T_n u - T_n v\| \leq (1 - d_n) \cdot \frac{1}{n+1}\sum_{j=1}^{n} k_j \|u - v\|$$

for each $u, v \in C$.

4. Strong Convergence Theorems

Our main result is the following, which is a generalization of Shioji and Takahashi's result [12]:

Theorem 4.1. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E whose norm is uniformly Gâteaux differentiable and let T be an asymptotically nonexpansive mapping on C with Lipschitz constants $\{k_n\}$ such that $F(T) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences of real numbers such that $0 \leq \alpha_n \leq 1, 0 \leq \beta_n \leq 1$,*

$$\lim_{n\to\infty} \alpha_n = 0, \ \sum_{n=0}^{\infty} \alpha_n = \infty$$

and

$$\sum_{n=0}^{\infty} (1 - \alpha_n)(M_n - 1) < \infty, \tag{15}$$

where $M_n = \left(\frac{1}{n+1}\sum_{j=0}^{n} k_j\right)\left(\beta_n + (1-\beta_n)\left(\frac{1}{n+1}\sum_{j=0}^{n} k_j\right)\right)$. *Let* $x \in C$ *and let* $\{x_n\}$ *be the sequence defined by*

$$\begin{cases} x_0 \in C \\ x_{n+1} = \alpha_n x + (1-\alpha_n)\dfrac{1}{n+1}\displaystyle\sum_{j=0}^{n} T^j y_n, \\ y_n = \beta_n x_n + (1-\beta_n)\dfrac{1}{n+1}\displaystyle\sum_{j=0}^{n} T^j x_n \end{cases} \tag{16}$$

for each $n \in \mathbb{N}$. *Then,* $\{x_n\}$ *converges strongly to* Px, *where* P *is the sunny nonexpansive retraction from* C *onto* $F(T)$.

Proof. Set $M_n = \left(\frac{1}{n+1}\sum_{j=0}^{n} k_j\right)\left(\beta_n + (1-\beta_n)\left(\frac{1}{n+1}\sum_{j=0}^{n} k_j\right)\right)$ and set $K_0 = \sup_n k_n$. Since $F(T) \neq \emptyset$ and $\sum_{n=0}^{\infty}(1-\alpha_n)(M_n - 1) < \infty$, from Lemma 3.1, we see that $\{x_n\}$, $\{y_n\}$, $\{T^j x_n\}$ and $\{T^j y_n\}$ are bounded for each $j \in \mathbb{N}$.

Since $\overline{\lim}_{n\to\infty} k_n \leq 1$, we can choose a sequence $\{d_n\}$ of real numbers such that $d_n > 0, \lim_{n\to\infty} d_n = 0$,

$$\overline{\lim_{n\to\infty}} \frac{\dfrac{1}{n+1}\displaystyle\sum_{j=0}^{n} k_j - 1}{d_n} < 1 \tag{17}$$

and

$$\left(\frac{1}{n+1}\sum_{j=0}^{n} k_j\right)^2 \leq 1 + {d_n}^2 \tag{18}$$

for each $n \in \mathbb{N}$ (see also [13]). By the reason in Remark 3.4, there exists the unique point z_m of C satisfying $z_m = d_m x + (1-d_m)\dfrac{1}{m+1}\displaystyle\sum_{j=0}^{m} T^j z_m$ for all sufficiently large integer m. Without loss of generality, we may assume that $d_m \leq 1/2$ for all $m \in \mathbb{N}$ and z_m is defined for all $m \in \mathbb{N}$. We know that $\{z_n\}$ converges strongly to Px by Proposition 3.3. From (16), for each $m, n \in \mathbb{N}$, we have

$$
\begin{aligned}
&\left\| \frac{1}{m+1} \sum_{j=0}^{m} T^j x_{n+1} - x_{n+1} \right\| \\
&\leq \left\| \frac{1}{m+1} \sum_{j=0}^{m} T^j x_{n+1} - \frac{1}{m+1} \sum_{j=0}^{m} T^j \left(\frac{1}{n+1} \sum_{l=0}^{n} T^l y_n \right) \right\| \\
&\qquad + \left\| \frac{1}{m+1} \sum_{j=0}^{m} T^j \left(\frac{1}{n+1} \sum_{l=0}^{n} T^l y_n \right) - \frac{1}{n+1} \sum_{l=0}^{n} T^l y_n \right\| + \left\| \frac{1}{n+1} \sum_{l=0}^{n} T^l y_n - x_{n+1} \right\| \\
&\leq \left(\frac{1}{m+1} \sum_{j=0}^{m} k_j + 1 \right) \left\| x_{n+1} - \frac{1}{n+1} \sum_{l=0}^{n} T^l y_n \right\| \\
&\qquad + \frac{1}{m+1} \sum_{j=0}^{m} \left\| T^j \left(\frac{1}{n+1} \sum_{l=0}^{n} T^l y_n \right) - \frac{1}{n+1} \sum_{l=0}^{n} T^l y_n \right\| \\
&\leq (K_0 + 1) \cdot \alpha_n \left\| x - \frac{1}{n+1} \sum_{l=0}^{n} T^l y_n \right\| \\
&\qquad + \frac{1}{m+1} \sum_{j=0}^{m} \left\| T^j \left(\frac{1}{n+1} \sum_{l=0}^{n} T^l y_n \right) - \frac{1}{n+1} \sum_{l=0}^{n} T^l y_n \right\| \\
&\leq (K_0 + 1) \cdot \alpha_n \Big(\|x\| + \sup_{j,n} \|T^j y_n\| \Big) \\
&\qquad + \frac{1}{m+1} \sum_{j=0}^{m} \left\| T^j \left(\frac{1}{n+1} \sum_{l=0}^{n} T^l y_n \right) - \frac{1}{n+1} \sum_{l=0}^{n} T^l y_n \right\|.
\end{aligned}
$$

It follows from Lemma 3.2 that

$$
\varlimsup_{m\to\infty} \varlimsup_{n\to\infty} \frac{1}{m+1} \sum_{j=0}^{m} \left\| T^j \left(\frac{1}{n+1} \sum_{l=0}^{n} T^l y_n \right) - \frac{1}{n+1} \sum_{l=0}^{n} T^l y_n \right\| = 0.
$$

Hence by $\lim_{n\to\infty} \alpha_n = 0$, we have

$$
\varlimsup_{m\to\infty} \varlimsup_{n\to\infty} \left\| \frac{1}{m+1} \sum_{j=0}^{m} T^j x_{n+1} - x_{n+1} \right\| = \varlimsup_{m\to\infty} \varlimsup_{n\to\infty} \left\| \frac{1}{m+1} \sum_{j=0}^{m} T^j x_n - x_n \right\| = 0.
$$

Then, we may also assume that

$$
\varlimsup_{n\to\infty} \left\| \frac{1}{m+1} \sum_{j=0}^{m} T^j x_n - x_n \right\| \leq d_m^2 \tag{19}
$$

for each $m \in \mathbb{N}$. Set $R = \sup\big(\{\|T^j z_m\| : j, m \in \mathbb{N}\} \cup \{\|T^j x_n\| : j, n \in \mathbb{N}\}\big)$. From

$$
(1 - d_m)\left(\frac{1}{m+1} \sum_{j=0}^{m} T^j z_m - x_n \right) = (z_m - x_n) - d_m(x - x_n),
$$

we obtain

$$
\begin{aligned}
(1-d_m)^2\left\|\frac{1}{m+1}\sum_{j=0}^{m}T^jz_m-x_n\right\|^2 &\geq \|z_m-x_n\|^2-2d_m\langle x-x_n,J(z_m-x_n)\rangle\\
&=\|z_m-x_n\|^2-2d_m\langle x-z_m+z_m-x_n,J(z_m-x_n)\rangle\\
&=(1-2d_m)\|z_m-x_n\|^2+2d_m\langle x-z_m,J(x_n-z_m)\rangle
\end{aligned}
$$

for each $m,n\in\mathbb{N}$. Then, it follows from (18) that

$$
\begin{aligned}
&\langle x-z_m,J(x_n-z_m)\rangle\\
&\leq\frac{1}{2d_m}\left((1-d_m)^2\left\|\frac{1}{m+1}\sum_{j=0}^{m}T^jz_m-x_n\right\|^2-(1-2d_m)\|z_m-x_n\|^2\right)\\
&=\frac{1-2d_m}{2d_m}\left(\left\|\frac{1}{m+1}\sum_{j=0}^{m}T^jz_m-x_n\right\|^2-\|z_m-x_n\|^2\right)+\frac{d_m}{2}\left\|\frac{1}{m+1}\sum_{j=0}^{m}T^jz_m-x_n\right\|^2\\
&\leq\frac{1-2d_m}{2d_m}\left(\left\{\left\|\frac{1}{m+1}\sum_{j=0}^{m}T^jz_m-\frac{1}{m+1}\sum_{j=0}^{m}T^jx_n\right\|+\left\|\frac{1}{m+1}\sum_{j=0}^{m}T^jx_n-x_n\right\|\right\}^2\right.\\
&\qquad\qquad\left.-\|z_m-x_n\|^2\right)+2R^2d_m\\
&\leq\frac{1}{2d_m}\left(\left\|\frac{1}{m+1}\sum_{j=0}^{m}T^jz_m-\frac{1}{m+1}\sum_{j=0}^{m}T^jx_n\right\|^2+2\cdot 2R\left\|\frac{1}{m+1}\sum_{j=0}^{m}T^jx_n-x_n\right\|\right.\\
&\qquad\qquad\left.+\left\|\frac{1}{m+1}\sum_{j=0}^{m}T^jx_n-x_n\right\|^2-\|z_m-x_n\|^2\right)+2R^2d_m\\
&\leq\frac{1}{2d_m}\left(\left\{\left(\frac{1}{m+1}\sum_{j=0}^{m}k_j\right)^2-1\right\}\|z_m-x_n\|^2+6R\left\|\frac{1}{m+1}\sum_{j=0}^{m}T^jx_n-x_n\right\|\right)+2R^2d_m\\
&\leq\frac{1}{2d_m}\left(d_m^2\|z_m-x_n\|^2+6R\left\|\frac{1}{m+1}\sum_{j=0}^{m}T^jx_n-x_n\right\|\right)+2R^2d_m\\
&\leq 4R^2d_m+\frac{3R}{d_m}\left\|\frac{1}{m+1}\sum_{j=0}^{m}T^jx_n-x_n\right\|
\end{aligned}
$$

for each $m,n\in\mathbb{N}$. Hence by (19), we have

$$\overline{\lim_{n\to\infty}}\langle x-z_m,J(x_n-z_m)\rangle\leq(4R^2+3R)d_m$$

for each $m\in\mathbb{N}$. Since $\{z_m\}$ converges strongly to Px and the norm of E is uniformly Gâteaux differentiable, we have

$$\overline{\lim_{n\to\infty}}\langle x-Px,J(x_n-Px)\rangle\leq 0.$$

Let $\varepsilon>0$. Then, there exists $n_0\in\mathbb{N}$ such that $\langle x-Px,J(x_n-Px)\rangle<\frac{\varepsilon}{2}$ for each $n\geq n_0$. From

$$(1-\alpha_n)\left(\frac{1}{n+1}\sum_{j=0}^{n}T^jy_n-Px\right)=(x_{n+1}-Px)-\alpha_n(x-Px),$$

we also obtain

$$\|x_{n+1}-Px\|^2 \leq 2\alpha_n\langle x-Px, J(x_{n+1}-Px)\rangle + (1-\alpha_n)^2\left\|\frac{1}{n+1}\sum_{j=0}^{n} T^j y_n - Px\right\|^2 \tag{20}$$

for each $n \in \mathbb{N}$. So, we get

$$\begin{aligned}
&\|x_{n+1}-Px\|^2 \\
&\leq \alpha_n\varepsilon + (1-\alpha_n)^2\left(\frac{1}{n+1}\sum_{j=0}^{n} k_j\right)^2 \|y_n - Px\|^2 \\
&\leq \alpha_n\varepsilon + (1-\alpha_n)^2\left(\left(\frac{1}{n+1}\sum_{j=0}^{n} k_j\right)^2\left(\beta_n+(1-\beta_n)\left(\frac{1}{n+1}\sum_{j=0}^{n} k_j\right)\right)\right)^2 \|x_n - Px\|^2 \\
&= \alpha_n\varepsilon + (1-\alpha_n)^2 {M_n}^2 \|x_n - Px\|^2
\end{aligned} \tag{21}$$

for each $n \geq n_0$. Set $p_n = \|x_n - Px\|^2$, $L_n = {M_n}^2$ and $c_n = ((1-\alpha_n)L_n - 1)_+$. Then, for each $n \in \mathbb{N}$, we have

$$\begin{aligned}
c_n &= ((1-\alpha_n)L_n - 1)_+ \leq (1-\alpha_n)(L_n - 1) \\
&= (1-\alpha_n)(M_n+1)(M_n-1) \leq ({K_0}^2+1)(1-\alpha_n)(M_n-1).
\end{aligned}$$

Hence by (15), $\sum_{i=0}^{\infty} c_i < \infty$. Let $n \in \mathbb{N}$ with $n \geq n_0$. Then, for each $m \in \mathbb{N}$, we have

$$\begin{aligned}
p_{n+m} &\leq \alpha_{n+m-1}\varepsilon + (1-\alpha_{n+m-1})^2 L_{n+m-1} p_{n+m-1} \\
&\leq \{\alpha_{n+m-1} + (1-\alpha_{n+m-1})^2 L_{n+m-1}\alpha_{n+m-2}\}\varepsilon \\
&\qquad + (1-\alpha_{n+m-1})^2 L_{n+m-1}(1-\alpha_{n+m-2})^2 L_{n+m-2} p_{n+m-2} \\
&\qquad \vdots \\
&\leq \left\{\alpha_{n+m-1} + \sum_{j=n}^{n+m-2}\left(\alpha_j \prod_{i=j+1}^{n+m-1}\left[(1-\alpha_i)^2 L_i\right]\right)\right\}\varepsilon + \left(\prod_{i=n}^{n+m-1}\left[(1-\alpha_i)^2 L_i\right]\right) p_n \\
&\leq \prod_{i=n+1}^{n+m-1}(1+c_i)\left\{\alpha_{n+m-1} + \sum_{j=n}^{n+m-2}\left(\alpha_j \prod_{i=j+1}^{n+m-1}(1-\alpha_i)\right)\right\}\varepsilon \\
&\qquad + \prod_{i=n}^{n+m-1}(1+c_i)\cdot\prod_{i=n}^{n+m-1}(1-\alpha_i)\cdot p_n \\
&\leq \prod_{i=n+1}^{n+m-1}(1+c_i)\left(1-\prod_{i=n}^{n+m-1}(1-\alpha_i)\right)\varepsilon + \prod_{i=n}^{n+m-1}(1+c_i)\cdot\prod_{i=n}^{n+m-1}(1-\alpha_i)\cdot p_n \\
&\leq \varepsilon\cdot\exp\left(\sum_{i=n+1}^{n+m-1} c_i\right) + \exp\left(\sum_{i=n}^{n+m-1} c_i\right)\cdot\exp\left(-\sum_{i=n}^{n+m-1}\alpha_i\right)\cdot p_n \\
&\leq \exp\left(\sum_{i=0}^{\infty} c_i\right)\left\{\varepsilon + \exp\left(-\sum_{i=n}^{n+m-1}\alpha_i\right)\cdot p_n\right\}.
\end{aligned}$$

By $\sum\limits_{i=0}^{\infty} \alpha_i = \infty$, we get

$$\overline{\lim_{m\to\infty}}\, p_m = \overline{\lim_{m\to\infty}}\, p_{n+m} \leq \varepsilon \cdot \exp\left(\sum_{i=0}^{\infty} c_i\right).$$

Since $\exp\left(\sum\limits_{i=0}^{\infty} c_i\right) < \infty$ and $\varepsilon > 0$ is arbitrary, $\{x_n\}$ converges strongly to $Px \in F(T)$. $\square$

Remark 4.2. $\sum_{n=0}^{\infty}(1-\alpha_n)(M_n-1) < \infty$ yields $\sum_{n=0}^{\infty} c_n < \infty$. So, by the proofs of Lemma 3.1 and Theorem 4.1, we see the following: Let E, C, T, x and $\{k_n\}$ be as in Theorem 4.1. Let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \leq \alpha_n \leq 1$, $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$, and let $\{\beta_n\}$ be a sequence of real numbers such that $0 \leq \beta_n \leq 1$. Assume

$$\sum_{n=0}^{\infty} \big((1-\alpha_n){M_n}^2 - 1\big)_+ < \infty,$$

where $M_n = \left(\frac{1}{n+1}\sum_{j=0}^{n} k_j\right)\left(\beta_n + (1-\beta_n)\left(\frac{1}{n+1}\sum_{j=0}^{n} k_j\right)\right)$. Let $\{x_n\}$ be the sequence defined by (16). Then, $\{x_n\}$ converges strongly to a fixed point of T if and only if $\{x_n\}$ is bounded.

Since $\sum_{n=0}^{\infty}(1-\alpha_n)\left(\frac{1}{n+1}\sum_{j=0}^{n} k_j - 1\right) < \infty$ yields $\sum_{n=0}^{\infty}(1-\alpha_n)(M_n-1) < \infty$, we get the following.

Corollary 4.3. *Let E, C, T, x and $\{k_n\}$ be as in Theorem 4.1. Let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \leq \alpha_n \leq 1$, $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$, and let $\{\beta_n\}$ be any sequence of real numbers such that $0 \leq \beta_n \leq 1$. Assume*

$$\sum_{n=0}^{\infty}(1-\alpha_n)\left(\frac{1}{n+1}\sum_{j=0}^{n} k_j - 1\right) < \infty.$$

Let $\{x_n\}$ be the sequence defined by (16). Then, $\{x_n\}$ converges strongly to Px, where P is the sunny nonexpansive retraction from C onto $F(T)$.

In the case when T is nonexpansive, by $\sum_{n=0}^{\infty}(1-\alpha_n)(M_n-1) = 0$, we can directly obtain the following.

Theorem 4.4. *Let E be a uniformly convex Banach space whose norm is uniformly Gâteaux differentiable and let C be a nonempty closed convex subset of E. Let T be a nonexpansive mapping of C into itself such that $F(T) \neq \emptyset$. Let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \leq \alpha_n \leq 1$, $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$, and let $\{\beta_n\}$ be a sequence of real numbers such that $0 \leq \beta_n \leq 1$. Let $x \in C$ and let $\{x_n\}$ be the sequence defined by (16). Then, $\{x_n\}$ converges strongly to Px, where P is the sunny nonexpansive retraction from C onto $F(T)$.*

References

[1] S. Atsushiba and W. Takahashi, *Approximating common fixed points of two nonexpansive mappings in Banach spaces,* Bull. Austral. Math. Soc., **57** (1998), 117–127.

[2] R. E. Bruck, *Nonexpansive retracts of Banach spaces*, Bull. Amer. Math. Soc., **76** (1970), 384–386.

[3] K. Goebel, W.A.kirk, *A fixed point theorems for asymptotically nonexpansive mappings,* Proc. Amer. Math. Soc., **35** (1972), 171–174.

[4] B. Halpern, *Fixed points of nonexpansive maps*, Bull. Amer. Math. Soc., **73** (1967), 957–961.

[5] N. Hirano and W. Takahashi, *Nonlinear ergodic theorems for nonexpansive mappings in Hilbert spaces*, Kodai Math. J., **2** (1979), 11-25.

[6] W. R. Mann, *Mean value methods in iteration*, Proc. Amer. Math. Soc., **4** (1953), 506–510.

[7] S. Reich, *Asymptotic behavior of contractions in Banach spaces,* J. Math. Anal. Appl., **44** (1973), 57–70.

[8] S. Reich, *Weak convergence theorems for nonexpansive mappings in Banach spaces,* J. Math. Anal. Appl., **67** (1979), 274-276.

[9] S. Reich, *Strong convergence theorems for resolvents of accretive operators in Banach spaces*, J. Math. Anal. Appl., **75** (1980), 287–292.

[10] N. Shioji and W. Takahashi, *Strong convergence of approximated sequences for nonexpansive mappings in Banach spaces*, Proc. Amer. Math. Soc., **125** (1997), 3641–3645.

[11] N. Shioji and W. Takahashi, *Strong convergence of averaged approximants for asymptotically nonexpansive mappings in Banach spaces*, J. Approximation Theory, **97** (1999), 53–64.

[12] N. Shioji and W. Takahashi, *A Strong convergence theorem for asymptotically nonexpansive mappings in Banach spaces*, Arch. Math., **72** (1999), 354–359.

[13] T. Shimizu and W. Takahashi, *Strong convergence theorem for asymptotically nonexpansive mappings*, Nonlinear Anal., **26** (1996), 265–272.

[14] T. Shimizu and W. Takahashi, *Strong convergence to common fixed points of families of nonexpansive mappings*, J. Math. Anal. Appl., **211** (1997), 71–83.

[15] R. Wittmann, *Approximation of fixed points of nonexpansive mappings,* Arch. Math., **58** (1992), 486–491.

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