ON THE APPROXIMATIONS OF SOLUTIONS TO FUZZY DIFFERENTIAL EQUATIONS*

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ABSTRACT. In this paper, the existence and uniqueness of Caratheodory solutions for fuzzy differential equations of one dimension are established. Furthermore, under some conditions, the approximations of solutions for the above differential equations are disscused.

1 Introduction

The notion of fuzzy number originated from [1] in which fuzzy sets with some properties on the field of real numbers are called fuzzy numbers. Because of their applications on fuzzy control and fuzzy approximation (see for example [17, 18]), there are more and more studies on the algebraic structure and analytic property of fuzzy numbers (see for example [3, 11, 12, 13]). Goetschel and Voxman [5] described fuzzy numbers with the following reference functions $\{(a(r), b(r), r) : r \in [0, 1]\}$. Later in [10] Wu and Ma got a series of results on the calculus of fuzzy numbers by embedding fuzzy numbers into the Banach space $\bar{C}[0, 1] \times \bar{C}[0, 1]$.

Fuzzy differential equations were introduced by Kandel and Bytt in [8, 9] and later applied to fuzzy processes and fuzzy dynamical systems. In [6, 7] Kaleva studied the classical solutions of Cauchy problem for fuzzy differential equations. Recently, Friedman, Ma and Kandel [4] studied the numerical solutions of fuzzy differential equations. In this paper we study the Caratheodory solutions (which is called C-solutions for the sake of simplicity) for a class of fuzzy differential equations, and obtain more general existence and uniqueness of C-solutions, and continuous depedence of solutions on initial values and stability of solutions.

2 Preliminaries

First let us recall some notions and facts about fuzzy numbers and fyzzy functions.

Let $\mathbf{R}$ be the field of real numbers. Denote $E^1 = \{u : \mathbf{R} \to [0, 1]\}$ where u has the following properties:

(1) u is normal. i. e., there exists an $x_0 \in \mathbf{R}$ with $u(x_0) = 1$;

(2) u is convex, i. e., $u(rx + (1 - r)y) \geq \min(u(x), u(y))$ whenever $x, y \in \mathbf{R}$ and $r \in [0, 1]$;

(3) $u(x)$ is upper semicontinuous;

(4) $[u]^0 = \mathrm{cl}\{x \in \mathbf{R} : u(x) > 0\}$ is a compact set.

For any $u \in E^1$, u is called a fuzzy number. Obviously, $[u]^r$ is bounded closed interval for $r \in [0, 1]$ where $[u]^r = \{x \in \mathbf{R} : u(x) \geq r\}$. For $u \in E^1$, there are two functions $\underline{u}, \overline{u} : [0, 1] \to R$ such that $[u]^r = [\underline{u}(r), \overline{u}(r)]$ and the two functions satisfy the following properties (i)-(iv):

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(i) $\underline{u}$ is a bounded, left continuous, nondecreasing function;

(ii) $\overline{u}$ is a bounded, left continuous, nonincreasing function;

(iii) $\underline{u}$, $\overline{u}$ are right continuous at $r = 0$;

(iv) $\underline{u}(r) \leq \overline{u}(r)$ for $0 \leq r \leq 1$.

For any $u, v \in E^1$ and $k > 0$ we define the addition $u + v$ and the multiplication by k as: $\underline{(u+v)}(r) = \underline{u}(r) + \underline{v}(r)$, $\overline{(u+v)} = \overline{u}(r) + \overline{v}(r)$, $\underline{(ku)}(r) = k\underline{u}(r)$, $\overline{(ku)}(r) = k\overline{u}(r)$.

We call θ the null element of E^1 if $\theta : \mathbf{R} \to [0, 1]$ satisfies:

$$\theta(x) = \begin{cases} 1, & \text{if } x = 0; \\ 0, & \text{otherwise.} \end{cases}$$

Define $D : E^1 \times E^1 \to [0, +\infty)$ by the following:

$$D(u, v) = \sup_{r \in [0,1]} \max(|\underline{u}(r) - \underline{v}(r)|, |\overline{u}(r) - \overline{v}(r)|),$$

then

(a) (E^1, D) is a complete metric space;

(b) $D(ku, kv) = kD(u, v)$ for any $u, v \in E^1$ and $k > 0$;

(c) $D(u + w, v + w) = D(u, v)$ for any $u, v, w \in E^1$.

Denote $\overline{C}[0, 1]$ the set of functions which are bounded, left continuous on $[0, 1]$, and have right limits for $t \in [0, 1)$, and are right continuous at $t = 0$. For $f \in \overline{C}[0, 1]$, endow it with the norm $\|f\| = \sup_{t \in [0,1]} |f(t)|$, then $\overline{C}[0, 1]$ is a Banach space with respect to the above norm.

In [10], Wu and Ma obtained the following Theorem:

Theorem 2.1 *For $u \in E^1$, denote $j(u) = (\underline{u}, \overline{u})$, then $j(E^1)$ is a closed convex cone with vertex θ in $\overline{C}[0, 1] \times \overline{C}[0, 1]$ and $j : E^1 \to \overline{C}[0, 1] \times \overline{C}[0, 1]$ satisfies statements (a) and (b).*

(a) *For any $u, v \in E^1, s, t \geq 0$, $j(su + tv) = sj(u) + tj(v)$;*

(b) *$D(u, v) = \|j(u) - j(v)\|$ for any $u, v \in E^1$, that is to say, j is an isometric isomorphism embedding from E^1 to $\overline{C}[0, 1] \times \overline{C}[0, 1]$ where $\overline{C}[0, 1] \times \overline{C}[0, 1]$ is endowed with the norm $\|(f, g)\| = \max\{\|f\|, \|g\|\}$.*

Denote $T = [a, b]$. $F : T \to E^1$ is measurable if $\forall r \in [0, 1]$ the set-valued mapping $F^\alpha = [F(\cdot)]^r : T \to \mathcal{P}_{KC}(\mathbf{R})$ is measurable where $\mathcal{P}_{KC}(\mathbf{R})$ is the set of bounded closed convex sets in $\mathbf{R}$. And F is integrably bounded if F is measurable and there exists an integrable function $h : T \to [0, +\infty)$ such that for each $x \in [F(t)]^0$, $|x| \leq h(t)$. For each integrably bounded function F define its integral as:

$$[\int_T F(t)dt]^r = \{\int_T f(t)dt : f(t) \in [F(t)]^r \text{ is a measurable selector}\}.$$

Now there exists $u \in E^1$ such that $[u]^r = [\int_T F(t)dt]^r$, $r \in [0, 1]$, then F is integrable on T and $\int_T F(t)dt = u$.

We refer to [6, 10, 14, 15, 16] for the measurability and properties of the integrals of fuzzy mappings.

Let $u, v \in E^1$. If there exists $w \in E^1$ such that $u = v + w$, then we call w the H-difference of x and y and denote it by $u - v$.We call a mapping $F : [a, b] \to E^1$ differentiable at $t_0 \in [a, b]$, if there exists $F'(t_0) \in E^1$ such that the following limits

$$\lim_{h \to 0^+} \frac{F(t_0 + h) - F(t_0)}{h} \quad \text{and} \quad \lim_{h \to 0^+} \frac{F(t_0) - F(t_0 - h)}{h}$$

exist and equal $F'(t_0)$ where the limits are taken in (E^1, D). The above definition is due to Puri and Ralescu [14].

Remark 2.1 *That the function $F : T \to E^1$ is integrable does not guarantee that $j \circ F$ is Bochner integrable (See Note 1 in Part II of [10]). For the function $F : T \to CE^1$, where $CE^1 = \{u \in E^1 : \underline{u}, \overline{u}$ are continuous on $[0, 1]\}$, from Theorem 5.3 in Part II of [10], F is integrable iff $j \circ F$ is Bochner integrable. Therefore the differential equations in this paper are discussed on CE^1. By Theorem 2.1, $j(CE^1) \subset C[0, 1] \times C[0, 1]$ ($C[0, 1]$ denotes the set of continuous functions on $[0, 1]$).*

3 Main Results

We consider the following Cauchy problem of differential equations:

$$\begin{cases} \dot{x} = F(t, x(t)), & t \in [a, b]; \\ x(a) = x_0, & x_0 \in CE^1. \end{cases} \tag{3.1}$$

Suppose that $F(t, u) : [a, b] \times CE^1 \to CE^1$ satisfies:

(A) For each $u \in CE^1$, $F(t, u)$ is measurable with respect to t;

(B) There exists integrable function $m \in L([a, b], \mathbf{R})$ such that

$$D(F(t, u), F(t, v)) \leq m(t)D(u, v);$$

(C) There exist integrable functions $\alpha, \beta \in L([a, b], \mathbf{R})$ such that for $u \in CE^1$

$$D(F(t, u), \theta) \leq \alpha(t) + \beta(t)D(u, \theta).$$

Remark 3.1 *Let a be a Lebesgue integrable function on $[a, b]$ and $f : [a, b] \times \mathbf{R} \to \mathbf{R}$ be defined by*

$$f(t, x) = \begin{cases} a(t), & x \leq 0; \\ \frac{a(t)}{1+x^2}, & x > 0. \end{cases}$$

For each $u \in CE^1$, denote

$$\underline{v}(t, r) = \min\{f(t, x) : x \in [u]^r\}, \ r \in [0, 1],$$

$$\overline{v}(t, r) = \max\{f(t, x) : x \in [u]^r\}, \ r \in [0, 1],$$

then the r level set of $F(t, u)$ is $[F(t, u)]^r = [\underline{v}(t, r), \overline{v}(t, r)]$, which means that $F(t, u) : [a, b] \times CE^1 \to CE^1$ satisfies (A)-(C).

Definition 3.1 *$x(t) : [a, b] \to CE^1$ is absolutely continuous if $\forall \varepsilon > 0$ there exists $\delta > 0$ such that whenever $\sum_{i=1}^{n} (b_i - a_i) < \delta$ we have $\sum_{i=1}^{n} D(x(b_i), x(a_i)) < \varepsilon$ where $\{(a_i, b_i)\}_{i=1}^{n}$ are disjoint open subintervals of $[a, b]$.*

Definition 3.2 *$x(t):[a,b]\to CE^1$ is called a Caratheodory solution of (3.1) if $x(t)$ is absolutely continuous, and differentiable a. e. on $[a,b]$, and $x(a)=x_0$, $x'(t)=F(t,x(t))$ a.e..*

Denote $L([a,b],CE^1)$ the set of integrably bounded functions $x:[a,b]\to CE^1$. Let m be a nonnegative function and $m\in L([a,b],\mathbf{R})$. For $x,y\in L([a,b],CE^1)$ we introduce the following metric:

$$d(x,y)=\int_a^b D(x(t),y(t))d\mu$$

where μ is an absolutely continuous measure on $[a,b]$ and we define $d\mu=e^{-2h(t)}dt$, $h(t)=\int_a^t m(s)ds$.

Lemma 3.1 *$j(CE^1)$ is closed in $C[0,1]\times C[0,1]$.*

Proof Suppose that $u_k\in CE^1$ satisfies that $j\circ u_k\to f$ is in $C[0,1]\times C[0,1]$. Then $\{u_k\}$ is a Cauchy sequence in (CE^1,D). As (E^1,D) is complete, there exists $u\in E^1$ such that $D(u_k,u)\to 0$, or $\sup\limits_{r\in[0,1]}|\underline{u_k}(r)-\underline{u}(r)|\to 0$, $\sup\limits_{r\in[0,1]}|\overline{u_k}(r)-\overline{u}(r)|\to 0$. In view of $\underline{u_k},\overline{u_k}\in C[0,1]$, we know that $\underline{u},\overline{u}\in C[0,1]$ and further $u\in CE^1$, or $f=j\circ u\in j(CE^1)$. $\square$

Lemma 3.2 *(see Wu and Ma [10]) If $x:[a,b]\to CE^1$, then the following conditions are equivalent:*

(1) *$x\in L([a,b],CE^1)$;*

(2) *$j\circ x$ is Bochner integrable;*

(3) *For any $r\in[0,1]$, $\underline{x(\cdot)}(r),\overline{x(\cdot)}(r)$ are all Lebesgue integrable functions. Furthermore, for $t\in[a,b]$ we have*

$$[\int_a^t x(s)ds]^r=[(L)\int_a^t \underline{x(s)}(r)ds,(L)\int_a^t\overline{x(s)}(r)ds]\quad r\in[0,1].$$

Lemma 3.3 *$L([a,b],CE^1)$ is complete with respect to the metric d.*

Proof Let $\{x_n\}$ be a Cauchy sequence in $L([a,b],CE^1)$. As

$$\begin{aligned} d(x_n,x_m) &= \int_a^b D(x_n(t),x_m(t))d\mu \\ &= \int_a^b \|j\circ x_n(t)-j\circ x_m(t)\|d\mu, \end{aligned}$$

$\{j\circ x_n\}$ is a Cauchy sequence in the Bochner integrable function space $L_\mu([a,b],C[0,1]\times C[0,1])$ where $L_\mu([a,b],C[0,1]\times C[0,1])$ is a Banach space of all Bochner integrable functions $f:[a,b]\to C[0,1]\times C[0,1]$ endowed with the following norm

$$\|f\|=\int_a^b\|f(t)\|d\mu.$$

Then there exists $f\in L_\mu([a,b],C[0,1]\times C[0,1])$ such that

$$\int_a^b\|j\circ x_n(t)-f(t)\|d\mu\to 0. \tag{3.2}$$

Further there exists a subsequence $\{j \circ x_{n_k}\}$ such that $j \circ x_{n_k} \to f$ a. e. in $C[0,1] \times C[0,1]$. By Lemma 3.1, there exists $x : [a,b] \to CE^1$ such that $f = j \circ x$. As f is Bochner integrable, by Lemma 3.2, $x \in L([a,b], CE^1)$. Finally by (3.2), we have

$$d(x_n, x) = \int_a^b \|j \circ x_n(t) - f(t)\| d\mu \to 0. \ \square$$

For simplicity, denote the complete metric space formed by $L([a,b], CE^1)$ with respect to the metric d by $L_\mu([a,b], CE^1)$.

Lemma 3.4 *Let $x : [a,b] \to CE^1$ be integrably bounded. Denote $y(t) = \int_a^t x(s)ds$. Then $y : [a,b] \to CE^1$ is absolutely continuous and differentiable a. e. on $[a,b]$, and $y'(t) = x(t)$ a. e..*

Proof For $t_1, t_2 \in [a,b], t_1 < t_2$, $y(t_2) = y(t_1) + \int_{t_1}^{t_2} x(s)ds$, then

$$\begin{aligned} D(y(t_2), y(t_1)) &= D(y(t_1) + \textstyle\int_{t_1}^{t_2} x(s)ds, y(t_1)) \\ &= D(\textstyle\int_{t_1}^{t_2} x(s)ds, \theta) \\ &= \textstyle\int_{t_1}^{t_2} D(x(s), \theta)ds. \end{aligned}$$

By the Lebesgue integrability of $D(x(s), \theta)$, y is absolutely continuous on $[a,b]$. For $h > 0$,

$$\begin{aligned} D(\tfrac{y(t+h)-y(t)}{h}, x(t)) &= D(\tfrac{1}{h}\textstyle\int_t^{t+h} x(s)ds, x(t)) \\ &= D(\tfrac{1}{h}\textstyle\int_t^{t+h} x(s)ds, \tfrac{1}{h}\int_t^{t+h} x(t)ds) \\ &\le \tfrac{1}{h}\textstyle\int_t^{t+h} D(x(s), x(t))ds \\ &= \tfrac{1}{h}\textstyle\int_t^{t+h} \|j \circ x(s) - j \circ x(t)\| ds. \end{aligned}$$

As $j \circ x$ is Bochner integrable, by [2, P. 49, Theorem 9] we get

$$\lim_{h \to 0^+} \frac{1}{h} \int_t^{t+h} \|j \circ x(s) - j \circ x(t)\| ds \to 0 \text{ a. e..}$$

Therefore $\lim\limits_{h \to 0^+} D(\frac{y(t+h)-y(t)}{h}, x(t)) = 0$ a. e.. Similarly we can obtain $\lim\limits_{h \to 0^+} D(\frac{y(t)-y(t-h)}{h}, x(t)) = 0$. Thus y is differentiable a. e. on $[a,b]$ and $y'(t) = x(t)$ a. e.. $\square$

Remark 3.2 *In the Proof of Lemma 3.4, $\frac{1}{h}\int_t^{t+h} x(t)ds = x(t)$. See the result of Example 4.1 in [6] for it.*

Theorem 3.1 *If $F(t,u)$ satisfies conditions (A)-(C), then there exists unique C-solution to (3.1).*

Proof Existence. For $x \in L_\mu([a,b], CE^1)$ define a mapping by $(Tx)(t) = F(t, x_0 + \int_a^t x(s)ds)$. Denote $y(t) = x_0 + \int_a^t x(s)ds$, then $y : [a,b] \to CE^1$ is continuous and there exists a sequence of simple functions $\{y_n\}$ such that $y_n \to y$ uniformly. Here $y_0 = y$, $y_n = T^n(y_0) = T(T^{n-1}(y_0))$ for $n = 1, 2, \cdots$. According to condition (B), $F(t, y(t)) = \lim\limits_{n \to \infty} F(t, y_n(t))$. Further by condition (A) and Lemma 3.2, $F(\cdot, y(\cdot))$ is measurable. In view of condition (C),

$$\begin{aligned} D(F(t, y(t)), \theta) &\le \alpha(t) + \beta(t) D(y(t), \theta) \\ &\le \alpha(t) + \beta(t) M \end{aligned}$$

where $M = \max_{t\in[a,b]} D(y(t),\theta)$. Then $F(\cdot, y(\cdot))$ is integrably bounded and T is a mapping from $L_\mu([a,b], CE^1)$ to $L_\mu([a,b], CE^1)$. Next we prove that T is a contraction mapping. In fact, for $x_1, x_2 \in L_\mu([a,b], CE^1)$, by condition (B) we have

$$\begin{aligned} d(Tx_1, Tx_2) &= \int_a^b D((Tx_1)(t), (Tx_2)(t))d\mu \\ &= \int_a^b D(F(t, x_0 + \int_a^t x_1(s)ds), F(t, x_0 + \int_a^t x_2(s)ds))d\mu \\ &\le \int_a^b m(t) D(\int_a^t x_1(s)ds, \int_a^t x_2(s)ds)d\mu \\ &\le \int_a^b m(t)(\int_a^t D(x_1(s), x_2(s))ds)d\mu \\ &= -\tfrac{1}{2}\int_a^b (\int_a^t D(x_1(s), x_2(s))ds)de^{-2h(t)}. \end{aligned}$$

Integrating by parts, we get

$$\begin{aligned} & -\tfrac{1}{2}\int_a^b (\int_a^t D(x_1(s), x_2(s))ds)de^{-2h(t)} \\ = & \ (-\tfrac{1}{2}\int_a^t D(x_1(s), x_2(s))ds)e^{-2h(t)}|_a^b + \tfrac{1}{2}\int_a^b D(x_1(t), x_2(t))e^{-2h(t)}dt \\ = & -\tfrac{1}{2}e^{-2h(b)}\int_a^b D(x_1(s), x_2(s))ds + \tfrac{1}{2}d(x_1, x_2) \\ \le & \ \tfrac{1}{2}d(x_1, x_2). \end{aligned}$$

By Lemma 3.3 and the contraction principle, there exists $x_*(t) \in L_\mu([a,b], CE^1)$ such that $Tx_* = x_*$, or

$$x_* = F(t, x_0 + \int_a^t x_*(s)ds) \text{ a. e.} \tag{3.3}$$

Denote $y_*(t) = x_0 + \int_a^t x_*(s)ds$, then $y_*(a) = x_0$. By (3.3) and Lemma 3.4, y_* is a solution to (3.1).

Uniqueness. Let y_1 be a solution of (3.1), then y_1' is integrably bounded. Denote $z(t) = x_0 + \int_a^t y_1'(s)ds$. By Lemma 3.2, for each $r \in [0,1]$ we have

$$[x_0 + \int_a^t y_1'(s)ds]^r = [\underline{x_0}(r) + (L)\int_a^t \underline{y_1'(s)}(r)ds, \overline{x_0}(r) + (L)\int_a^t \overline{y_1'(s)}(r)ds]. \tag{3.4}$$

By the properties of Lebesgue integrals and $y_1(a) = x_0$, we know

$$\begin{aligned} \underline{x_0}(r) + (L)\int_a^t \underline{y_1'(s)}(r)ds &= \underline{y_1(t)}(r), \\ \overline{x_0}(r) + (L)\int_a^t \overline{y_1'(s)}(r)ds &= \overline{y_1(t)}(r). \end{aligned} \tag{3.5}$$

From (3.4) and (3.5), it is immediate that $[y_1(t)]^r = [z(t)]^r$, or $y_1(t) = z(t), t \in [a,b]$. Therefore $(Ty_1')(t) = F(t, y_1(t))$, that is to say, y_1' is a fixed point of T. By the uniqueness of fixed point of T, $y_1'(t) = x_*(t)$ a. e., or $y(t) = y_*(t)$ $(t \in [a,b])$. $\square$

Theorem 3.2 *If F satisfies conditions (A)-(C), then corresponding to any initial values $x_0, y_0 \in CE^1$ respectively, the solutions $y(\cdot, x_0), y(\cdot, y_0)$ of (3.1) satisfy*

$$D(y(t, x_0), y(t, y_0)) \le e^{2h(b)} D(x_0, y_0)$$

for each $t \in [a,b]$.

Proof Let $x_0, y_0 \in L_\mu([a,b], CE^1)$. Define two mappings T_{x_0}, T_{y_0} as the following:

$$\begin{aligned} (T_{x_0}u)(t) &= F(t, x_0 + \int_a^t u(s)ds), \\ (T_{y_0}u)(t) &= F(t, y_0 + \int_a^t u(s)ds). \end{aligned}$$

According to the Proof of Theorem 3.1, T_{x_0}, T_{y_0} are contraction mappings and have fixed points x_*, y_* respectively. Then

$$\begin{array}{rcl} x_*(t) & = & F(t, x_0 + \int_a^t x_*(s)ds), \\ y_*(t) & = & F(t, y_0 + \int_a^t y_*(s)ds). \end{array}$$

As

$$\begin{array}{rl} & d(x_*, y_*) \\ = & \int_a^b D(F(t, x_0 + \int_a^t x_*(s)ds), F(t, y_0 + \int_a^t y_*(s)ds))d\mu \\ \leq & \int_a^b m(t)D(x_0 + \int_a^t x_*(s)ds, y_0 + \int_a^t y_*(s)ds)d\mu \\ \leq & \int_a^b m(t)[D(x_0, y_0) + D(\int_a^t x_*(s)ds, \int_a^t x_*(s)ds)]d\mu \\ \leq & (\int_a^b m(t)e^{-2h(t)}dt)D(x_0, y_0) + \int_a^b m(t)(\int_a^t D(x_*(s), y_*(s))ds)e^{-2h(t)}dt, \end{array}$$

integrating by parts we have

$$\begin{array}{rl} & \int_a^b m(t)(\int_a^t D(x_*(s), y_*(s))ds)e^{-2h(t)}dt \\ = & -\frac{1}{2}e^{-2h(b)}\int_a^b D(x_*(t), y_*(t))dt + \frac{1}{2}\int_a^b D(x_*(t), y_*(t))e^{-2h(t)}dt \\ \leq & \frac{1}{2}d(x_*, y_*). \end{array}$$

Therefore

$$d(x_*, y_*) \leq \frac{1}{2}(1 - e^{-2h(b)})D(x_0, y_0) + \frac{1}{2}d(x_*, y_*),$$

or

$$d(x_*, y_*) \leq (1 - e^{-2h(b)})D(x_0, y_0).$$

Now the two solutions of (3.1) corresponding to initial values x_0, y_0 can be respectively written as

$$\begin{array}{rcl} y(t, x_0) & = & x_0 + \int_a^t x_*(s)ds, \\ y(t, y_0) & = & y_0 + \int_a^t y_*(s)ds. \end{array}$$

So for $t \in [a, b]$, we get, by (c) in Section 2,

$$\begin{array}{rl} & D(y(t, x_0), y(t, y_0)) \\ = & D(x_0 + \int_a^t x_*(s)ds, y_0 + \int_a^t y_*(s)ds) \\ \leq & D(x_0, y_0) + \int_a^t D(x_*(s), y_*(s))ds \\ \leq & D(x_0, y_0) + e^{2h(b)}\int_a^b D(x_*(s), y_*(s))e^{-2h(s)}ds \\ = & D(x_0, y_0) + e^{2h(b)}d(x_*, y_*) \\ \leq & D(x_0, y_0) + e^{2h(b)}(1 - e^{-2h(b)})D(x_0, y_0) \\ = & e^{2h(b)}D(x_0, y_0).\square \end{array}$$

Theorem 3.3 *Suppose that $\{F_n(t, u)\}$ and $F(t, u)$ satisfy the following conditions:*

(A') *For each $u \in CE^1$, $F_n(t, u), F(t, u)$ are measurable with respect to t;*

(B') *There exists integrable function $m \in L([a, b], \mathbf{R})$ such that for all $u, v \in CE^1$ and $G \in \{F_n\} \bigcup \{F\}$,*

$$D(G(t, u), G(t, v)) \leq m(t)D(u, v);$$

(C') *There exist integrable functions $\alpha, \beta \in L([a, b], \mathbf{R})$ such that for $G \in \{F_n\} \bigcup \{F\}$,*

$$D(G(t, u), \theta) \leq \alpha(t) + \beta(t)D(u, \theta);$$

(D') *For* $(t,u) \in [a,b] \times CE^1$,

$$\lim_{n\to\infty} D(F_n(t,u), F(t,u)) = 0.$$

Then the solutions y_n *to*

$$\left\{ \begin{array}{l} \dot{y}(t) = F_n(t, y(t)); \\ y(a) = x_0 \end{array} \right. \tag{3.6}$$

and the solution y *to*

$$\left\{ \begin{array}{l} \dot{y} = F(t, y(t)); \\ y(a) = x_0 \end{array} \right. \tag{3.7}$$

satisfy

$$\lim_{n\to\infty} D(y_n(t), y(t)) = 0$$

uniformly for $t \in [a,b]$.

Proof On $L_\mu([a,b], CE^1)$ define the mappings T_n, T as the following:

$$\begin{array}{rcl} (T_n x)(t) & = & F_n(t, x_0 + \int_a^t x(s)ds), \\ (Tx)(t) & = & F(t, x_0 + \int_a^t x(s)ds). \end{array}$$

Denote the fixed points of T_n, T by x_n, x respectively. Then

$$\begin{array}{rcl} (T_n x_n)(t) & = & F_n(t, x_0 + \int_a^t x_n(s)ds), \\ (Tx)(t) & = & F(t, x_0 + \int_a^t x(s)ds). \end{array}$$

As

$$\begin{array}{rl} d(x_n, x) & = d(T_n x_n, Tx) \\ & \leq d(T_n x_n, T_n x) + d(T_n x, Tx) \\ & \leq \frac{1}{2} d(x_n, x) + d(T_n x, Tx), \end{array}$$

we have

$$d(x_n, x) \leq 2d(T_n x, Tx). \tag{3.8}$$

Next we consider

$$d(T_n x, Tx) = \int_a^b D(F_n(t, x_0 + \int_a^t x(s)ds), F(t, x_0 + \int_a^t x(s)ds)) d\mu. \tag{3.9}$$

Denote $y(t) = x_0 + \int_a^t x(s)ds$. By condition (C'),we get

$$\begin{array}{rl} & D(F_n(t, y(t)), F(t, y(t))) \\ \leq & D(F_n(t, y(t)), \theta) + D(F(t, y(t)), \theta) \\ \leq & 2\alpha(t) + 2\beta(t) D(y(t), \theta) \\ \leq & 2\alpha(t) + 2\beta(t) M \end{array}$$

where $M = \max\limits_{t\in[a,b]} D(y(t), \theta)$. From condition (D') we know

$$\lim_{n\to\infty} D(F_n(t, y(t)), F(t, y(t))) = 0,$$

further by Dominated Convergence Theorem we have

$$\lim_{n\to\infty}\int_a^b D(F_n(t,y(t)),F(t,y(t)))d\mu = 0.$$

In view of (3.9) we conclude

$$\lim_{n\to\infty} d(T_n x, Tx) = 0.$$

Denote $y_n(t) = x_0 + \int_a^t x_n(s)ds$. By (3.8) we obtain

$$\begin{aligned}
& D(y_n(t), y(t)) \\
= \;& D(x_0 + \textstyle\int_a^t x_n(s)ds, x_0 + \int_a^t x(s)ds) \\
\leq \;& D(\textstyle\int_a^t x_n(s)ds, \int_a^t x(s)ds) \\
\leq \;& \textstyle\int_a^t D(x_n(s), x(s))ds \\
\leq \;& e^{2h(b)} \textstyle\int_a^b D(x_n(s), x(s))d\mu \\
= \;& e^{2h(b)} d(x_n, x) \\
\leq \;& 2e^{2h(b)} d(T_n x, Tx).
\end{aligned}$$

Therefore

$$\lim_{n\to\infty} D(y_n(t), y(t)) = 0$$

uniformly with respect to $t \in [a,b]$. $\square$

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