C-SEMIGROUPS AND (P,Q)-SUMMING OPERATORS*

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Abstract. Let $T(t), 0 \le t < \infty$, be a one parameter C-semigroup of bounded linear operators on a Banach space X, and A be the generator of $T(t)$. Let $R(\lambda, A)$ be the resolvent operator of A. It is known that for exponentially bounded C-semigroups, $\|R(\lambda, A)C\| \le \frac{M}{\lambda-\omega}$ for $\lambda > \omega$. The object of this paper is to study such an inequality for the (p,q)-summing norms. Further, we give some conditions for a C-semigroup to be in the ideal of (p,q)-summing operators.

0.Introduction. Let X^* be the dual of the Banach space X, and $L(X)$ be the space of all bounded linear operators from X into X. For $T \in L(X)$, $\|T\|$ denotes the operator norm of T.

An operator $T \in L(X)$ is called (p,q)**-summing** if there exists $\lambda > 0$, such that

$$\left(\sum_{i=1}^{n} \|T(x_i)\|^p\right)^{\frac{1}{p}} \le \lambda \sup_{\|x^*\| \le 1} \left(\sum_{i=1}^{n} |\langle x_i, x^*\rangle|^q\right)^{\frac{1}{q}} \tag{1}$$

for every finite set $\{x_1, x_2, ..., x_n\} \subseteq X$ and $x^* \in X^*$. Let $\Pi_{p,q}(X)$ denote the set of all (p,q)-summing operators in $L(X)$. For $T \in \Pi_{p,q}(X)$, the (p,q)-summing norm of T is $\|T\|_{\Pi(p,q)} = \inf\{\lambda : (1) \text{ holds}\}$. It is known that $\Pi_{p,q}(X)$ is an ideal of operators in $L(X)$, and $\|.\|_{\Pi(p,q)}$ is an ideal norm on $\Pi_{p,q}(X)$. If $p = q$ we write $\Pi_p(X)$ for $\Pi_{p,q}(X)$.

A one parameter family $T(t)$, $t \in [0, \infty)$, of bounded linear operators from X into X is called a one parameter C-semigroup of operators on X if : (i) $T(0) = C$ and (ii) $CT(s+t) = T(s)T(t)$ for all s, t in $[0, \infty)$, where C is an injective bounded linear operator on X. A C-semigroup, $T(t)$ is called strongly continuous if $\lim_{t \to 0^+} T(t)x = Cx$ for every $x \in X$. A C-semigroup for which there exist constants $M > 0$ and $\omega \in R$ (the set of real numbers) such that $\|T(t)\| \le Me^{\omega t}$ is called an exponentially bounded C-semigroup. The linear operator A defined by:

$D(A) = \{x \in X : C^{-1} \lim_{t \to 0^+} \frac{T(t)x - Cx}{t} \text{ exists}\}$ and $Ax = C^{-1} \lim_{t \to 0^+} \frac{T(t)x - Cx}{t}$

for $x \in D(A)$, is called the generator of the C-semigroup $T(t)$ and $D(A)$ is the domain of A. The resolvent set of A is denoted by $\rho(A)$ and for $\lambda \in \rho(A)$, the operator $R(\lambda, A) = (\lambda - A)^{-1}$ is the resolvent operator of A. It is known,[3] , for exponentially bounded C-semigroups, that $D(A)$ is dense in $Range(C)$ and A is a closed operator, and the resolvent operator $R(\lambda, A)$ is a bounded operator for all $\lambda \in \rho(A)$. We refer to [2] and [3] for excellent monographs on C-semigroups.

It is known,[2] , that the resolvent operator $R(\lambda, A)$ of the generator A of an exponentially bounded C-semigroup $T(t)$ satisfies the inequality:

$$\|R(\lambda, A)C\| \le \frac{M}{\lambda - \omega} \quad \text{for } \omega \text{ and } M \text{ as above and for } \lambda > \omega \text{ with } \lambda \in \rho(A) \tag{2}$$

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The norm of the resolvent operator in (2) is the usual operator norm. However, there are many important norms on different classes of bounded linear operators on X. It is natural to ask: **Does inequality (2) hold true for norms other than the operator norm ?**

In this paper we address two questions:

(i) If $T(t) \in \Pi_{p,q}(X) \subseteq L(X)$, can we prove $\|R(\lambda, A)C\|_{\Pi(p,q)} \leq \frac{M}{\lambda-\omega}$ for ω and M as above and for $\lambda > \omega$ with $\lambda \in \rho(A)$?

(ii) When can a C-semigroup be in the ideal $\Pi_{p,q}(X)$?

Pazy,[10] , studied the problem for c_0-semigroups and the case of the ideal of compact operators, $K(X) \subseteq L(X)$ for any Banach space X and Khalil and Deeb,[7], studied the problem for the ideal of Schatten Classes $C_p(H) \subseteq L(H)$, where H is a Hilbert space.

Throughout this paper, the dual of a Banach space X is denoted by X^*, and $B_1(X)$ is the open unit ball of X. For $x^* \in X^*$ and $x \in X$ the value of x^* at x is denoted by $< x^*, x >$. The set of real numbers will be denoted by R, and the set of positive integers by N.

I. C-Semigroups and (p,q)-Summing Operators.

Let X be a Banach space and C be an injective bounded linear operator on X with dense range. Thus by Theorem 2.4 in [3] , $D(A)$ is dense in X.

Lemma 1.1. Let $T_n \in \Pi_{p,q}(X)$ for which $\sup\limits_n \|T_n\|_{\Pi_{(p,q)}} \leq \xi$ for some $\xi > 0$. If $\lim\limits_{n\to\infty} T_n x = Tx$ for all $x \in X$, then $T \in \Pi_{p,q}(X)$ and $\|T\|_{\Pi_{(p,q)}} \leq \xi$.

Proof. Let (T_n) be a sequence in $\Pi_{p,q}(X)$ such that $\sup\limits_n \|T_n\|_{\Pi(p,q)} \leq \xi$ for some $\xi > 0$ and $\lim\limits_{n\to\infty} T_n x = Tx$ for all $x \in X$. Then for all finite sequences $(x_1, x_2 , ..., x_m)$ and all $n \in N$ we have :

$$\left(\sum_{i=1}^m \|T_n(x_i)\|^p\right)^{\frac{1}{p}} \leq \|T_n\|_{\Pi(p,q)} \sup_{\|x^*\|\leq 1} \left(\sum_{i=1}^m |\langle x_i, x^*\rangle|^q\right)^{\frac{1}{q}}$$
$$\leq \xi \sup_{\|x^*\|\leq 1} \left(\sum_{i=1}^m |\langle x_i, x^*\rangle|^q\right)^{\frac{1}{q}}.$$

Since $\lim\limits_{n\to\infty} T_n x = Tx$ for all $x \in X$, we get :

$$\left(\sum_{i=1}^m \|T(x_i)\|^p\right)^{\frac{1}{p}} = \left(\sum_{i=1}^m \left\|\lim_{n\to\infty} T_n(x_i)\right\|^p\right)^{\frac{1}{p}}$$
$$= \lim_{n\to\infty} \left(\sum_{i=1}^m \|T_n(x_i)\|^p\right)^{\frac{1}{p}}$$
$$\leq \xi \sup_{\|x^*\|\leq 1} \left(\sum_{i=1}^m |\langle x_i, x^*\rangle|^q\right)^{\frac{1}{q}}.$$

This implies $T \in \Pi_{p,q}(X)$ and $\|T\|_{\Pi(p,q)} \leq \xi$. ■

Lemma 1.2. Let $T(t)$ be a strongly continuous C-semigroup of bounded linear operators on X. If $T(t_0) \in \Pi_{p,q}(X)$ for some $t_0 > 0$, then $CT(t) \in \Pi_{p,q}(X)$ for all $t > t_0$.

Proof. Suppose $T(t_0) \in \Pi_{p,q}(X)$ for some $t_0 > 0$. Then,

$$CT(t) = CT(t - t_0 + t_0) = T(t - t_0)T(t_0).$$

Thus $CT(t) \in \Pi_{p,q}(X)$ for all $t > t_0$. ■

Lemma 1.3. Let $T(t)$ be a strongly continuous exponentially bounded C-semigroup of bounded linear operators on X with generator A. If for $\lambda \in \rho(T)$, $\lambda > \omega$, $\lim\limits_{\lambda\to\infty} \|R(\lambda, A)\| = 0$ then, $\lim\limits_{\lambda\to\infty} \lambda R(\lambda, A)T(t)x = T(t)x$ for all $x \in X$.

Proof. Let $\lambda \in \rho(A)$, and $x \in D(A)$. Then,

$$\begin{aligned}\|\lambda R(\lambda,A)T(t)x - T(t)x\| &= \|AR(\lambda,A)T(t)x\| \\ &= \|R(\lambda,A)AT(t)x\| \\ &\leq \|R(\lambda,A)\|\,\|AT(t)x\|.\end{aligned}$$

But $D(A)$ is dense in X, and $\lim_{\lambda\to\infty}\|R(\lambda,A)\| = 0$. Hence;

$$\lim_{\lambda\to\infty}\lambda R(\lambda,A)T(t)x = T(t)x$$

for all $x \in X$. ■

Now we prove one of the main results of this paper.

Theorem 1.4. Let $T(t)$ be an exponentially bounded strongly continuous C-semigroup of bounded linear operators on X with generator A. If $T(t) \in \Pi_{p,q}(X)$ and $\|T(t)\|_{\Pi_{(p,q)}} \leq \xi$ in $[0,\epsilon)$ for some $\epsilon > 0$, then $C^2R(\lambda,A) \in \Pi_{p,q}(X)$ for all $\lambda \in \rho(A)$ and for $\lambda > \omega > 0$, $\left\|C^2R(\lambda,A)\right\|_{\Pi_{(p,q)}} \leq \frac{\beta}{\lambda-\omega}$ for some $\beta > 0$.

Proof. Let $x \in X$ and $\lambda \in \rho(A)$, $\lambda \in R$, $\lambda > \omega > 0$. Then by Theorem 3.3, [3] we have :

$$CR(\lambda,A)x = R(\lambda,A)Cx = \int_0^\infty e^{-\lambda s}T(s)x\,ds.$$

For $t \in (0,\epsilon)$ and $\lambda > \omega > 0$, define

$$\begin{aligned}R_t(\lambda,A) &= C\int_t^\infty e^{-\lambda s}T(s)\,ds \\ &= \int_t^\infty e^{-\lambda s}CT(s-t+t)\,ds \\ &= \int_t^\infty e^{-\lambda s}T(s-t)T(t)\,ds \\ &= T(t)\int_t^\infty e^{-\lambda s}T(s-t)\,ds.\end{aligned}$$

Since $\int_t^\infty e^{-\lambda s}T(s-t)\,ds$ is a bounded operator in $L(X)$ for $\lambda > \omega$ and $T(t) \in \Pi_{p,q}(X)$, the operators $R_t(\lambda,A) \in \Pi_{p,q}(X)$ for $\lambda \in R$, $\lambda > \omega > 0$ and $t \in (0,\epsilon)$. Further :

$$\begin{aligned}\left\|R_t(\lambda,A) - C^2R(\lambda,A)\right\|_{\Pi_{(p,q)}} &= \left\|C\int_t^\infty e^{-\lambda s}T(s)\,ds - C\int_0^\infty e^{-\lambda s}T(s)\,ds\right\|_{\Pi_{(p,q)}} \\ &= \left\|C\int_0^t e^{-\lambda s}T(s)\,ds\right\|_{\Pi_{(p,q)}} \\ &\leq \|C\|\int_0^t e^{-\lambda s}\,\|T(s)\|_{\Pi_{(p,q)}}\,ds \\ &\leq \|C\|\int_0^t e^{-\lambda s}\xi\,ds.\end{aligned}$$

Since $\lim_{t\to 0^+}\|C\|\int_0^t e^{-\lambda s}\xi\,ds = 0$, $R_t(\lambda,A) \in \Pi_{p,q}(X)$ for all $t \in (0,\epsilon)$, and $\Pi_{p,q}(X)$ is a Banach space, then $C^2R(\lambda,A) \in \Pi_{p,q}(X)$ for all $\lambda \in R$, $\lambda > \omega > 0$. Further:

$$\begin{aligned}\|R_t(\lambda,A)\|_{\Pi_{(p,q)}} &= \left\|C\int_t^\infty e^{-\lambda s}T(s)\,ds\right\|_{\Pi_{(p,q)}} \\ &= \left\|\int_t^\infty e^{-\lambda s}CT(s-t+t)\,ds\right\|_{\Pi_{(p,q)}}\end{aligned}$$

$$= \left\| \int_t^\infty e^{-\lambda s} T(s-t)T(t)\, ds \right\|_{\Pi_{(p,q)}}$$
$$\leq \int_t^\infty e^{-\lambda s} \|T(s-t)T(t)\|_{\Pi_{(p,q)}}\, ds$$
$$\leq \int_t^\infty e^{-\lambda s} \|T(s-t)\| \, \|T(t)\|_{\Pi_{(p,q)}}\, ds.$$

Since $T(t)$ is an exponentially bounded C-semigroup, then $\|T(s-t)\| \leq Me^{\omega(s-t)}$. Thus
$$\|R_t(\lambda, A)\|_{\Pi_{(p,q)}} \leq \xi\, e^{-\omega t} \int_t^\infty e^{-\lambda s} M e^{\omega s}\, ds = \frac{M\xi}{\lambda-\omega}\, e^{-\lambda t}.$$

Consequently,
$$\left\|C^2 R(\lambda, A)\right\|_{\Pi_{(p,q)}} = \left\| \lim_{t\to 0^+} R_t(\lambda, A) \right\|_{\Pi_{(p,q)}}$$
$$= \lim_{t\to 0^+} \|R_t(\lambda, A)\|_{\Pi_{(p,q)}}$$
$$\leq \lim_{t\to 0^+} \frac{M\xi}{\lambda-\omega} e^{-\lambda t} = \frac{\beta}{\lambda-\omega}.$$

Now let $\lambda, \mu \in \rho(A)$ and $\lambda > \omega > 0$. Then the resolvent identity

$$C^2 R(\mu, A) = C^2 R(\lambda, A) + (\lambda - \mu) C^2 R(\lambda, A) R(\mu, A)$$

and the fact that $\Pi_{p,q}(X)$ is an ideal in $L(X)$ implies $C^2 R(\mu, A) \in \Pi_{p,q}(X)$ for all $\mu \in \rho(A)$. ■

Theorem 1.5. Let $T(t)$ be an exponentially bounded strongly continuous C-semigroup of bounded linear operators on X with generator A. If $R(\lambda, A) \in \Pi_{p,q}(X)$ for all $\lambda \in \rho(A)$ and $\|R(\lambda, A)\|_{\Pi_{(p,q)}} \leq \frac{\beta}{\lambda-\omega}$ for $\lambda > \omega$, then $T(t) \in \Pi_{p,q}(X)$ and $\|T(t)\|_{\Pi_{(p,q)}} \leq \xi$ in $(0, \varepsilon)$ for some $\varepsilon > 0$. Further $CT(t) \in \Pi_{p,q}(X)$ for all $t > 0$.

Proof. Let $\lambda \in \rho(A)$, $\lambda > \omega > 0$. Since $R(\lambda, A) \in \Pi_{p,q}(X)$ and $T(t) \in L(X)$ for all $t > 0$, it follows that $\lambda R(\lambda, A) T(t) \in \Pi_{p,q}(X)$. But $\|R(\lambda, A)\|_{\Pi_{(p,q)}} \leq \frac{\beta}{\lambda-\omega}$. Thus by Lemma 1.3 we get :
$$\lim_{\lambda\to\infty} \lambda R(\lambda, A) T(t) x = T(t) x,$$
for all $x \in X$ and so,

$$\|\lambda R(\lambda, A) T(t)\|_{\Pi_{(p,q)}} \leq \|\lambda R(\lambda, A)\|_{\Pi_{(p,q)}} \|T(t)\| \leq \frac{\beta\lambda}{\lambda - \omega} \|T(t)\|.$$

Consequently, since $T(t)$ is exponentially bounded there exist $\gamma > 0$ and $\varepsilon > 0$, such that $\|\lambda R(\lambda, A) T(t)\|_{\Pi_{(p,q)}} \leq \gamma \frac{\beta\lambda}{\lambda-\omega}$ for all $t \in (0, \varepsilon)$, and $\lambda > \omega$. Lemma 1.1 implies that $T(t) \in \Pi_{p,q}(X)$ for all $t \in (0, \varepsilon)$ and Lemma 1.2 then implies $CT(t) \in \Pi_{p,q}(X)$ for all $t > 0$. Further, since $\{\frac{\lambda}{\lambda-\omega} : \lambda > \omega \geq 0\}$ is a bounded set, it follows that $\|T(t)\|_{\Pi_{(p,q)}} \leq \gamma \sup_\lambda \frac{\beta\lambda}{\lambda-\omega}$ for $t \in (0, \varepsilon)$. ■

Theorem 1.6. Let $T(t)$ be a differentiable strongly continuous exponentially bounded C-semigroup on X with generator A. If there exists $\lambda_0 \in \rho(A)$ such that $R(\lambda_0, A) \in \Pi_{p,q}(X)$, then $T(t) \in \Pi_{p,q}(X)$ for all $t > 0$.

Proof. Let $\lambda_0 \in \rho(A)$ and $\lambda_0 = 0$. Define $B(t)x = \int_0^t T(s)x\, ds$. Then $B \in L(X)$ and
$$AB(t)x = A\int_0^t T(s)x\, ds = T(t)x - Cx = (T(t) - C)x$$
for all $x \in X$, (Lemma 2.7,[3]). Hence
$$-AB(t)x = (0 - A)B(t)x = (C - T(t))x.$$

So $B(t)x = R(0,A)(C-T(t))x$ for all $x \in X$. Thus $B(t) = R(0,A)(C-T(t))$. But $R(0,A) \in \Pi_{p,q}(X)$. So $B(t) \in \Pi_{p,q}(X)$ for all $t > 0$.

Now, since $T(t)$ is strongly continuous, then for $x \in D(A)$, $B^{'}(t)x$ exists and

$$\begin{aligned} B^{'}(t)x &= \lim_{h\to 0} \tfrac{B(t+h)x-B(t)x}{h} \\ &= \lim_{n\to\infty} n\left(B(t+\tfrac{1}{n})x - B(t)x\right) \\ &= \lim_{n\to\infty} n\left(R(0,A)(C-T(t+\tfrac{1}{n}))x - R(0,A)(C-T(t))x\right) \\ &= \lim_{n\to\infty} nR(0,A)\left(T(t)x - T(t+\tfrac{1}{n})x\right). \end{aligned}$$

Define $D_n(t)x = nR(0,A)\left(T(t)x - T(t+\frac{1}{n})x\right)$. Since $R(0,A) \in \Pi_{p,q}(X)$, it follows that $D_n(t) \in \Pi_{p,q}(X)$ for all $t > 0$ and all $n \in N$. But

$$B^{'}(t)x = \tfrac{d}{dt}\int_0^t T(s)x\,ds = T(t)x.$$

Consequently, since $T(t)$ is differentiable, then $\lim_{n\to\infty} n\left(T(t) - T(t+\frac{1}{n})\right) = -T^{'}(t)$ and

$$T(t)x = \lim_{n\to\infty} D_n(t)x = -R(0,A)T^{'}(t)x.$$

But $D(A)$ is dense in X. Thus $T(t) \in \Pi_{p,q}(X)$. Further :

$$\|T(s)\|_{\Pi_{(p,q)}} \le \left\|-R(0,A)T^{'}(t)\right\|_{\Pi_{(p,q)}} \le \|R(0,A)\|_{\Pi_{(p,q)}} \left\|T^{'}(t)\right\| < \infty.$$

For $\lambda_0 \neq 0$, define $S(t) = e^{\lambda_0 t}T(t)$. Then if G is the generator of $T(t)$, then $G - \lambda_0$ is the generator of $e^{-\lambda_0 t}T(t)$. So if $\lambda_0 \in \rho(G-\lambda_0)$, then $0 \in \rho(G)$. ■

II. C-semigroups and Uniformly Dominated Sets of $\Pi_p(X)$.

Let X be a Banach space and C be an injective bounded linear operator on X. We start with the following definition.

Definition 2.1. A subset $E \subseteq \Pi_p(X)$ is called uniformly dominated, if there exists a probability measure μ on $B_1(X^*)$, such that :

$$\|Tx\| \le \lambda_T \left(\int_{B_1(X^*)} |\langle x, x^*\rangle|^p \, d\mu\,(x^*) \right)^{\frac{1}{p}}$$

for all $x \in X$ and all $T \in E$.

Now we prove another main result of this paper.

Theorem 2.2. Let $T(t)$ be an exponentially bounded strongly continuous C-semigroup on X with generator A. If $\{T(t),\ t \in (0,\infty)\}$ is uniformly dominated set in $\Pi_p(X)$, and $\int_0^\infty e^{-\lambda s}\,\|T(s)\|_{\Pi(p)}\,ds < \infty$ for all $\lambda > \omega$, then $\{CR(\lambda, A),\ \lambda \in \rho(A)\}$ is uniformly dominated set in $\Pi_p(X)$.

Proof. Let $x \in X$ and $\lambda \in \rho(A)$, $\lambda \in R$, $\lambda > \omega > 0$. Then by Theorem 3.3, [3] we have :

$$\begin{aligned}
\|CR(\lambda, A)x\| &= \left\| \int_0^\infty e^{-\lambda s} T(s)x\, ds \right\| \\
&\leq \int_0^\infty e^{-\lambda s} \|T(s)x\|\, ds \\
&\leq \int_0^\infty e^{-\lambda s} \|T(s)\|_{\Pi(p)} \left(\int_{B_1(X^*)} |\langle x, x^* \rangle|^p \, d\mu(x^*) \right)^{\frac{1}{p}} ds \\
&= \int_0^\infty e^{-\lambda s} \|T(s)\|_{\Pi(p)}\, ds \left(\int_{B_1(X^*)} |\langle x, x^* \rangle|^p \, d\mu(x^*) \right)^{\frac{1}{p}}.
\end{aligned}$$

But $\int_0^\infty e^{-\lambda s} \|T(s)\|_{\Pi(p)}\, ds < \infty$. Thus using Pietsch Dominated Theorem, [12], for p-summing operators we get: $CR(\lambda, A) \in \Pi_p(X)$ and $\{CR(\lambda, A),\ \lambda \in \rho(A), \lambda > \omega > 0\}$ is uniformly dominated set in $\Pi_p(X)$.

Now let $\lambda, \mu \in \rho(A)$ and $\lambda > \omega > 0$. Then the resolvent identity

$$CR(\mu, A) = CR(\lambda, A) + (\lambda - \mu)CR(\lambda, A)R(\mu, A)$$

and the fact that $\Pi_p(X)$ is an ideal in $L(X)$ implies $CR(\mu, A) \in \Pi_p(X)$ for all $\mu \in \rho(A)$. Further :

$$\begin{aligned}
\|CR(\mu, A)x\| &\leq \|CR(\lambda, A)x\| + |\mu - \lambda|\, \|R(\mu, A)CR(\lambda, A)x\| \\
&\leq \|CR(\lambda, A)x\| + |\mu - \lambda|\, \|R(\mu, A)\|\, \|CR(\lambda, A)x\| \\
&= (1 + |\mu - \lambda|\, \|R(\mu, A)\|)\, \|CR(\lambda, A)x\| \\
&\leq (1 + |\mu - \lambda|\, \|R(\mu, A)\|)\, \|CR(\lambda, A)\|_{\Pi(p)} \left(\int_{B_1(X^*)} |\langle x, x^* \rangle|^p \, d\mu(x^*) \right)^{\frac{1}{p}},
\end{aligned}$$

which implies that $\{CR(\mu, A),\ \mu \in \rho(A)\}$ is uniformly dominated set in $\Pi_p(X)$. ∎

Theorem 2.3. Let $T(t)$ be a differentiable strongly continuous exponentially bounded C-semigroup on X with generator A. If $Range(C)$ is dense and $\{R(\lambda, A),\ \lambda \in \rho(A)\}$ is uniformly dominated set in $\Pi_p(X)$, then $\{T(t),\ t \in (0, \infty)\}$ is uniformly dominated set in $\Pi_p(X)$.

Proof. With no loss of generality, assume $0 \in \rho(A)$. For $x \in X$ and $t > 0$ define, $B(t)x = \int_0^t T(s)x\, ds$. Then $B \in L(X)$ and

$$AB(t)x = A\int_0^t T(s)x\, ds = T(t)x - Cx = (T(t) - C)x$$

for all $x \in X$, (Lemma 2.7,[3]). Hence

$$-AB(t)x = (0 - A)B(t)x = (C - T(t))x.$$

So $B(t)x = R(0,A)(C-T(t))x$ for all $x \in X$. Thus $B(t) = R(0,A)(C-T(t))$. But $R(0,A) \in \Pi_{p,q}(X)$. So $B(t) \in \Pi_{p,q}(X)$ for all $t > 0$. Consequently, $B(t)$ is uniformly bounded in $\Pi_p(X)$ for $t \in (0, t_0)$ for some $t_0 > 0$.

Now, since $T(t)$ is strongly continuous, then for $x \in D(A)$, $B^{'}(t)x$ exists and

$$\begin{aligned} B^{'}(t)x &= \lim_{h\to 0} \tfrac{B(t+h)x - B(t)x}{h} \\ &= \lim_{n\to\infty} n\left(B(t+\tfrac{1}{n})x - B(t)x\right) \\ &= \lim_{n\to\infty} n\left(R(0,A)(C - T(t+\tfrac{1}{n}))x - R(0,A)(C-T(t))x\right) \\ &= \lim_{n\to\infty} nR(0,A)\left(T(t)x - T(t+\tfrac{1}{n})x\right). \end{aligned}$$

Define $D_n(t)x = nR(0,A)\left(T(t)x - T(t+\frac{1}{n})x\right)$. Since $R(0,A) \in \Pi_p(X)$, it follows that $D_n(t) \in \Pi_p(X)$ for all $t > 0$ and all $n \in N$. But

$$B^{'}(t)x = \tfrac{d}{dt}\int_0^t T(s)x\,ds = T(t)x.$$

it follows that

$$T(t)x \;=\; \tfrac{d}{dt}\int_0^t T(s)x\,ds \;=\; \lim_{n\to\infty} n\left(B(t+\tfrac{1}{n})x - B(t)x\right) = -R(0,A)T^{'}(t)x.$$

But $Range(C)$ is dense. So by Theorem 2.4,[3], $D(A)$ is dense. Consequently, since $T(t)$ is differentiable we get $T(t) \in \Pi_p(X)$ and

$$\begin{aligned} \|T(t)x\| &= \left\|-R(0,A)T^{'}(t)x\right\| \\ &\le \left\|T^{'}(t)\right\| \|R(0,A)x\| \\ &\le \left\|T^{'}(t)\right\| \|R(0,A)\|_{\Pi(p)} \left(\int_{B_1(X^*)} |\langle x, x^*\rangle|^p \, d\mu\,(x^*) \right)^{\frac{1}{p}}. \end{aligned}$$

Hence : $\{T(t),\ t \in (0,\infty)\}$ is uniformly dominated in $\Pi_p(X)$. ∎

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