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INTUITIONISTIC FUZZY K-IDEALS OF IS-ALGEBRAS

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Abstract. In this paper, we introduce the notion of intuitionistic fuzzy K-ideals of IS-algebras and investigate some of their properties.

1. Introduction and Preliminaries

In 1966, Iseki [1] introduced the notion of BCI-algebras. For the general development of BCK/BCI-algebras, the ideal theory plays an important role. In 1993, Jun et al. [2] introduced a new class of algebras related to BCI-algebras and semigroups, called a BCI-semigroup. In 1998, for the convenice of study, Jun et al. [3] renamed the BCI-semigroups as the IS-algebra and studied further properties. In [4], we introduced the concept of K-ideals of BCI-algebras. In this paper, we consider the fuzzification of K-ideals of IS-algebras and study their properties.

By a BCI-algebra we mean algebra $(X;*,0)$ of type $(2,0)$ satisfying the following conditions:

(I) $((x*y)*(x*z))*(z*y)=0$

(II) $(x*(x*y))*y=0$

(III) $x*x=0$

(IV) $x*y=0$ and $y*x=0$ imply $x=y$.

In any BCI-algebra X one can define a partial order $\leq$ by putting $x\leq y$ if and only if $x*y=0$.

A nonempty subset I of a BCI-algebra X is called an ideal of X if it satisfies (i) $0\in I$, (ii) $x*y\in I$ and $y\in I$ imply $x\in I$ for all $x,y\in I$.

By an IS-algebra we mean a nonempty set X with two binary operation "$*$" and "$\cdot$" and constant 0 satisfying the axioms:

(I) $I(X)=(X;*,0)$ is a BCI-algebra.

(II) $S(X)=(X;\cdot)$ is a semigroup.

(III) The operation "$\cdot$" is distribute over the operation "$*$", that is, $x\cdot(y*z)=(x\cdot y)*(x\cdot z)$ and $(x*y)\cdot z=(x\cdot z)*(y\cdot z)$ for all $x,y,z\in X$.

A nonempty subset A of a semigroup $S(X)=(X;\cdot)$ is said to be stable if $xa\in A$ whenever $x\in S(X)$ and $a\in A$.

We now review some fuzzy logic concepts. A fuzzy set in a set X is a function $\mu:X\to[0,1]$ and the complement of μ, denoted by $\overline{\mu}$, is the fuzzy set in X given by $\overline{\mu}(x)=1-\mu(x)$. For $t\in[0,1]$, the set $U(\mu;t)=\{x\in X\mid \mu(x)\geq t\}$ is called an upper t-level cut of and the

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. set $L(\mu;t) = \{x \in X \mid \mu(x) \leq t\}$ is called a lower t-level cut of μ. We shall write $a \wedge b$ for $min\{a,b\}$ and $a \vee b$ for $max\{a,b\}$, where a and b are any real numbers.

An intuitionistic fuzzy set (bridfly, IFS) A in a nonempty set X is an object having the form

$$A = \{(x, \alpha_A(x), \beta_A(x)) \mid x \in X\}$$

where the functions $\alpha_A : X \to [0,1]$ and $\beta_A : X \to [0,1]$ denote the degree of membership and the degree of non membership respectively, and $0 \leq \alpha_A(x) + \beta_A(x) \leq 1, \quad \forall x \in X$.

An intuitionistic fuzzy set $A = \{(x, \alpha_A(x), \beta_A(x)) \mid x \in X\}$ in X can be identified to an ordered pair (α_A, β_A) in $I^X \times I^X$. For the sake of simplicity, we shall use the symbol $A = (\alpha_A, \beta_A)$ for the $IFSA = \{(x, \alpha_A(x), \beta_A(x)) \mid x \in X\}$.

2. Intuitionistic Fuzzy K-ideals

Definition 2.1 ([4]). Let k be any positive integer. A nonempty subset I of a BCI-algebra X is called a K-ideal of X if

(i) $0 \in I$,

(ii) $x * y^k \in I$ and $y \in I$ imply $x \in I$.

Definition 2.2. A nonempty subset I of an $IS-$algebra X is called a K-ideal of X if

(i) $xa \in I$ for any $x \in S(X)$ and $a \in I$

(ii) $x * y^k \in I$ and $y \in I$ imply $x \in I$

Definition 2.3. A fuzzy set μ in an IS-algebra X is called a fuzzy K-ideal(briefly, FK-ideal) of X if

(i) $\mu(x \cdot y) \geq \mu(y)$,

(ii) $\mu(x) \geq \mu(x * y^k) \wedge \mu(y)$

for all $x, y \in X$.

Definition 2.4. An $IFSA$=(α_A, β_A) in an IS-algebra X is called an intuitionistic fuzzy K-ideals (briefly, IFK-ideal) of X if

(I) $\alpha_A(x \cdot y) \geq \alpha_A(y)$,

(II) $\beta_A(x \cdot y) \leq \beta_A(y)$,

(III) $\alpha_A(x) \geq \alpha_A(x * y^k) \wedge \alpha_A(y)$,

(IV) $\beta_A(x) \leq \beta_A(x * y^k) \vee \beta_A(y)$

for all $x, y \in X$.

Example 2.5. Consider an IS-algebra $X = \{0, a, b, c\}$ with cayley tables as follows:

$*$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

$\cdot$	0	a	b	c
0	0	0	0	0
a	0	a	b	c
b	0	a	b	c
c	0	0	0	0

Define an $IFSA = (\alpha_A, \beta_A)$ in X as follows:
$\alpha_A(0) = \alpha_A(a) = 1$ and $\alpha_A(b) = \alpha_A(c) = t$
$\alpha_A(0) = \beta_A(a) = 0$ and $\beta_A(b) = \beta_A(c) = s$
where $t, s \in [0,1]$ and $t + s \leq 1$.
Hence $A = (\alpha_A, \beta_A)$ is an IFK-ideal of X.

Lemma 2.6. An $IFSA$=(α_A, β_A) is an IFK-ideal of IS-algebra X if and only if the fuzzy sets α_A and $\overline{\beta}_A$ are a FK-ideal of X.

Proof. Let $IFSA = (\alpha_A, \beta_A)$ be an IFK-ideal of X, Clearly α_A is a FK-ideal of X. For any $x, y \in X$, we have $\overline{\beta_A}(x \cdot y) \geq 1 - \beta_A(x \cdot y) = 1 - \beta_A(y) = \overline{\beta}_A(y)$ and $\overline{\beta}_A(x) \geq 1 - \overline{\beta}_A(x * y^k) \vee \beta_A(y) = (1 - \beta_A(x * y^k)) \wedge (1 - \beta_A(y)) = \overline{\beta}_A(x * y^k) \wedge \overline{\beta}_A(y)$. Hence $\overline{\beta}_A$ is a FK-ideal of X.

Conversely, assume that α_A and $\overline{\beta}_A$ are FK-ideal of X. For any $x, y \in X$, we get $\overline{\beta}_A(x \cdot y) \geq \overline{\beta}_A(y)$ and that $\beta_A(x \cdot y) \leq \beta_A(y)$. Moreover, $\overline{\beta}_A(x) \geq \overline{\beta}_A(x * y^k) \wedge \overline{\beta}_A(x)$ and that $1 - \beta_A(x) \geq (1 - \beta_A(x * y^k)) \wedge (1 - \beta_A(y)) = 1 - \beta_A(x * y^k) \vee \beta_A(y)$, that is, $\beta_A(x) \leq \beta_A(x * y^k) \vee \beta_A(y)$. Hence $IFSA$=(α_A, β_A) is an IFK-ideal of X.

Theorem 2.7. $IFSA = (\alpha_A, \beta_A)$ is an IFK-ideal of IS-algebra X if and only if $\Box A = (\alpha_A, \overline{\alpha}_A)$ and $\diamond A = (\overline{\beta}_A, \beta_A)$ are IFK-ideals of X.

Proof. If $IFSA$=(α_A, β_A) is an IFK-ideal of X, then $\alpha_A = \overline{\overline{\alpha}}_A A$ and β_A are FK-ideals of X from Lemma 2.6, hence $\Box A = (\alpha_A, \overline{\alpha}_A)$ and $\diamond A = (\overline{\beta}_A, \beta_A)$ are IFK-ideals of X. Conversely, if $\Box A = (\alpha_A, \overline{\alpha}_A)$ and $\diamond A = (\overline{\beta}_A, \beta_A)$ are IFK-ideals of X, then α_A and $\overline{\alpha}_A$ are FK-ideals of X, hence $IFSA$=(α_A, β_A) is an IFK-ideal of X.

Theorem 2.8. An $IFSA$=(α_A, β_A) is an IFK-ideal of IS-algebra X if and only if for all $s, t \in [0,1]$, the nonempty sets $U(\alpha_A; t)$ and $L(\beta_A; s)$ are K-ideals of X.

Proof. Let $x \in S(X)$ and $y \in U(\alpha_A; t)$. If $IFSA = (\alpha_A, \beta_A)$ is an IFK-ideal of X, then $\alpha_A(y) \geq t$ and that $\alpha_A(x \cdot y) \geq \alpha_A(y) \geq t$, which implies that $x \cdot y \in U(\alpha_A; t)$. Let $x, y \in I(X)$ be such that $x * y^k \in U(\alpha_A; t)$ and $y \in U(\alpha_A; t)$. Then $\alpha_A(x * y^k) \geq t$ and $\alpha_A(y) \geq t$. It follows that $\alpha_A(x) \geq \alpha_A(x * y^k) \wedge \alpha_A(y) \geq t$, so that $x \in U(\alpha_A; t)$. Hence $U(\alpha_A; t)$ is a K-ideal of X. Now let $x \in S(X)$ and $y \in L(\beta_A; s)$, then $\beta_A(y) \leq s$ and so $\beta_A(x \cdot y) \leq \beta_A(y) \leq s$, which implies that $x \cdot y \in L(\beta_A; s)$. Let $x, y \in I(X)$ be such that $x * y^k \in L(\beta_A; s)$ and $y \in L(\beta_A; s)$, then $\beta_A(x * y^k) \leq s$ and $\beta_A(y) \leq s$. It follows that $\beta_A(x) \leq \beta_A(x * y^k) \vee \beta_A(y) \leq s$, so that $x \in L(\beta_A; s)$. Hence $L(\beta_A; s)$ is a K-ideal of X.

Conversely, assume that for each $s, t \in [0,1]$, the nonempty sets $U(\alpha_A; t)$ and $L(\beta_A; s)$ are K-ideals of X. If there are $x_0, y_0 \in S(X)$ such that $\alpha_A(x_0 \cdot y_0) < \alpha_A(y_0)$, then taking $t_0 = (\alpha_A(x_0 \cdot y_0) + \alpha_A(y_0))/2$, we have $\alpha_A(x_0 \cdot y_0) < t_0 < \alpha_A(y_0)$. It follows that $y_0 \in U(\alpha_A; t_0)$ and $x_0 \cdot y_0 \notin U(\alpha_A; t_0)$. This is a contradiction. Therefore α_A is a fuzzy stable set in $S(X)$. If there are $x_0, y_0 \in S(X)$ such that $\beta_A(x_0 \cdot y_0) < \beta_A(y_0)$, then taking $s_0 = (\beta_A(x_0 \cdot y_0) + \beta_A(y_0))/2$, we have $\beta_A(x_0 \cdot y_0) > s_0 > \beta_A(y_0)$, it follows that $y_0 \in L(\beta_A; s_0)$ and $x_0 \cdot y_0 \notin L(\beta_A; s_0)$. This is a contradiction. Therefore β_A is a fuzzy stable set in $S(X)$. Suppose that $\alpha_A(x_0) < \alpha_A(x_0 * y_0^k) \wedge \beta_A(y_0)$ for some $x_0, y_0 \in X$, putting $t_0 = (\alpha_A(x_0) + \alpha_A(x_0 * y_0^k) \wedge \beta_A(y_0))/2$, we have $\alpha_A(x_0) < t_0 < \alpha_A(x_0 * y_0^k) \wedge \beta_A(y_0)$, which shows that $x_0 * y_0^k, y_0 \in U(\alpha_A; t_0)$ and $x_0 \notin U(\alpha_A; t_0)$. This is impossible. Finally, assume that $a, b \in X$ such that $\beta_A(a) > \beta_A(a * b^k) \vee \beta_A(b)$. Taking $s_0 = (\beta_A(a) + \beta_A(a * b^k) \vee \beta_A(b))/2$, then $\beta_A(a * b^k) \vee \beta_A(b) < s_0 < \beta_A(a)$. Therefore $a * b^k$ and $b \in L(\beta_A; s_0)$, but $a \notin L(\beta_A; s_0)$, which is a contradiction. This completes the proof.

3. On homomorphism of IS-algebras

Definition 3.1. ([4]) A mapping $f : X \to Y$ of IS-algebras is called a homomorphism if

(i) $f(x * y) = f(x) * f(y)$ for all $x, y \in I(X)$;

(ii) $f(x \cdot y) = f(x) \cdot f(y)$ for all $x, y \in S(X)$.

For any $IFSA = (\alpha_A, \beta_A)$ in Y, we define a new $IFSA^f = (\alpha_A^f, \beta_A^f)$ in X by $\alpha_A^f(x) = \alpha_A(f(x)), \beta_A^f(x) = \beta_A(f(x)) \quad \forall x \in X$

Theorem 3.2. Let $f : X \to Y$ be a homomorphism of IS-algebras. If an $IFSA$=(α_A, β_A) is an IFK-ideal of Y, then $IFSA^f = (\alpha_A^f, \beta_A^f)$ in X is an IFK-ideal of X.

Proof. Suppose an $IFSA$=(α_A, β_A) is an IFK-ideal of Y, then $\alpha_A^f(x \cdot y) = \alpha_A(f(x \cdot y)) = \alpha_A(f(x) \cdot f(y)) \geq \alpha_A(f(y)) = \alpha_A^f(y)$ and $\beta_A^f(x \cdot y) = \beta_A(f(x \cdot y)) = \beta_A(f(x) \cdot f(y)) \leq \beta_A(f(y)) = \beta_A^f(y)$. Now let $x, y, z \in X$, then $\alpha_A^f(x) = \alpha_A(f(x)) \geq \alpha_A(f(x) * f(g)^k) \wedge \alpha_A(f(y)) = \alpha_A(f(x * y^k)) \wedge \alpha_A(f(y)) = \alpha_A^f(x * y^k) \wedge \alpha_A^f(y)$ and $\beta_A^f(x) = \beta_A(f(x)) \leq \beta_A(f(x) * (y)^k) \vee \beta_A(f(y)) = \beta_A(f(x * y^k)) \vee \beta_A(f(y)) = \beta_A^f(x * y^k) \vee \beta_A^f(y)$. This completes the proof.

If we strengthen the condition f, then the converse of Theorem 3.2 is obtained as follows:

Theorem 3.3. Let $f : X \to Y$ be an epimorphism of IS-algebras and let $IFSA$=(α_A, β_A) be in Y. If $IFSA^f = (\alpha_A^f, \beta_A^f)$ is an IFK-ideal of X, then $IFSA$=(α_A, β_A) is an IFK-ideal of Y.

Proof. For any $x, y \in Y$, there exist $a, b \in X$ such that $f(a) = x$ and $f(b) = y$. Then $\alpha_A(x \cdot y) = \alpha_A(f(a) \cdot f(b)) = \alpha_A^f(a \cdot b) \geq \alpha_A^f(b) = \alpha_A(f(b)) = \alpha_A(y)$ and $\beta_A(x \cdot y) = \beta_A(f(a) \cdot f(b)) = \beta_A^f(a \cdot b) \leq \beta_A^f(b) = \beta_A(f(b)) = \beta_A(y)$. Moreover, $\alpha_A(x) = \alpha_A(f(a)) = \alpha_A^f(a) \geq \alpha_A^f(a * b^k) \wedge \alpha_A^f(b) = \alpha_A(f(a * b^k)) \wedge \alpha_A(f(b)) = \alpha_A(f(a) * f(b)^k) \wedge \alpha_A(f(b)) = \alpha_A(x * y^k) \wedge \alpha_A(y)$ and $\beta_A(x) = \beta_A(f(a)) = \beta_A^f(a) \leq \beta_A^f(a * b^k) \vee \beta_A^f(b) = \beta_A(f(a * b^k)) \vee \beta_A(f(b)) = \beta_A(f(a) * f(b)^k) \vee \beta_A(f(b)) = \beta_A(x * y^k) \vee \beta_A(y)$. This completes the proof.

References

[1] K. Kseki, *An algebra related with a propositional calculus*, Proc. Japan Acad **42** (1966), 26-29.

[2] Y. B. Jun, S.M.Hong, and E. H. Roh, *BCI-semigroups*, Honam Math. J **15** (1993), 59-64.

[3] Y. B. Jun, X. L. Xin, and E. H. Roh, *A class of algebras related to BCI-algebras and semigroups*, Soochow J. Math **24** (1998), 309-321.

[4] Zhan Jianming and Tan Zhisong, *On the BCI-KG part of BCI-algebras*, Sci. Math. Japon **55** (2002), 149-152.

[5] Zhan Jianming and Tan Zhisong, *Intuitionstic fuzzy associative ϕ-ideals of IS-algebras*, Int. J. Math. & Math. Sci., submitted.

[6] E. H. Roh, Y. B. Jun & W. H. Shim, *Fuzzy associative ϕ-ideals of IS-algebras*, Int. J. Math. & Math. Sci **24** (2000), 839-849.

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