CONVEX STOCHASTIC GAMES OF CAPITAL ACCUMULATION WITH NONDIVISIBLE MONEY UNIT

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Abstract. We consider a nonsymmetric infinite-horizon discounted stochastic game of capital accumulation with discrete state and action spaces. We show that, under strong convexity condition on transition law cumulative distribution and with bounded one-period consumption capacities, the game has an equilibrium. The optimal strategies have Lipschitz property and are nondecreasing. Moreover, in every state they are concentrated in at most two adjoining points of players' action spaces.

1 Introduction The game theory (and the theory of dynamic/stochastic games in particular) provides us with a possibility of modelling different kinds of economic interaction. Two of them: capital accumulation and resource extraction are traditionally modelled in the same setting. The resource extraction game was introduced by Levhari and Mirman [5]. Existence of a stationary equilibrum in deterministic version of this class of games was established by Sundaram [9]. His result was extended to the stochastic case by Majumdar and Sundaram [6] and Dutta and Sundaram [3]. All of them considered models where the symmetry of the players was assumed.

Further extension of their works to the nonsymmetric case was given by Amir [1]. This generalization was achieved in expense of some additional structural assumptions (continuity and convexity of law of motion between states, bounded spaces of players' actions). However, this enabled the author to show some important features of stationary equilibrium strategies, such as continuity, monotonicity and Lipschitz property.

All of the above papers treated the game with state and action spaces being intervals (not necessarily of finite length) of the real line. The main objective of our paper is to present a model of stochastic game of capital accumulation similar to that of Amir's, but with countable state space. Such reformulation of the model is motivated by the fact, that in real-life economies there always exists some nondivisible unit of money, and therefore the players on the market can't use all of the strategies available in continuous models. From this point of view, continuous model can be seen as too strong simplification, at least in some cases. In our paper we discuss a fully discrete model, where the state and action spaces are countable (represented by natural numbers) and investigate the impact that the lack of continuity assumption has on players' strategies. The result is, that optimal stationary strategies of the players preserve most of the desired properties that were established in the Amir's paper, such as monotonicity. Moreover, they remain "almost pure", i.e. in every state they are concentrated in at most two neighbouring points of the players' action spaces. In fact, it appears that this result doesn't require assumptions as strong as in model of Amir. In

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our paper we have weakened the assumptions about strict monotonicity and strict concavity of players' utility functions and transition probability distribution function. Many of the techniques used by Amir didn't work effectively for our discrete model. Therefore we had to introduce a lot of new ones. However, the general scheme of the proof from [1] was sustained, along with a couple of lemmata.

The organization of this paper is as follows. In section 2 we present the assumptions of the model along with the main theorem, while section 3 contains its proof.

2 The model and the main result The game-model we discuss is the following: Two players jointly own a productive asset characterized by a stochastic input-output technology. At each of infinitely many periods of the game, they decide independently and simultaneously, what part of the available stock should be utilized for consumption and what part for investment. The objective of each player is the maximization of the discounted sum of utilities from his own consumption over infinite horizon. The players have different utility functions, discount factors and one-period consumption capacities.

The model is described in the form of a nonzero-sum two-person stochastic game G by five items below.

1. The game is played at discrete moments $t = 0, 1, \dots$
2. The state space for the game is the set of all natural numbers, $S = \mathrm{N} = \{0, 1, 2, \dots\}$. The state at moment t, interpreted as current available stock, will be denoted by x_t.
3. The sets of actions available to players 1 and 2 in state $x \in \mathrm{N}$ are $\{0, 1, \dots, K_1(x)\}$ and $\{0, 1, \dots, K_2(x)\}$ respectively, where $K_i(x)$ is player i's one-period consumption capacity, as a function of available stock x.
4. Player i's payoff is given by
$$E \sum_{t=0}^{\infty} \beta_i^t u_i(c_t^i),$$
where c_t^i is his action in period t, u_i his utility function and $\beta_i \in [0, 1)$ his discount factor. The expectation here is taken over the induced probability measure on all histories, described below.
5. The transition law is described by
$$x_{t+1} \sim q(\cdot \mid x_t - c_t^1 - c_t^2),$$
where q is a conditional probability distribution given current joint investment $x_t - c_t^1 - c_t^2$.

A general *strategy* for player 1 in game G is a sequence $\pi = (\pi_1, \pi_2, \dots)$, where π_n is a conditional probability $\pi_n(\cdot \mid h_n)$ on the set $A^1 = \bigcup_{x \in \mathrm{N}} \{0, \dots, K_1(x)\}$ of his possible actions, depending on all the histories of the game up to its n-th stage $h_n = (x_1, c_1^1, c_1^2, \dots, x_{n-1}, c_{n-1}^1, c_{n-1}^2, x_n)$, such that $\pi_n(\{0, \dots, K_1(x_n)\} \mid h_n) = 1$. The class of all strategies for player 1 is denoted by Π^1.

Let F^1 be the set of all transition probabilities $f : \mathrm{N} \to P(A^1)$ such that $f(x)(\cdot) \in P(\{0, \dots, K_1(x)\})$ for each $x \in \mathrm{N}$. (Here and in the sequel $P(S)$ denotes the set of all probability measures on S). Then a strategy of the form $\pi = (f, f, \dots)$, where $f \in F^1$ will be called *stationary* and identified with f. We will interpret f as a strategy for player 1 that prescribes him to take, at any moment t, action c_t^1 being a realization of $f(x)$, provided x is a state at that moment. Similarly, we define the set Π^2 (F^2) of all strategies (stationary strategies) for player 2. A strategy $\pi = (\pi_1, \pi_2, \dots)$ is called *pure* if each conditional probability $\pi_n(\cdot \mid h_n)$ is concentrated at exactely one point.

Let $H = \mathrm{N} \times A^1 \times A^2 \times \mathrm{N} \times \cdots$ be the space of all infinite histories of the game. For every initial state $x_0 = x \in \mathrm{N}$ and all strategies $\pi \in \Pi^1$ and $\gamma \in \Pi^2$ we define, with the help of Ionescu-Tulcea's theorem(Proposition V.1.1 in [7]) the unique probability measure $P_x^{\pi\gamma}$ defined on subsets of H consisting of histories starting at x. Then, for each initial state $x \in \mathrm{N}$, any strategies $\pi \in \Pi^1$ and $\gamma \in \Pi^2$ and the discount factor $\beta_i \in (0,1)$ the *expected discounted reward* for the player i is

$$J^i(x, \pi, \gamma) = E_x^{\pi\gamma}\left[\sum_{t=0}^{\infty} \beta^t u_i(c_t^i)\right].$$

A pair of (stationary) strategies (f^1, f^2) is called the *(stationary) Nash equilibrium* for the discounted stochastic game, iff for every $\pi \in \Pi^1$, $\gamma \in \Pi^2$ and $x \in \mathrm{N}$ we have:

$$J^1(x, f^1, f^2) \geq J^1(x, \pi, f^2) \quad \text{and} \quad J^2(x, f^1, f^2) \geq J^2(x, f^1, \gamma).$$

The functions $V_{f^2}^1$ and $V_{f^1}^2$ are called the players' *value functions* for optimally responding to f_2 and f_1 respectively (sometimes we will call them simply "value functions corresponding to (f_1, f_2)".)

Before listing the necessary assumptions we will need to introduce one more definition. Because our model of game G can be seen as a discrete counterpart of the Amir's one, with the state space $[0, \infty)$ replaced by its discrete counterpart N, it seems very natural to "restrict" his assumptions to the set of natural numbers N here. Consequently, for the notion of concavity and convexity on $[0, \infty)$ used there, we propose this fairly natural version.

Definition 2.1 A function $W : N \to R$ is said to be convex (concave) if there is a convex (concave) function $\overline{W} : [0, \infty) \to R$ such that $W(n) = \overline{W}(n)$ for all $n \in N$.

It is immediately seen that the above definition can be equivalently rewritten in the following form:
A function $W : N \to R$ is convex if and only if for all $i \in \mathrm{N}$

$$W(i+1) - W(i) \leq W(i+2) - W(i+1), \tag{1}$$

and for the concavity we have the reverse inequality.

The restriction of Amir's assumptions for our discrete model leads to the following list of conditions that will be assumed for the game G throughout whole of this paper.

(A1) $u_i : \mathrm{N} \to [0, \infty)$ is nondecreasing concave function, $i = 1, 2$.

(A2) q is a transition probability from N to itself. Let $F(\cdot \mid y)$ denote the cumulative distribution function associated with $q(\cdot \mid y)$ by the formula $F(x \mid y) = \sum_{i \leq x} q(i \mid y)$ for $x, y \in \mathrm{N}$. We assume that:

(a) For each $x \in \mathrm{N}$, $F(x \mid \cdot)$ is a nonincreasing function (F is first-order stochastically increasing in y.)

(b) For each $x \in \mathrm{N}$, $F(x \mid \cdot)$ is a convex function.

(c) $F(0 \mid 0) = 1$.

(A3) For $i = 1, 2$ the function $K_i(\cdot)$ is nondecreasing, uniformly bounded above by some constant $C_i \in \mathrm{N}$, and satisfies $K_i(0) = 0$,

$$\frac{K_i(x_1) - K_i(x_2)}{x_1 - x_2} \leq 1, \quad \forall x_1, x_2 \in \mathrm{N}, x_1 \neq x_2$$

and $K_1(x) + K_2(x) \leq x$ for all $x \in \mathrm{N}$.

To express our main result about game G we must introduce two next definitions. The effective strategy space for player i, as it will be shortly seen, is the space of two-adjoining-point strategies, satisfying Lipschitz property and nondecreasing in their expected value:

$$\begin{aligned} LTM_i \quad = \quad & \Big\{ f : \mathrm{N} \to P(\{0, 1, \ldots, C_i\}) : \text{ for all } x \in \mathrm{N} \\ & f(x) = \alpha_x \delta[a_x] + (1 - \alpha_x)\delta[a_x + 1] \text{ for some } 0 \leq \alpha_x \leq 1 \\ & \text{and } a_x \in \mathrm{N},\ 0 \leq a_x < K_i(x), \\ & \text{and } 0 \leq \frac{E(\widetilde{f}(x_1)) - E(\widetilde{f}(x_2))}{x_1 - x_2} \leq 1 \text{ for all distinct } x_1, x_2 \in \mathrm{N} \Big\}. \end{aligned}$$

Here and in the sequel, $\delta[a]$ denotes the probability measure with total mass concentrated in point a, while for $x \in \mathrm{N}$ and for all f, $\widetilde{f}(x)$ means a random variable with distribution described by $f(x)$.

The corresponding space of value functions in the game G for player i using strategies in LTM_i will be:

$$\begin{aligned} M_i \quad = \quad & \Big\{ v : \mathrm{N} \to [0, \infty) \text{ such that } 0 \leq v \leq \frac{u_i(C_i)}{1 - \beta_i} \\ & \text{and } v \text{ is nondecreasing} \Big\}. \end{aligned}$$

These two spaces, LTM_i and M_i for $i = 1, 2$, can be clearly viewed as counterparts of effective strategy space LCM_i and value-function space CM_i considered in Amir's work.

Now we are ready to formulate our main result.

Theorem 2.1 *The game G has a stationary equilibrium which is an element of $LTM_1 \times LTM_2$. Furthermore, the corresponding value functions $(V_1, V_2) \in M_1 \times M_2$.*

3 Proof of Theorem 2.1 Our proof contains only some elements (Lemma 3.1 and partially Lemma 3.4) taken from Amir's one. In prevailing part, it essentially modifies those ideas or is based on quite different constructions. The proof is rather complex, hence ten lemmata will be needed.

We begin with three technical ones which will be used in different parts of our analysis.

Lemma 3.1 *Let F_1 and F_2 be probability distributions over N, with its Borel subsets. Then F_2 first order stochastically dominates F_1, or $F_1(x) \geq F_2(x)$ for all $x \in N$ if and only if $\int v\, dF_1 \leq \int v\, dF_2$ for all real-valued nondecreasing functions v on N.*

Proof: The result is well known. See e.g. Stoyan [8] (Theorem 1.2.2 p. 5). ■

Lemma 3.2 *Let w and w_n $(n = 1, 2, \dots)$ be measurable real-valued functions defined on a metric space A and let D be the set of all $x \in A$ such that $w_n(x_n) \not\to w(x)$ for some sequence $x_n \to x$ in A. If a sequence $\{\mu_n\}$ of measures on A converges weakly to μ then*

$$\lim_{n\to\infty} \int w_n\, d\mu_n = \int w\, d\mu$$

if only $\mu(D) = 0$.

Proof: A more general version of this theorem can be found in Billingsley [2] (Theorem 5.5). ■

Lemma 3.3 *Assume that a function $\phi : N \to R$ is nondecreasing. Then for $x_1 < x_2$, $x_1, x_2 \in N$ and $h \in LTM_2$*

$$E\phi(\widetilde{h}(x_1)) \leq E\phi(\widetilde{h}(x_2)) \tag{2}$$

and

$$E\phi(x_1 - \widetilde{h}(x_1)) \leq E\phi(x_2 - \widetilde{h}(x_2)). \tag{3}$$

Proof: Assume that $h(x_1) = p_1\delta[y_1] + (1 - p_1)\delta[y_1 + 1]$ and $h(x_2) = p_2\delta[y_2] + (1 - p_2)\delta[y_2 + 1]$, where $y_i \in \mathrm{N}, p_i \in [0, 1)$. Since $h \in LTM_2$,

$$y_2 + 1 - p_2 = E(\widetilde{h}(x_2)) \geq E(\widetilde{h}(x_1)) = y_1 + 1 - p_1.$$

This implies that either $y_2 > y_1$ or $y_2 = y_1$ and $p_1 \geq p_2$.

In both cases $h(x_2)$ stochastically dominates $h(x_1)$, and thus, by Theorem 2.2.2a in [8] also $\phi(h(x_2))$ stochastically dominates $\phi(h(x_1))$. Using Lemma 3.1 with $v = \phi$ and F_i, $i = 1, 2$ being the cumulative distribution function of $h(x_i)$ we obtain (2).

To verify the second inequality (3), it is enough to notice that the relation $h \in LTM_2$ implies

$$E\left[x_2 - \widetilde{h}(x_2)\right] \geq E\left[x_1 - \widetilde{h}(x_1)\right].$$

Now we can use the same argument as in the first part of the proof to prove (3).
■

The first important step of the proof of Theorem 2.1 will be made in the next lemma. Before we express it, we must define the associated optimization problem.

Suppose that player 2 uses a stationary strategy $h \in LTM_2$. Then player 1 faces the problem of finding a sequence of his best choices $\{c_t^1\}_{t=0}^{\infty}$ (as realizations of some strategy

$\pi = (\pi_1, \pi_2, \dots))$, and the value function V_h^1 for optimally responding to h, satisfying for all initial states x

$$V_h^1(x) = \sup E \sum_{t=0}^{\infty} \beta_1^t u_1(c_t^1)$$
$$\text{where } x_{t+1} \sim q(\cdot \mid x_t - c_t^1 - h(x_t)), \quad t = 0, 1, \dots \text{ and } x_0 = x$$
$$\text{with } c_t^1 \in \{0, \dots, K_1(x_t)\},$$

where "sup" runs over all strategies π of player 1, and the expectation is over the unique probability measure induced by x, h and π.

Lemma 3.4 *Assume that $h \in LTM_2$. Then V_h^1 is the unique solution of the functional equation*

$$V_h^1(x) = \max_{c \in \{0, \dots, K_1(x)\}} \left[u_1(c) + \beta_1 \int V_h^1(x')\, dF(x' \mid x - c - \widetilde{h}(x)) \right] \tag{4}$$

and $V_h^1 \in M_1$.

Proof: Let us fix $h \in LTM_2$ and define the map $T : M_1 \to M_1$ by

$$T(v)(x) = \sup_{c \in \{0, \dots, K_1(x)\}} \left[u_1(c) + \beta_1 \int v(x')\, dF(x' \mid x - c - \widetilde{h}(x)) \right]. \tag{5}$$

We start by showing that T indeed maps M_1 into itself. First, note that the inequality $v \leq \frac{u_1(C_1)}{1-\beta_1}$ implies

$$T(v) \leq u_1(C_1) + \beta_1 \frac{u_1(C_1)}{1-\beta_1} = \frac{u_1(C_1)}{1-\beta_1}.$$

Therefore, to show $T(v) \in M_1$, it is enough to show that $T(v)$ is nondecreasing. Define the function

$$\phi(y) = u_1(c) + \beta_1 \int v(x') dF(x' \mid y - c), \quad y \geq c,$$

where c is a natural parameter. Notice that by Lemma 3.1 and (a) of Assumption (A2) ϕ is nondecreasing.

Now fix two natural $x_1 < x_2$ and let $a \leq K_1(x_1)$. With the help of (3) we can deduce as follows:

$$\begin{aligned}
&E\left[u_1(c) + \beta_1 \int v(x') dF(x' \mid x_1 - c - \widetilde{h}(x_1)) \right] && (6)\\
=\;& E\phi(x_1 - \widetilde{h}(x_1)) \leq E\phi(x_2 - \widetilde{h}(x_2)) &&\\
=\;& E\left[u_1(c) + \beta_1 \int v(x') dF(x' \mid x_2 - c - \widetilde{h}(x_2)) \right] && (7)
\end{aligned}$$

Since $T(v)(x_1)$ is the sup of (6) over $c \in \{0, \dots, K_1(x_1)\}$ and $T(v)(x_2)$ of (7) over $c \in \{0, \dots, K_1(x_2)\}$, and $\{0, \dots, K_1(x_1)\} \subset \{0, \dots, K_1(x_2)\}$ by (A3), we get $T(v)(x_1) \leq T(v)(x_2)$. Thus T maps M_1 into itself.

Now observe, that M_1 endowed with uniform distance is a closed subset of Banach space of all bounded functions from N to $[0,+\infty)$, and thereby a complete metric space. Standard dynamic programming arguments show that T is a contraction with unique fixed-point $V_h^1 \in M_1$ which thus satisfies (4). ■

The next lemma considers properties of the following auxiliary function of natural variable c,

$$\begin{aligned}\Psi_1^{xh}(c) &= E\Big\{[u_1(c)-u_1(c-1)]+\beta_1\Big[\int V_h^1(x')\,dF(x' \mid x-c-\widetilde{h}(x)) \\ &- \int V_h^1(x')\,dF(x' \mid x-c+1-\widetilde{h}(x))\Big]\Big\}.\end{aligned}$$

Lemma 3.5 *Let $h \in LTM_2$. Then for natural $x_1 < x_2$ and $0 < c \le K_1(x_1)$*

(i) $\Psi_1^{x_1h}(c) \le \Psi_1^{x_2h}(c)$

(ii) $\Psi_1^{x_1h}(c) \ge \Psi_1^{x_2h}(c+x_2-x_1)$

(iii) $\Psi_1^{xh}(c)$ *is nonincreasing in c for all $x \in N$.*

Proof: *(i)* Let $y', y_1, y_2 \in$ N and $c \le y_1 < y_2$. By (b) of Assumption (A2) and (1) we easily get

$$\frac{F(y' \mid y_2-c+1)+F(y' \mid y_1-c)}{2} \ge \frac{F(y' \mid y_1-c+1)+F(y' \mid y_2-c)}{2}.$$

Note that both sides of this inequality are probability distributions, whence by Lemma 3.1,

$$\begin{aligned}&\int V_h^1(y')\,d\left[\frac{F(y' \mid y_2-c+1)+F(y' \mid y_1-c)}{2}\right] \\ \le\;& \int V_h^1(y')\,d\left[\frac{F(y' \mid y_1-c+1)+F(y' \mid y_2-c)}{2}\right].\end{aligned}$$

But this can be rewritten as

$$\begin{aligned}&\int V_h^1(y')\,dF(y' \mid y_1-c)-\int V_h^1(y')\,dF(y' \mid y_1-c+1) \\ \le\;& \int V_h^1(y')\,dF(y' \mid y_2-c)-\int V_h^1(y')\,dF(y' \mid y_2-c+1).\end{aligned} \tag{8}$$

Hence, the function

$$\phi_1(y)=\int V_h^1(y')\,dF(y' \mid y-c)-\int V_h^1(y')\,dF(y' \mid y-c+1) \tag{9}$$

is nondecreasing. Using the second part of Lemma 3.3 we can conclude as follows:

$$\begin{aligned}&E\left[\int V_h^1(x')\,dF(x' \mid x_1-c-\widetilde{h}(x_1))-\int V_h^1(x')\,dF(x' \mid x_1-c+1-\widetilde{h}(x_1))\right] \\ =\;& E\phi_1(x_1-\widetilde{h}(x_1)) \le E\phi_1(x_2-\widetilde{h}(x_2)) \qquad (10) \\ =\;& E\left[\int V_h^1(x')\,dF(x' \mid x_2-c-\widetilde{h}(x_2))-\int V_h^1(x')\,dF(x' \mid x_2-c+1-\widetilde{h}(x_2))\right].\end{aligned}$$

If we multiply both sides of (10) by β_1 and add $u_1(c) - u_1(c-1)$ we obtain the desired inequality.

(ii) A simple analysis of inequality (8) with y_1 and y_2 replaced by $x_1 - y_1$ and $x_1 - y_2$ respectively shows that the function

$$\phi_2(y) = \int V_h^1(y')\, dF(y' \mid x_1 - c + 1 - y) - \int V_h^1(y')\, dF(y' \mid x_1 - c - y)$$

is nondecreasing ($y \leq x_1 - c$), whence, by the first part of Lemma 3.3 we obtain

$$\begin{aligned} & E\left[\int V_h^1(x')\, dF(x' \mid x_1 - c - \widetilde{h}(x_1)) - \int V_h^1(x')\, dF(x' \mid x_1 - c + 1 - \widetilde{h}(x_1))\right] \\ (11) \quad = \;& -E\phi_2(\widetilde{h}(x_1)) \geq -E\phi_2(\widetilde{h}(x_2)) \\ = \;& E\left[\int V_h^1(x')\, dF(x' \mid x_1 - c - \widetilde{h}(x_2)) - \int V_h^1(x')\, dF(x' \mid x_1 - c + 1 - \widetilde{h}(x_2))\right]. \end{aligned}$$

On the other hand, Assumption (A1) implies that

$$u_1(c) - u_1(c-1) \geq u_1(c + x_2 - x_1) - u_1(c + x_2 - x_1 - 1) \tag{12}$$

Adding LHS of inequality (12) to LHS of (11) multiplied by β_1 and RHS of inequality (12) to RHS of (11) multiplied by β_1, we obtain inequality *(ii)*.

(iii) The statement is an easy consequence of (12) and the fact that the function $\phi_1(y)$ of form (9) is nondecreasing in y. ■

The *best response* of player 1 to $h \in LTM_2$ is defined as any argmax of (4) for $x \in \mathrm{N}$. In the remaining part of this section we shall formally write it down as "the best response multifunction" denoted $c_1(h)(x)$, which attaches to x all of the best responses of player 1 to h in x.

Remark 3.1 As one can easily see, Ψ_1^{xh} is a discrete counterpart of derivative of the maximand in (4), and so an analysis of its behaviour gives us a simple method for finding $c_1(h)$. From this point of view, the fact that Ψ_1^{xh} is nonincreasing in c (Lemma 3.5) means that values of the best-response multifunction $c_1(h)(\cdot)$ for player 1 $c_1(h)(\cdot)$ have always form $[y_1, y_2] \cap \mathrm{N}$, for some $y_1, y_2 \in \mathrm{N}$. The next lemma gives some further characteristics of $c_1(h)$ found in this way.

Lemma 3.6 *If $h \in LTM_2$ then minimum $c_{min}(x) = \min(c_1(h))(x)$ of the best-response-multifunction for player 1 is nondecreasing in x and*

$$c_{min}(x_2) - c_{min}(x_1) \leq x_2 - x_1 \tag{13}$$

for any natural $x_1 < x_2$.

Proof: Fix $h \in LTM_2$. We will start by showing that c_{min} is nondecreasing. Let $x_1 < x_2$ be any two natural numbers. Now considering the definition of function Ψ_1^{xh} and the fact that $c_{min}(x)$ realizes maximum in (4) we can easily deduce with the help of statement (iii) of Lemma 3.5 that

$$\Psi_1^{x_1 h}(c_{min}(x_1)) > 0, \quad \Psi_1^{x_1 h}(c_{min}(x_1)+1) \leq 0 \tag{14}$$

and

$$\Psi_1^{x_2 h}(c_{min}(x_2)) > 0, \quad \Psi_1^{x_2 h}(c_{min}(x_2)+1) \leq 0. \tag{15}$$

But by (i) of Lemma 3.5,

$$\Psi_1^{x_1 h}(c_{min}(x_1)) \leq \Psi_1^{x_2 h}(c_{min}(x_1)),$$

whence

$$\Psi_1^{x_2 h}(c_{min}(x_1)) > 0,$$

which means that, in view of (15) and (iii) of Lemma 3.5, $c_{min}(x_2) \geq c_{min}(x_1)$. Thus c_{min} in nondecreasing.

Now we show inequality (13). As before, let $x_1 < x_2$. Assume that $c_{min}(x_1) < K_1(x_1)$. (Otherwise, by Assumption (A3) $c_{min}(x_2) - c_{min}(x_1) = c_{min}(x_2) - K_1(x_1) \leq K_1(x_2) - K_1(x_1) \leq x_2 - x_1$).

Using (ii) of Lemma 3.5, we get

$$\Psi_1^{x_1 h}(c_{min}(x_1)+1) \geq \Psi_1^{x_2 h}(c_{min}(x_1)+1+x_2-x_1).$$

Therefore, by the second inequality of (14),

$$\Psi_1^{x_2 h}(c_{min}(x_1)+1+x_2-x_1) \leq 0.$$

Hence, in view of (15),

$$c_{min}(x_2) < c_{min}(x_1)+1+x_2-x_1,$$

which is equivalent to (13). ∎

Fix $h \in LTM_2$ and let $c_{min}(x) = \min c_1(h)(x)$ for $x \in \mathrm{N}$. Notice now, that Lemma 3.6 implies that strategy for player 1 defined by the formula

$$g_0(x) = \delta[c_{min}(x)] \tag{16}$$

belongs to the set LTM_1. However this, together with Lemma 3.4 leads to

$$\begin{aligned} V_h(x) &= \max_{f \in LTM_1} E\left[u_1(\widetilde{f}(x)) + \beta_1 \int V_h(x')\, dF(x' \mid x - \widetilde{f}(x) - \widetilde{h}(x))\right] \\ &= E\left[u_1(\widetilde{g_0}(x)) + \beta_1 \int V_h(x')\, dF(x' \mid x - \widetilde{g_0}(x) - \widetilde{h}(x))\right] \end{aligned} \tag{17}$$

To complete the proof of Theorem 2.1, we need to formulate last couple of lemmata, considering an auxiliary one-stage game Γ, closely related to game G. Before the definition of Γ, we will need to introduce some additional notation first.

For real $a \geq 0$ $\lfloor a \rfloor$ will denote the biggest natural number, which is not bigger than a, while $\lceil a \rceil$ will denote the smallest natural number which is not smaller than a.

We define the game Γ in the following way:

1. N is the state space for game Γ (the same as in G).
2. When game is in a state x, player i, $i = 1, 2$, chooses a real number from the interval $[0, K_i(x)]$; so his strategy in Γ in a state x is any function $f^\Gamma : \mathrm{N} \to [0, C_i]$ satisfying $0 \le f^\Gamma(x) \le K_i(x)$ for all x.
3. Reward in Γ for player 1 using strategy g^Γ, in the situation when player 2 uses strategy h^Γ, is defined in each state x as follows:

$$R_1(x, g^\Gamma, h^\Gamma) = E\left[u_1(\widetilde{g^G}(x)) + \beta_1 \int V^1_{h^G}(x')\, dF(x' \mid x - \widetilde{g^G}(x) - \widetilde{h^G}(x))\right] \tag{18}$$

where, by definition

$$g^G(x) = p_g \delta[\lfloor g^\Gamma(x)\rfloor] + (1 - p_g)\delta[\lceil g^\Gamma(x)\rceil] \text{ with } p_g = \lceil g^\Gamma(x)\rceil - g^\Gamma(x)$$

and

$$h^G(x) = p_h \delta[\lfloor h^\Gamma(x)\rfloor] + (1 - p_h)\delta[\lceil h^\Gamma(x)\rceil] \text{ with } p_h = \lceil h^\Gamma(x)\rceil - h^\Gamma(x).$$

Reward R_2 for player 2 is defined in similiar way.

Strategies of the players in game Γ which are essential in our considerations, correspond with those in LTM_i in game G. Therefore, for $i = 1, 2$, we define

$$\begin{aligned} LM_i^\Gamma \;=\; & \Big\{ f^\Gamma : \mathrm{N} \to [0, C_i],\;\; f^\Gamma(x) \le K_i(x) \text{ for each } x \in \mathrm{N} \\ & \text{and } \; 0 \le \frac{f^\Gamma(x_1) - f^\Gamma(x_2)}{x_1 - x_2} \le 1 \text{ for all distinct } x_1, x_2 \in \mathrm{N} \Big\}. \end{aligned}$$

It is not difficult to check that for any $g^\Gamma \in LM_1^\Gamma$ and $h^\Gamma \in LM_2^\Gamma$, g^G and h^G defined above are unique solutions in LTM_1 and LTM_2, respectively, of the equations

$$E(\widetilde{g^G}(x)) = g^\Gamma(x) \quad \text{and} \quad E(\widetilde{h^G}(x)) = h^\Gamma(x) \tag{19}$$

for each $x \in \mathrm{N}$.

Therfore, for the sake of simplicity we will use the notation:

$$u_1(g^\Gamma(x)) = E u_1(\widetilde{g^G}(x)) \tag{20}$$

and

$$F(x' \mid x - g^\Gamma(x) - h^\Gamma(x)) = E[F(x' \mid x - \widetilde{g^G}(x) - \widetilde{h^G}(x)]. \tag{21}$$

Lemma 3.7 *Let g^Γ and g_n^Γ ($n = 1, 2, \dots$) be strategies for player 1 and h^Γ and h_n^Γ ($n = 1, 2, \dots$) for player 2 in game Γ. If $g_n^\Gamma \to g^\Gamma$ and $h_n^\Gamma \to h^\Gamma$ then*

$$\lim_{n\to\infty} F(x' \mid x - g_n^\Gamma(x) - h_n^\Gamma(x)) = F(x' \mid x - g^\Gamma(x) - h^\Gamma(x))$$

for all $x', x \in N$.

Proof: The proof is straightforward and is left to the reader. ■

The next lemma considers some properties of the best response strategies in the game Γ.

Lemma 3.8 *For every $h^\Gamma \in LM_2^\Gamma$ there exists $g^\Gamma \in LM_2^\Gamma$ such that for every $x \geq 0$*

$$R_1(x, g^\Gamma, h^\Gamma) = \max_{f^\Gamma \in LM_1^\Gamma} R_1(x, f^\Gamma, h^\Gamma) = V_{h^G}(x). \tag{22}$$

Proof: Since $f^\Gamma \in LM_1^\Gamma$ is equivalent to $f^G \in LTM_1$ for every f^Γ, the definition of R_1 leads to

(23)
$$\max_{f^\Gamma \in LM_1^\Gamma} R_1(x, f^\Gamma, h^\Gamma) = \max_{f \in LTM_1} E\left[u_1(\widetilde{f}(x)) + \beta_1 \int V_{h^G}(x')\, dF(x' \mid x - \widetilde{f}(x) - \widetilde{h^G}(x))\right].$$

Now let g^Γ be defined in such a way that $g^G \equiv g_0$ of the form (16). Then, comparing the last equality with (17), we get (22). ■

Now we can define B, the best response map for the game Γ:

$$B : LM_1^\Gamma \times LM_2^\Gamma \to 2^{LM_1^\Gamma} \times 2^{LM_2^\Gamma},$$

$$B(g, h) = c_1^\Gamma(h) \times c_2^\Gamma(g),$$

where

$$c_1^\Gamma(h) = \left\{ g' \in LM_1^\Gamma : V_h^1(x) = u_1(g'(x)) + \beta_1 \int V_h^1(x')\, dF(x' \mid x - g'(x) - h(x))\, \forall x \in \mathrm{N} \right\}$$

and

$$c_2^\Gamma(g) = \left\{ h' \in LM_2^\Gamma : V_g^2(x) = u_2(h'(x)) + \beta_2 \int V_g^2(x')\, dF(x' \mid x - g(x) - h'(x))\, \forall x \in \mathrm{N} \right\}$$

(recall, that we use the notation (20) and (21)).

In the next lemma we establish some further properties of the best response strategies in LM_i^Γ. In the subsequent considerations we will use the following notation for $g \in LTM_1$ and $h \in LTM_2$:

$$S_1(x, g, h) = E\left[u_1(\widetilde{g}(x)) + \beta_1 \int V_h^1(x')\, dF(x' \mid x - \widetilde{g}(x) - \widetilde{h}(x))\right],$$

and analogously for $S_2(x, g, h)$.

Lemma 3.9 *Let $x \in N$, $h^\Gamma \in LM_2^\Gamma$ and $g^\Gamma \in c_1^\Gamma(h^\Gamma)$. Then the following implication holds:*

$$b \in \operatorname{supp}(g^G(x)) \Longrightarrow b \in c_1(h^G)(x). \tag{24}$$

Proof: By definition, $g^G(x) = p\delta[a] + (1-p)\delta[a+1]$ for some $0 \le p \le 1$ and $0 \le a < K_1(x)$. Hence, we have

$$R_1(x, g^\Gamma, h^\Gamma) = S_1(x, g^G, h^G) = pS_1(x, a, h^G) + (1-p)S_1(x, a+1, h^G). \tag{25}$$

On the other hand, by (23) and Lemmata 3.4 and 3.8,

$$\begin{aligned} V^1_{h^G}(x) &= \max_{c \in \{0,\dots,K_1(x)\}} S_1(x, c, h^G) = \max_{g \in LTM_1} S_1(x, g, h^G) \\ &= \max_{g' \in LM_1^\Gamma} R_1(x, g', h^\Gamma). \end{aligned}$$

A simple analysis of the last equalities and (25) leads to the following conclusion: $V^1_{h^G}(x) = S_1(x, a, h^G)$ if $p > 0$, and $V^1_{h^G}(x) = S_1(x, a+1, h^G)$ if $1 - p > 0$. But this is equivalent to (24). ■

Let $(g^\Gamma, h^\Gamma) \in LM_1^\Gamma \times LM_2^\Gamma$. Note, that by Lemmata 3.9 and 3.4 it follows that

$$g^\Gamma \in c_1^\Gamma(h^\Gamma) \quad \text{iff} \quad g^G(x) \in c_1(h^G)(x) \text{ for all } x \in \mathrm{N} \tag{26}$$

and

$$h^\Gamma \in c_2^\Gamma(g^\Gamma) \quad \text{iff} \quad h^G(x) \in c_2(g^G)(x) \text{ for all } x \in \mathrm{N}. \tag{27}$$

Lemma 3.10 *The map B has a fixed point.*

Proof: To prove that map B has a fixed point, it is enough to check, that it satisfies the assumptions of Kakutani-Glicksberg fixed point theorem [4].

Using the diagonal method we can easily show, that LM_i^Γ, $i = 1, 2$, are compact in pointwise-convergence topology.

Convexity of LM_i^Γ is obvious.

Next we will show that for each g^Γ and h^Γ, the set $B(g^\Gamma, h^\Gamma)$ is convex. Fix $x \in \mathrm{N}$ and $h^\Gamma \in LM_2^\Gamma$. Now, let $g_1^\Gamma, g_2^\Gamma \in c_1^\Gamma(h^\Gamma)$ and $0 \le \alpha \le 1$. Let $g_3^\Gamma = \alpha g_1^\Gamma + (1-\alpha) g_2^\Gamma$. It is easily seen that $g_3^\Gamma \in LM_1^\Gamma$. For the functions g_1^Γ, g_2^Γ, g_3^Γ, we have

$$g_i^G(x) = p_i\delta[a_i] + (1-p_i)\delta[a_i+1], \tag{28}$$

for some $p_i \in [0,1]$ and natural $a_i < K_i(x)$.

Let us denote for $i = 1, 2, 3$,

$$\underline{b_i} = \begin{cases} a_i & \text{if } p_i > 0 \\ a_i + 1 & \text{otherwise} \end{cases} \quad \text{and} \quad \overline{b_i} = \begin{cases} a_i + 1 & \text{if } 1 - p_i > 0 \\ a_i & \text{otherwise.} \end{cases}$$

We easily deduce that

$$\min(\underline{b_1}, \underline{b_2}) \leq \underline{b_3}, \overline{b_3} \leq \max(\overline{b_1}, \overline{b_2})$$

and

$$g_3^G(x) = p_3 \delta[\underline{b_3}] + (1-p_3)\delta[\overline{b_3}].$$

By Lemma 3.9, $\underline{b_1}, \underline{b_2} \in c_1(h^G)(x)$ and $\overline{b_1}, \overline{b_2} \in c_1(h^G)(x)$. Hence, by Remark 3.1, $\underline{b_3}$ and $\overline{b_3} \in c_1(h^G)(x)$, whence

$$V_{h^G}(x) = S_1(x, \underline{b_3}, h^G) = S_1(x, \overline{b_3}, h^G).$$

But this finally implies, $V_{h^G}(x) = S_1(x, g_3^G, h^G)$, ending the proof of convexity of the set $c_1^\Gamma(h^\Gamma)$. Therefore $B(g^\Gamma, h^\Gamma)$ is convex for all $(g^\Gamma, h^\Gamma) \in LM_1^\Gamma \times LM_2^\Gamma$.

Now we are left with showing that graph of map B is closed. It is enough to restrict our attention only to one coordinate. For $n = 1, 2, \ldots$, let $h_n^\Gamma, h^\Gamma \in LM_2^\Gamma$ such that $h_n^\Gamma \to h^\Gamma$ and let $\gamma_n^\Gamma, \gamma^\Gamma \in LM_2^\Gamma$, $\gamma_n^\Gamma \in c_1^\Gamma(h_n^\Gamma)$ and $\gamma_n^\Gamma \to \gamma^\Gamma$. The proof will be completed if we show $\gamma^\Gamma \in c_1^\Gamma(h^\Gamma)$. By definition of c_1^Γ and Lemma 3.8 we have for $x \in \mathrm{N}$:

$$\begin{aligned} V^1_{h_n^G}(x) &= u_1(\gamma_n^\Gamma(x)) + \beta_1 \int V^1_{h_n^G}(x')\, dF(x' \mid x - \gamma_n^\Gamma(x) - h_n^\Gamma(x)) \\ &= \max_{c \in [0, K_1(x)]} \left[u_1(c) + \beta_1 \int V^1_{h_n^G}(x')\, dF(x' \mid x - c - h_n^\Gamma(x)) \right] \end{aligned} \tag{29}$$

$V^1_{h_n^G}$ are uniformly bounded (by Lemma 3.4) and have N as their domain so we can use the diagonal method to show that there exists a subsequence $V^1_{h^G_{n_k}}$ pointwise convergent to some $V^1 \in M_1$. Without loss of generality we may assume that $V^1_{h_n^G} \to V^1$. Showing that for $x \in \mathrm{N}$

$$\begin{aligned} V^1(x) &= u_1(\gamma^\Gamma(x)) + \beta_1 \int V^1(x')\, dF(x' \mid x - \gamma^\Gamma(x) - h^\Gamma(x)) \\ &= \max_{c \in [0, K_1(x)]} \left[u_1(c) + \beta_1 \int V^1(x')\, dF(x' \mid x - c - h^\Gamma(x)) \right] \end{aligned}$$

will be sufficient to prove $\gamma^\Gamma \in c_1^\Gamma(h^\Gamma)$.

Clearly, $V^1_{h_n^G}(x'_n) \to V^1(x')$ if $x'_n \to x'$. Hence, using Lemmata 3.7 and 3.2, we can deduce as follows:

$$\begin{aligned} & u_1(\gamma^\Gamma(x)) + \beta_1 \int V^1(x')\, dF(x' \mid x - \gamma^\Gamma(x) - h^\Gamma(x)) \\ = & \lim_{n \to \infty} \left[u_1(\gamma_n^\Gamma(x)) + \beta_1 \int V^1_{h_n^G}(x')\, dF(x' \mid x - \gamma_n^\Gamma(x) - h_n^\Gamma(x)) \right] \\ = & \lim_{n \to \infty} V^1_{h_n^G}(x) = V^1(x). \end{aligned}$$

Fix $x \in \mathrm{N}$. All that we have to check now is that $c = \gamma^\Gamma(x)$ maximizes $u_1(c) + \beta_1 \int V^1(x')\, dF(x' \mid x - c - h^\Gamma(x))$ on $[0, K_1(x)]$. Let

$$w_n(c) = u_1(c) + \beta_1 \int V^1_{h_n^G}(x')\, dF(x' \mid x - c - h_n^\Gamma(x))$$

and see that again by Lemmata 3.7 and 3.2 w_n converges to

$$w(c) = u_1(c) + \beta_1 \int V^1(x')\, dF(x' \mid x - c - h^\Gamma(x)).$$

Notice now, that by Lemma 3.9, whenever some $c \in (l, l+1)$, where $l \in \mathrm{N}$, maximizes w_n, $l, l+1 \in c_1(h_n^G)(x)$. However, in view of the definitions of c_1 and R_1 together with Lemmata 3.4 and 3.8, it means that every point of interval $[l, l+1]$ maximizes w_n. Therefore, we may consider two cases:

Case 1. $\gamma^\Gamma(x) \in \mathrm{N}$: Then for all n big enough $\gamma_n^\Gamma(x) \in (\gamma^\Gamma(x) - 1, \gamma^\Gamma(x) + 1)$. By (29) $\gamma_n^\Gamma(x) \in \arg\max w_n$ and by argument presented above w_n attains its maximum also in $\gamma^\Gamma(x)$.

But $w_n \to w$ and so $\gamma^\Gamma(x) \in \arg\max w$.

Case 2. $\gamma^\Gamma(x) \in (l, l+1)$ for some $l \in \mathrm{N}$: Then for n big enough $\gamma_n^\Gamma(x) \in (l, l+1)$ and therefore each of such w_n-s attains its maximum also in $\gamma^\Gamma(x)$. The same argument as in Case 1 shows that $\gamma^\Gamma(x) \in \arg\max w$.

Therefore, the graph of B is closed, and thereby B has a fixed point. ■

Proof of Theorem 2.1: It has been shown in Lemma 3.10 that map B has a fixed point, which is equivalent to saying that game Γ has a Nash equilibrium in $LM_1^\Gamma \times LM_2^\Gamma$. However, notice that R_i, $i = 1, 2$, were constructed in such a way, that the rewards R_i and S_i for players using corresponding strategies, g^G, h^G in game G and g^Γ, h^Γ in game Γ are equal. This, together with Lemma 3.8 implies that there exists a pair of strategies in $LTM_1 \times LTM_2$ which is a stationary equilibrium in game G. ■

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