ON THE NUMBER OF THE NON-EQUIVALENT 1-REGULAR SPANNING SUBGRAPHS OF THE COMPLETE GRAPHS OF EVEN ORDER

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Abstract. The Dihedral group D_n acts on the complete graph K_n naturally. This action of D_n induces the action on the set of the 1-regular spanning subgraphs of the complete graph K_n of even order n. In this paper we calculate the number of the equivalence classes of the 1-regular spanning subgraphs of the complete graph K_n of even order n by this action by using Burnside's Lemma. This problem was presented by Dr. Shun-ichiro Koh who is a physicist of Kochi University. Also we calculate the number of the equivalence classes of the maximal matchings of the complete graph K_n with odd order n by the group action of the Dihedral group D_n.

Let n be even and be greater than or equal to 2. Let $\{v_0, v_1, v_2, \cdots, v_{n-1}\}$ be the vertices of the complete graph K_n. The action to K_n of the Dihedral group $D_n = \{\rho_0, \rho_1, \cdots, \rho_{n-1}, \sigma_0, \sigma_1, \cdots, \sigma_{n-1}\}$ is defined by

$$\rho_i(v_k) = v_{(k+i) \pmod{n}} \quad for \;\; 0 \leq i \leq n-1, \; 0 \leq k \leq n-1$$

$$\sigma_i(v_k) = v_{(n+i-k) \pmod{n}} \quad for \;\; 0 \leq i \leq n-1, \; 0 \leq k \leq n-1$$

Let X_n be the set of the 1-regular spanning subgraphs of K_n. Then the above action induces the action on X_n of the Dihedral group D_n.

The equivalence classes of X_4 are given with the next figure.

The equivalence classes of X_6 are given with the next figure.

The equivalence classes of X_8 are given with the next figure.

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We calculate the number of the equivalence classes by this group action. This problem was presented by Dr. Shun-ichiro Koh who is a physicist of Kochi University. These computations can be done by using Burnside's lemma.

Definition 1. *Let P be a nonempty collection of permutations on the same finite set of objects Y such that P is a group. Then the mathematical structure $[P : Y]$ is a permutation group.*

Definition 2. *Let $P = [P : Y]$ be a permutation group, and let $\pi \in P$. The fixed-point set of the permutation π is the subset $Fix(\pi) = \{y \in Y | \pi(y) = y\}$.*

Definition 3. *Let $P = [P : Y]$ be a permutation group. The orbit of an object $y \in Y$ is the set $\{\pi(y) | \pi \in P\}$ of all the objects onto which y is permuted.*

Theorem 1. *(Burnside's lemma) Let $P = [P : Y]$ be a permutation group with n orbits. Then*

$$n = \frac{1}{|P|} \sum_{\pi \in P} |Fix(\pi)|$$

Notation 1. *Let $(2k+1)!!$ be $\prod_{d=0}^{k}(2d+1)$ for $k \geq 0$ and $(-1)!!$ be 1.*

Our main Theorem is the following:

Theorem 2. *The number of the non-equivarent 1-regular spanning subgraphs of the complete graph K_n of even order n is*

$$\frac{1}{2n}\{\sum_{i=0}^{n-1} R_i^n + \frac{n}{2}(S_n + S_{n-2})\}$$

Here R_i^n is given by

1. *in the case $(n, i) = 2d+1$:*

$$\sum_{k=0}^{d} \binom{2d+1}{2k+1} \times (2d-2k-1)!! \times (\frac{n}{2d+1})^{d-k}$$

2. *in the case $(n, i) = 2d$:*
   if $n/2d \equiv 1 \pmod 2$ then

$$(2d-1)!! \times (\frac{n}{2d})^{d}$$

   if $n/2d \equiv 0 \pmod 2$ then

$$\sum_{k=0}^{d} \binom{2d}{2k} \times (2d-2k-1)!! \times (\frac{n}{2d})^{d-k}$$

And S_n is given by the following recursive formula:

$$S_0 = 1, S_2 = 1, S_n = S_{n-2} + (n-2)S_{n-4} \; for \; n \geq 4$$

We must determine the numbers of the fixed points of each permutation ρ_i and σ_i to prove the Theorem by using Burnside's Lemma.

Lemma 1. *The number of the 1-regular spanning subgraphs of K_n is (n-1)!!. This is the number of the fixed points of ρ_0.*

Proof. We prove this lemma by the induction on n. The number of the 1-regular spanning subgraphs of K_2 is one. We surpose that the number of the 1-regular spanning subgraphs of K_{n-2} is $(n-3)!!$. For each edge (v_0, v_i) of K_n , $1 \leq i \leq n-1$, there are $(n-3)!!$ 1-regular spanning subgraphs of $K_n - \{v_0, v_i\}$. Then totally there are $(n-1)!!$ 1-regular spanning subgraphs of K_n. □

Remark 1. *It is easily checked that R_0^n is equal to $(n-1)!!$.*

Lemma 2. *If (n,i)=1 then the number of the fixed points of ρ_i is one.*

Proof. If $H = \{v_\alpha v_{n/2+\alpha} | 0 \leq \alpha \leq n/2-1\}$ then H is a 1-regular spanning subgraph of K_n and $\rho_i(H) = H$. Conversely, let H be a 1-regular spanning subgraph of K_n which is fixed by ρ_i and let $v_0 v_m$ be an edge of H. Since (n,i)=1, there is an integer α such that $\alpha i \equiv m \pmod{n}$. Then $\rho_i^\alpha(v_0) = v_m$ and $\rho_i^\alpha(v_m) = v_{(m+i\alpha) \pmod{n}}$. Since $\rho_i(H) = H$, we have $v_0 v_m = v_m v_{(m+i\alpha) \pmod{n}}$. Then we have $m + i\alpha \equiv 0 \pmod{n}$ and $2m \equiv 0 \pmod{n}$ and therefore $m = n/2$ and $v_0 v_{n/2} \in H$. Since $\{\rho_i^\alpha(0) | 0 \leq \alpha \leq n-1\} = \{0, 1, 2, \cdots, n-1\}$, H is uniquely determined by $v_0 v_{n/2}$ and $H = \{v_\alpha v_{n/2+\alpha} | 0 \leq \alpha \leq n/2-1\}$. Then the number of the fixed points of ρ_i is one. □

Notation 2. *Let M_n be the 1-regular spanning subgraph $\{v_\alpha v_{n/2+\alpha} | 0 \leq \alpha \leq n/2-1\}$ of K_n.*

Lemma 3. *If (n,i)=2 and $n \equiv 2 \pmod{4}$ then the number of the fixed points of ρ_i is n/2 and if (n,i)=2 and $n \equiv 0 \pmod{4}$ then the number of the fixed points of ρ_i is n/2+1.*

Proof. Since $(n.i) = 2$, the equation $xi \equiv m \pmod{n}$ has a solution if and only if m is even. Then if $V_0 = \{v_0, v_2, v_4, \cdots, v_{n-2}\}$ and $V_1 = \{v_1, v_3, v_5, \cdots, v_{n-1}\}$ then $\rho_i(V_0) = V_0$ and $\rho_i(V_1) = V_1$. Let H be a 1-regular spanning subgraph of K_n such that $\rho_i(H) = H$ and let $v_0 v_m \in H$. If m is even then the edge $v_0 v_m$ induces a 1-regular spanning subgraph of $K_{n/2}$ that is fixed by $\rho_{i/2}$. Since (n/2, i/2)=1, the subgraph is uniquely determined by Lemma 2. Similarly, the induced subgraph $H|V_1$ is also unique 1-regular spanning subgraph of $K_{n/2}$ that is fixed by $\rho_{i/2}$ by Lemma 2. Then we have that $H = M_n$. Let m be odd. Since $\rho_i(V_0) = V_0$ and $\rho_i(V_1) = V_1$, edge $v_0 v_m$ determines unique 1-regular spanning subgraph $H = \{v_{i\alpha} v_{(m+i\alpha) \pmod{n}} | 0 \leq \alpha \leq n/2-1\}$.

Therefore if $n \equiv 2 \pmod{4}$ then there are n/2 1-regular spanning subgraph of K_n which are fixed by ρ_i and if $n \equiv 0 \pmod{4}$ then there are n/2+1 1-regular spanning subgraph of K_n which are fixed by ρ_i. We have the results. □

Lemma 4. *The number of the way of dividing $2m$ objects into m sets which contain two objects is $(2m-1)!!$.*

Proof. This is easily verified by the induction on m and this number is essentially same the number given in Lemma 1. □

Lemma 5. *If (n,i)=2d+1 then the number of the fixed points of ρ_i is*

$$\sum_{k=0}^{d} \binom{2d+1}{2k+1} \times (2d-2k-1)!! \times (\frac{n}{2d+1})^{d-k}$$

Proof. Let $V_0 = \{v_0, v_{2d+1}, v_{4d+2}, \cdots, v_{n-2d-1}\}, V_1 = \{v_1, v_{2d+2}, v_{4d+3}, \cdots, v_{n-2d}\}$, $V_2 = \{v_2, v_{2d+3}, v_{4d+4}, \cdots, v_{n-2d+1}\}, \cdots, V_{2d} = \{v_{2d}, v_{4d+1}, v_{4d+2}, \cdots, v_{n-1}\}$.

Since $(n, i) = 2d + 1$, the equation $xi \equiv m \pmod{n}$ has a solution if and only if $2d + 1$ divides m. Then we have $\rho_i(V_k) = V_k$ *for* $0 \leq k \leq 2d$. Let H be a 1-regular spanning subgraph of K_n which is fixed by ρ_i and let $v_\alpha v_\beta$ be an edge of H. If $v_\alpha \in V_k$ and $v_\beta \in V_k$ then the induced subgraph $H|V_k$ is a 1-regular spanning subgraph of $K_{n/(2d+1)}$ which is fixed by $\rho_{i/(2d+1)}$ and it is unique 1-regular spanning subgraph $M_{n/(2d+1)}$ by Lemma 2. If $v_\alpha \in V_{k_1}$ and $v_\beta \in V_{k_2}$ then the induced subgraph $H|V_{k_1} \cup V_{k_2}$ is a 1-regular spanning subgraph of $K_{2n/(2d+1)}$ which is fixed by $\rho_{i/(2d+1)}$. Since $(2n/(2d+1), i/(2d+1)) = 2$ and $2n/(2d+1) \equiv 0 \pmod{4}$, the number of the 1-regular spanning subgraphs of $K_{2n/(2d+1)}$ which is fixed by $\rho_{i/(2d+1)}$ is $n/(2d+1)+1$ by Lemma3 and one 1-regular spanning subgraph among these subgraphs is $M_{2n/(2d+1)}$. We calculate the number of the case that $2k+1$ sets of vertices make 1-regular spanning subgraph $M_{n/(2d+1)}$ and the remainning $2(d-k)$ sets of vertices make 1-regular spanning subgraph with pair. There are $\binom{2d+1}{2k+1} \times (2d-2k-1)!!$ combinations of the sets of vertices like these by Lemma 4. Then, if $k < d$ then the number of the 1-regular spanning subgraphs fixed by ρ_i which are not M_n is

$$\binom{2d+1}{2k+1} \times (2d-2k-1)!! \times (\frac{n}{2d+1})^{d-k}.$$

If $k = d$ then the number of the 1-regular spanning subgraphs fixed by ρ_i is one and this subgraph is M_n. Therefore the total number of the 1-regular spanning subgraphs fixed by ρ_i is given by

$$\sum_{k=0}^{d} \binom{2d+1}{2k+1} \times (2d-2k-1)!! \times (\frac{n}{2d+1})^{d-k}$$

We have the results. □

Lemma 6. *If* $(n, i) = 2d$ *and* $n/(2d) \equiv 1 \pmod{2}$ *then the number of the fixed points of* ρ_i *is*

$$(2d-1)!! \times (\frac{n}{2d})^d$$

and if $(n, i) = 2d$ *and* $n/(2d) \equiv 0 \pmod{2}$ *then the number of the fixed points of* ρ_i *is*

$$\sum_{k=0}^{d} \binom{2d}{2k} \times (2d-2k-1)!! \times (\frac{n}{2d})^{d-k}$$

Proof. Let $V_0 = \{v_0, v_{2d}, v_{4d}, \cdots, v_{n-2d}\}, V_1 = \{v_1, v_{2d+1}, v_{4d+1}, \cdots, v_{n-2d+1}\}$, $V_2 = \{v_2, v_{2d+2}, v_{4d+2}, \cdots, v_{n-2d+2}\}, \cdots, V_{2d-1} = \{v_{2d-1}, v_{4d-1}, v_{6d-1}, \cdots, v_{n-1}\}$. Since $(n, i) = 2d$, the equation $xi \equiv m \pmod{n}$ has a solution if and only if $2d$ divides m. Then $\rho_i(V_k) = V_k$ *for* $0 \leq k \leq 2d-1$.

Let $n/(2d)$ be odd. Since $|V_k| = n/(2d)$ is odd, $H|V_k$ is not 1-regular spanning subgraph of $K_{n/(2d)}$ for all k. Accordingly, two vertices of each edge of H are contained in two subsets of vertices. If $v_\alpha \in V_{k_1}$ and $v_\beta \in V_{k_2}$ for an edge $v_\alpha v_\beta$ of H then the induced subgraph $H|V_{k_1} \cup V_{k_2}$ is a 1-regular spanning subgraph of $K_{n/d}$ which is fixed by $\rho_{i/(2d)}$. Since $n/d \equiv 2 \pmod{4}$, the number of such 1-regular spanning subgraphs of $K_{n/d}$ which is fixed by $\rho_{i/(2d)}$ is $n/(2d)$. Since the number of the pairings of $V_0, V_1, \cdots, V_{2d-1}$ is $(2d-1)!!$, the total number of the 1-regular spanning subgraphs of K_n which is fixed by $\rho_{i/(2d)}$ is

$$(2d-1)!! \times (\frac{n}{2d})^d$$

Next let $n/(2d)$ be even. Since $|V_k| = n/(2d)$ is even, if there is some edge $v_\alpha v_\beta \in H$ such that v_α and v_β are both contained in some V_k then the induce subgraph $H|V_k$ is a 1-regular spanning subgraph of $K_{n/2d}$ fixed by $\rho_{i/(2d)}$. By the essentially same augments as above, in this case, we have that the number of the 1-regular spanning subgraphs of K_n which is fixed by ρ_i is

$$\sum_{k=0}^{d} \binom{2d}{2k} \times (2d-2k-1)!! \times (\frac{n}{2d})^{d-k}$$

We have the results. □

Lemma 7. *The number of the fixed points of σ_0 is equal to the number of the fixed points of σ_{2d} for all $1 \leq d \leq n/2-1$.*

Proof. Let H be a 1-regular spanning subgraph of K_n fixed by σ_0. Then it is easily verified that $\rho_d(H)$ is a 1-regular spanning subgraph of K_n fixed by σ_{2d}. Conversely, if H is a 1-regular spanning subgraph of K_n fixed by σ_{2d} then $\rho_d^{-1}(H)$ is a 1-regular spanning subgraph of K_n fixed by σ_0. Then we have the results. □

Similarly, we have the next Lemma.

Lemma 8. *The number of the fixed points of σ_1 is equal to the number of the fixed points of σ_{2d+1} for all $1 \leq d \leq n/2-1$.*

Lemma 9. *The number of the fixed points of σ_0 is equal to the number of the 1-regular spanning subgraphs of K_{n-2} fixed by σ_1.*

Proof. Let H be a 1-regular spanning subgraph of K_n fixed by σ_0 and $v_0v_m \in H$. Since $\sigma_0(v_0) = v_0$, $\sigma(v_m)$ must be v_m. Since $\sigma(v_m) = v_{(n+0-m) \pmod n}$, m must be $n/2$. We remove two vertices v_0 and $v_{n/2}$ from H and change the labels of the vertices of H from $v_1, v_2, \cdots, v_{n/2-1}$ to $v_0, v_1, \cdots, v_{n/2-2}$ and from $v_{n/2+1}, v_{n/2+2}, \cdots, v_{n-1}$ to $v_{n/2-1}, v_{n/2}, \cdots, v_{n-3}$. Let H' be the resulting graph. Since $\sigma_0(H) = H$, we have $\sigma_{n-3}(H') = H'$. Conversely, let H' be a 1-regular spanning subgraph of K_{n-2} fixed by σ_{n-3}. We change the labels of the vertices of H' from $v_0, v_1, \cdots, v_{n/2-2}$ to $v_1, v_2, \cdots, v_{n/2-1}$ and from $v_{n/2-1}, v_{n/2}, \cdots, v_{n-3}$ to $v_{n/2+1}, v_{n/2+2}, \cdots, v_{n-1}$ and add the edge $v_0v_{n/2}$ to it. Let H be the resulting graph. H is a 1-regular spanning subgraph of K_n fixed by σ_0. This correspondence is one to one correspondece between the set of the 1-regular spanning subgraphs of K_n fixed by σ_0 and the set of the 1-regular spanning subgraphs of K_{n-2} fixed by σ_{n-3}. Then we have the results by Lemma 8. □

Lemma 10. *Let S_n be the number of the fixed points of σ_1 for X_n. Then we have*

$$S_4 = 3, S_6 = 7 \text{ and } S_n = S_{n-2} + (n-2)S_{n-4} \text{ for all } n \geq 8.$$

Proof. By the direct computation, we can easily checked that $S_4 = 3$ and $S_6 = 7$. We study two kinds of constitutions that compose 1-regular spanning subgraphs of K_n fixed by σ_1 inductively.

The first method is the following:

Let H be a 1-regular spanning subgraph of K_{n-2} fixed by σ_1. We change the labels of vertices of H from v_0 to v_{n-1} and from $v_1, v_2, \cdots, v_{n-3}$ to $v_2, v_3, \cdots, v_{n-2}$ and add an edge v_0v_1 to it. Let H_0 be the resulting graph. Then H_0 is a 1-regular spanning subgraph of K_n such that $\sigma_1(H_0) = H_0$. We change the labels of vertices of H from $v_{n/2}, v_{n/2+1}, \cdots, v_{n-3}$

to $v_{n/2+2}, v_{n/2+3}, \cdots, v_{n-1}$ and add an edge $v_{n/2}v_{n/2+1}$ to it. Let H_1 be the resulting graph. Then H_1 is a 1-regular spanning subgraph of K_n such that $\sigma_1(H_1) = H_1$.

The second method is the following:
Let H be a 1-regular spanning subgraph of K_{n-4} fixed by σ_1. We change the labels of the vertices of H from v_0 to v_{n-2} and from $v_1, v_2, \cdots, v_{n-5}$ to $v_3, v_4, \cdots, v_{n-3}$. Let H_0 be the graph which is added edges v_1v_2 and v_0v_{n-1} to it and H_1 be the graph which is added edges v_0v_2 and v_1v_{n-1} to it. Then H_0 and H_1 are 1-regular spanning subgraphs of K_n fixed by σ_1. For each $1 \le i \le n/2 - 2$, we change the labels of the vertices of H from v_0 to v_{n-1} and from $v_1, v_2, \cdots, v_i$ to $v_2, v_3, \cdots, v_{i+1}$ and from $v_{i+1}, v_{i+2}, \cdots, v_{n-i-4}$ to $v_{i+3}, v_{i+4}, \cdots, v_{n-i-2}$ and from $v_{n-i-3}, v_{n-i-2}, \cdots, v_{n-5}$ to $v_{n-i}, v_{n-i+1}, \cdots, v_{n-2}$. Let H_{2i} be the graph which is added two edges v_0v_{i+2} and v_1v_{n-i-1} and H_{2i+1} be the graph which is added two edges v_1v_{i+2} and v_0v_{n-i-1}. Then H_{2i} and H_{2i+1} are 1-regular spanning subgraphs of K_n fixed by σ_1. We change the labels of the vertices of H from v_0 to v_{n-1} and from $v_1, v_2, \cdots, v_{n/2-2}$ to $v_2, v_3, \cdots, v_{n/2-1}$ and from $v_{n/2-1}, v_{n/2}, \cdots, v_{n-5}$ to $v_{n/2+2}, v_{n/2+3}, \cdots, v_{n-2}$. Let H'_0 be the graph which is added two edges $v_1v_{n/2}$ and $v_0v_{n/2+1}$ and H'_1 be the graph which is added two edges $v_0v_{n/2}$ and $v_1v_{n/2+1}$. For each $1 \le i \le n/2 - 2$, we change the labels of the vertices of H from $v_{i+1}, v_{i+2}, \cdots, v_{n/2-2}$ to $v_{i+2}, v_{i+3}, \cdots, v_{n/2-1}$ and from $v_{n/2-1}, v_{n/2}, \cdots, v_{n-i-4}$ to $v_{n/2+2}, v_{n/2+3}, \cdots, v_{n-i-1}$ and from $v_{n-i-3}, v_{n-i-2}, \cdots, v_{n-5}$ to $v_{n-i+1}, v_{n-i+2}, \cdots, v_{n-1}$. Let H'_{2i} be the graph which is added two edges $v_{n/2}v_{i+1}$ and $v_{n/2+1}v_{n-i}$ and H'_{2i+1} be the graph which is added two edges $v_{n/2}v_{n-i}$ and $v_{n/2+1}v_{i+1}$. Then H'_{2i} and H'_{2i+1} are 1-regular spanning subgraphs of K_n fixed by σ_1. By these constructions, we can construct $2S_{n-2} + 2 \times 2 \times (n/2 - 1) \times S_{n-4}$ 1-regular spanning subgraphs of K_n fixed by σ_1. Clearly there are doubling two pieces of each. Also, it is clear to be able to compose all the 1-regular spanning subgraphs of K_n fixed by σ_1 by these methods. Then the number of the 1-regular spanning subgraphs of K_n fixed by σ_1 is given by $S_{n-2} + (n-2)S_{n-4}$. We have the results. □

Remark 2. *Let $S_0 = 1$ and $S_2 = 1$. Then we have $S_n = S_{n-2} + (n-2)S_{n-4}$ for $n \ge 4$.*

Then we completely proved Theorem 2.

Remark 3. *We calculated the non-equivarent 1-regular spanning subgraphs of K_n, $n \le 12$ by computer. The numbers agreed with the numbers that are given by Theorem 2. The results is as follows:*

n=2	1
n=4	2
n=6	5
n=8	17
n=10	79
n=12	554

Next let n be odd and be greater than or equal to 3. Let $\{v_0, v_1, v_2, \cdots, v_{n-1}\}$ be the vertices of the complete graph K_n. The action to K_n of the Dihedral group $D_n = \{\rho_0, \rho_1, \cdots, \rho_{n-1}, \sigma_0, \sigma_1, \cdots, \sigma_{n-1}\}$ is defined by

$$\rho_i(v_k) = v_{(k+i) \pmod{n}} \quad for \quad 0 \le i \le n-1,\ 0 \le k \le n-1$$

$$\sigma_i(v_k) = v_{(n+2i-k) \pmod{n}} \quad for \quad 0 \le i \le n-1,\ 0 \le k \le n-1$$

Let Y_n be the set of the maximal matchings of K_n. Then the above action induces the action on Y_n of the Dihedral group D_n.We calculate the number of the equivalence classes by this group action.

Theorem 3. *The number of the non-equivalent maximal matcings of the complete graph K_n with odd order n is*

$$\frac{1}{2n}\{n!! + nS_{n-1}\}$$

Here S_n is given in Lemma 10.

Proof. This Theorem is also proved by Burnside's Lemma. To construct a maximal matching we choose an isolated vertex and then choose $(n-1)/2$ pairings of resulting $n-1$ vertices. There are $n \times (n-2)!!$ combinations like these by Lemma 4. Then the number of the maximal matcings of the complete graph K_n is $n!!$ and this number is the number of the fixed points of Y_n by ρ_0. Since there is only one isolated vertex, $\rho_i, 1 \leq i \leq n-1$, fixes no maximal matcings of the complete graph K_n. Let i be greater than 0 and less than n and H be a maximal matcings of the complete graph K_n such that $\sigma_i(H) = H$. Since $\sigma_i(v_i) = v_i$, v_i is an isolated vertex of H. If $i = 0$ then we remove the vertex v_0 from H and change the labels of the vertices of H from $v_1, v_2, \cdots, v_{(n-1)/2}$ to $v_0, v_1, \cdots, v_{(n-3)/2}$. Let H_0 be the resulting graph. Then H_0 is a 1-regular spanning subgraph of K_{n-1} such that $\sigma_{n-2}(H_0) = H_0$. By this construction, we can construct an one to one correspondence between the set of the maximal matcings of the complete graph K_n such that $\sigma_0(H) = H$ and the set of the 1-regular spanning subgraph of K_{n-1} such that $\sigma_{n-2}(H_0) = H_0$. If $1 \leq i \leq n-1$ then we remove the vertex v_i from H and change the labels of the vertices of H from $v_{i+1}, v_{i+2}, \cdots, v_{n-1}$ to $v_i, v_{i+1}, \cdots, v_{n-2}$. Let H_i be the resulting graph. Then H_i is a 1-regular spanning subgraph of K_{n-1} such that $\sigma_{2i-1 \pmod n}(H_i) = H_i$. By this construction, we can construct an one to one correspondence between the set of the maximal matcings of the complete graph K_n such that $\sigma_i(H) = H$ and the set of the 1-regular spanning subgraph of K_{n-1} such that $\sigma_{2i-1 \pmod n}(H_i) = H_i$. Then the number of the fixed points of σ_i is S_{n-1}. Then we have the results. □

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