ON THE NUMBER OF THE NON-EQUIVALENT KM-SPANNING SUBGRAPHS OF THE COMPLETE GRAPH WITH ORDER MK

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Abstract. Let m be greater than or equal to 2 and n be a multiple of m. We will call a spanning subgraph whose components are K_m of the complete graph K_n a K_m-spanning subgraph of K_n. The Dihedral group D_n acts on the complete graph K_n naturally. This action of D_n induces the action on the set of the K_m-spanning subgraphs of the complete graph K_n . In [3], we calculated the number of the equivalence classes of the 1-regular spanning subgraphs of the complete graph K_n of even order n by this action by using Burnside's Lemma. This is in the case m = 2. In this paper, we generalize this results and calculate the number of the non-equivalent K_m-spanning subgraphs of K_n for all m and n.

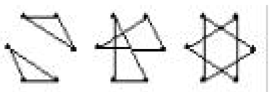
Let m be greater than or equal to 2 and let n be a multiple of m. Let $\{v_0; v_1; v_2; \cdots; v_{n-1}\}$ be the vertices of the complete graph K_n. The action to K_n of the Dihedral group $D_n = \{\rho_0; \rho_1; \cdots; \rho_{n-1}; \sigma_0; \sigma_1; \cdots; \sigma_{n-1}\}$ is defined by

$$\rho_i(v_k) = v_{(k+i) \pmod n} \quad \text{for } 0 \leq i \leq n-1;\ 0 \leq k \leq n-1$$

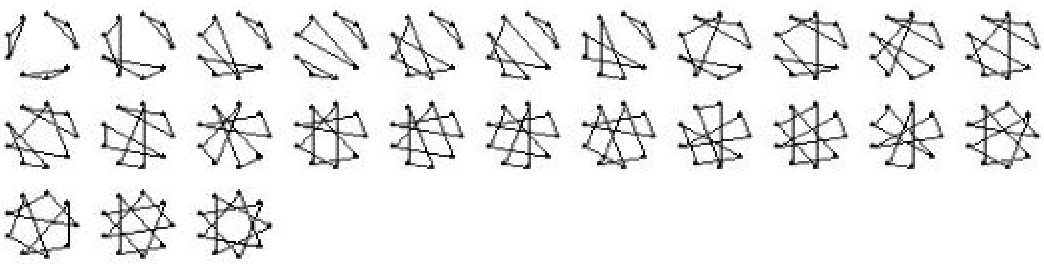
$$\sigma_i(v_k) = v_{(n+i-k) \pmod n} \quad \text{for } 0 \leq i \leq n-1;\ 0 \leq k \leq n-1$$

We call a spanning subgraph whose componenta are K_m of the complete graph K_n a K_m-spanning subgraph of K_n. Let X_n^m be the set of the K_m-spanning subgraphs of K_n. Then the above action induces the action on X_n^m of the Dihedral group D_n.

For example, the equivalence classes of X_6^3 are given with the next figure.



The equivalence classes of X_9^3 are given with the next figure.



We calculate the number of the equivalence classes by this group action. These computations can be done by using Burnside's lemma.

Theorem 1. (Burnside's lemma) Let G be a group of permutations acting on a set S. Then the number of orbits induced on S is given by

$$\frac{1}{|G|}\sum_{\pi \in G} |fix(\pi)|$$

where $fix(\pi) = \{x \in S \mid \pi(x) = x\}$.

Notation 1. An integer function $\mu(p, q)$ is defined by

$$\mu(p, q) = \begin{cases} 1 & \text{if } p \equiv 0 \pmod q \\ 0 & \text{otherwise} \end{cases}$$

Notation 2. For each integer i such that $0 \le i \le n-1$, let $d = (n, i)$ and $R^m_{n,i}$ be

$$R^m_{n,i} = \sum_{\substack{d = \sum_{j=1}^{l} s_j p_j \\ s_j \ge 1,\, p_j \mid m \text{ for } 1 \le j \le l}} \left(\frac{d!}{\prod_{j=1}^{l} (p_j!)^{s_j} s_j!} \prod_{j=1}^{l} \mu\left(\frac{n}{d}, \frac{m}{p_j}\right) \left(\frac{p_j n}{dm}\right)^{s_j(p_j-1)} \right)$$

Notation 3. $S^m_{n,i}, 0 \le i \le n-1$ is given by the following recursive formula:
If n is odd then
$S^m_{n,k} = S^m_{n,0}$ for $1 \le k \le n-1$.
If n is even then
$S^m_{n,2k} = S^m_{n,0}$ for $1 \le k \le \frac{n}{2}-1$ and $S^m_{n,2k+1} = S^m_{n,1}$ for $1 \le k \le \frac{n}{2}-1$.
If m is odd then

$$S^m_{m,0} = 1$$
$$S^m_{2m,1} = 2^{m-1}$$
$$S^m_{n,0} = \binom{\frac{n-1}{2}}{\frac{m-1}{2}} \times S^m_{n-m,1} \qquad \text{if } n \text{ is odd and } n \ge 2m$$
$$S^m_{n,0} = \binom{\frac{n-2}{2}}{\frac{m-1}{2}} \times S^m_{n-m,0} \qquad \text{if } n \text{ is even and } n \ge 2m$$
$$S^m_{n,1} = \left(\sum_{k=0}^{m-1} \frac{(\frac{n-2}{2})!}{k!(m-k-1)!(\frac{n-2m}{2})!}\right) \times S^m_{n-2m,1} \qquad \text{if } n \text{ is even and } n \ge 3m$$

If m is even then

$$S^m_{m,0} = S^m_{m,1} = 1$$
$$S^m_{2m,1} = 2^{m-1} + \binom{\frac{2m-2}{2}}{\frac{m-2}{2}}$$
$$S^m_{n,0} = \binom{\frac{n-2}{2}}{\frac{m-2}{2}} \times S^m_{n-m,1} \text{ if } n \ge 2m$$
$$S^m_{n,1} = \binom{\frac{n-2}{2}}{\frac{m-2}{2}} \times S^m_{n-m,1} + \left(\sum_{k=0}^{m-1} \frac{(\frac{n-2}{2})!}{k!(m-k-1)!(\frac{n-2m}{2})!}\right) \times S^m_{n-2m,1} \text{ if } n \ge 3m$$

Our main Theorem is the following:

Theorem 2. The number of the non-equivalent K_m-spanning subgraphs of the complete graph K_n is given by the following formula:
If n is odd then

$$\frac{1}{2n}\left\{\sum_{i=0}^{n-1} R^m_{n,i} + n \times S^m_{n,0}\right\}$$

If n is even then

$$\frac{1}{2n}\left\{\sum_{i=0}^{n-1} R^m_{n,i} + \frac{n}{2} \times (S^m_{n,0} + S^m_{n,1})\right\}$$

We must determine the numbers of the fixed points of each permutation ρ_i and σ_i to prove the Theorem by using Burnside's Lemma.

Lemma 1. The number of the K_m-spanning subgraphs of K_n is

$$\prod_{k=1}^{n/m}\binom{mk-1}{m-1}$$

This is the number of the fixed points of ρ_0.

Proof. Since the number of ways to select m items from a collection of n items is $\binom{n}{m}$, the number of ways to partition n items into subsets of size m is $\prod_{i=1}^{n/m}\binom{mi}{m}$. Then the number of ways to select $\frac{n}{m}$ groups of size m from a collection of n items is $\frac{\prod_{i=1}^{n/m}\binom{mi}{m}}{\left(\frac{n}{m}\right)!}$. Then we have the results. □

Remark 1. It is easily checked that $R^m_{n,0}$ is equal to $\prod_{k=1}^{n/m}\binom{mk-1}{m-1}$.

Notation 4. Let M^m_n be the union of $G_0, G_1, \cdots, G_{n/m-1}$, where G_j be the complete graph whose vertices are $\{v_j, v_{j+n/m}, v_{j+2n/m}, \cdots, v_{j+(m-1)n/m}\}$ for $0 \le j \le n/m-1$.

Lemma 2. If (n,i)=1 then the number of the fixed points of ρ_i is one.

Proof. M^m_n is a K_m-spanning subgraph of K_n and $\rho_i(M^m_n) = M^m_n$. Conversely, let H be a K_m-spanning subgraph of K_n which is fixed by ρ_i and contain a commponent C whose vertices are $\{v_0, v_{k_1}, v_{k_2}, \cdots, v_{k_{m-1}}\}; 0 < k_1 < k_2 < \cdots < k_{m-1}$. Since (n,i)=1, there is an integer α such that $\alpha i \equiv 1 \pmod n$. Then $\rho_i^{\alpha k_1}(v_0v_{k_1}) = v_{k_1}v_{2k_1} \in C$. If $2k_1$ is not equal to k_2 then $2k_1$ must be greater than k_2 by the assumption of $0 < k_1 < k_2 < \cdots < k_{m-1}$. Since $\rho_i^{\alpha(k_1-k_2)}(v_{k_1}v_{k_2}) = v_{2k_1-k_2}v_{k_1} \in C$ and $0 < 2k_1 - k_2 < k_1$, this is impossible. Then we have $k_2 = 2k_1$. We assume that $k_j = jk_1$ for $j \le s$ and prove that $k_{s+1} = (s+1)k_1$. Since $\rho_i^{\alpha s k_1}(v_0v_{k_1}) = v_{k_s}v_{(s+1)k_1} \in C$, $(s+1)k_1$ is greater than or equal to k_{s+1}. If $(s+1)k_1 > k_{s+1}$ then $\rho_i^{\alpha(k_s-k_{s+1})}(v_{k_s}v_{k_{s+1}}) = v_{2k_s-k_{s+1}}v_{k_s} \in C$ and $k_{s-1} < 2k_s - k_{s+1} < k_s$. This is impossible. Then we have $k_{s+1} = (s+1)k_1$. We finally prove that $mk_1 = n$. Since

$\rho_i^{\alpha(m-1)}(v_0v_{k_1}) = v_{k_{m-1}}v_{mk_1} \in C$ and $k_{m-1} = (m-1)k_1 < mk_1$, we have $mk_1 \ge n$ and $0 \le mk_1 \pmod n < k_1$. Then we have that $mk_1 \equiv 0 \pmod n$ and $mk_1 = n$ and $k_1 = n/m$. Then the set of the vertices of C is $\{v_0, v_{n/m}, v_{2n/m}, \cdots, v_{(m-1)n/m}\}$. Since H is determined by C, H must be M_n^m. Then the number of the fixed points of ρ_i is one. □

Lemma 3. The fixed points of ρ_i is $R^m_{n,i}$.

Proof. Let $d = (n, i)$ and $V_0 = \{v_0, v_d, v_{2d}, \cdots, v_{n-d}\}, V_1 = \{v_1, v_{d+1}, v_{2d+1}, \cdots, v_{n-d+1}\}$, $V_2 = \{v_2, v_{d+2}, v_{2d+2}, \cdots, v_{n-d+2}\}$, $\cdots$, $V_{d-1} = \{v_{d-1}, v_{2d-1}, v_{3d-1}, \cdots, v_{n-1}\}$.

Since $(n, i) = d$, the equation $xi \equiv m \pmod n$ has a solution if and only if d divides m. Then we have $\rho_i(V_k) = V_k$ for $0 \le k \le d-1$. Let H be a K_m-spanning subgraph of K_n which is fixed by ρ_i. We assume that each component of $H|V_{k_0} \cup V_{k_1} \cup \cdots \cup V_{k_{p-1}}$ is K_m and any component of the restriction to the proper subset of $\{V_{k_0}, V_{k_1}, \cdots, V_{k_{p-1}}\}$ of H is not K_m. Since $\rho_i(V_k) = V_k$ for $0 \le k \le d-1$, the vertices of each component K_m must be distributed equally to $V_{k_0}, V_{k_1}, \cdots, V_{k_{p-1}}$. Then $m \equiv 0 \pmod p$ and $\frac{n}{d} \equiv 0$ $(\mathrm{mod}\ \frac{m}{p})$ and each component of $H|V_j$ is $K_{\frac{m}{p}}$. If we change the name of the vertices of V_j to $v_0, v_1, v_2, \cdots, v_{\frac{n}{d}-1}$ then we have $\rho_{\frac{i}{d}}(H|V_j) = H|V_j$. Since $(n, i) = d$, we have that $(\frac{n}{d}, \frac{i}{d}) = 1$. By Lemma 1, we have that $H|V_j = M^{\frac{m}{p}}_{\frac{n}{d}}$. Since the number of components $K_{\frac{m}{p}}$ of $H|V_j$ is $\frac{pn}{dm}$, the number of the possible arrangements of $H|V_{k_0} \cup V_{k_1} \cup \cdots \cup V_{k_{p-1}}$ is $\left(\frac{pn}{dm}\right)^{p-1}$. We divide $\{V_0, V_1, V_2, \cdots, V_{d-1}\}$ into the subsets $W_1, W_2, W_3, \cdots, W_s$ which are satisfied the above conditions. If $|W_k|$ is equal to $p_k, 1 \le k \le s$ then the number of such H is $\prod_{k=1}^{s} \left(\frac{p_k n}{dm}\right)^{p_k-1}$. In this case we have that $\sum_{k=1}^{s} p_k = d$ and p_k is a divisor of m for $1 \le k \le s$. Let $d = \sum_{j=1}^{l} s_j p_j$ be a representation of d as the sum of divisors p_j of m. The number of ways to divide $\{V_0, V_1, V_2, \cdots, V_{d-1}\}$ into s_1 pieces of p_1-element set, s_2 pieces of p_2-element set, s_3 pieces of p_3-element set, $\cdots$, s_l pieces of p_l-element set is

$$\frac{d!}{\prod_{j=1}^{l} (p_j!)^{s_j} s_j!}.$$

Accordingly, the number of all the possibilities of H is

$$\sum_{\substack{d = \sum_{j=1}^{l} s_j p_j \\ s_j \ge 1,\ p_j | m \text{ for } 1 \le j \le l}} \left(\frac{d!}{\prod_{j=1}^{l} (p_j!)^{s_j} s_j!} \prod_{j=1}^{l} \phi\left(\frac{n}{d}, \frac{m}{p_j}\right) \left(\frac{p_j n}{dm}\right)^{s_j(p_j-1)} \right).$$

This number is $R^m_{n,i}$ given by Notaion 2. We have the results. □

Notation 5. Let $S^m_{n,i}$ be the number of the fixed points of σ_i for X^m_n.

Remark 2. By the following lemmas we will see that $S^m_{n,i}$ agrees with the one which is given in Notation 3.

Lemma 4. If n is odd then the number of the fixed points of σ_0 is equal to the number of the fixed points of σ_k for all $1 \le k \le n-1$.

Proof. We assume that k is even. Let H be a K_m-spanning subgraph of K_n fixed by σ_0. Then it is easily verified that $\rho_{\frac{k}{2}}(H)$ is a K_m-spanning subgraph of K_n fixed by σ_k. Conversely, if H is a K_m-spanning subgraph of K_n fixed by σ_k then $\rho_{\frac{k}{2}}^{-1}(H)$ is a K_m-spanning subgraph of K_n fixed by σ_0. Next we assume that k is odd. Let H be a K_m-spanning subgraph of K_n fixed by σ_0. Then it is easily verified that $\rho_{\frac{n+k}{2}}(H)$ is a K_m-spanning subgraph of K_n fixed by σ_k. Conversely, if H is a K_m-spanning subgraph of K_n fixed by σ_k then $\rho^{-1}_{\frac{n+k}{2}}(H)$ is a K_m-spanning subgraph of K_n fixed by σ_0. Then we have the results. □

Similarly, we have the next Lemma.

Lemma 5. If n is even then the number of the fixed points of σ_0 is equal to the number of the fixed points of σ_{2d} for all $1 \le d \le n/2-1$ and the number of the fixed points of σ_1 is equal to the number of the fixed points of σ_{2d+1} for all $1 \le d \le n/2-1$.

Lemma 6. If n is odd and m is odd then

$$S^m_{m;0} = 1 \qquad \text{and}$$
$$S^m_{n;0} = \binom{\frac{n-1}{2}}{\frac{m-1}{2}} \times S^m_{n-m;1} \qquad \text{if } n \ge 2m$$

Proof. The K_m-spanning subgraph of K_m is K_m and K_m is fixed by σ_0. Then we have $S^m_{m;0} = 1$. We assume that $n \ge 2m$. Let H be a K_m-spanning subgraph of K_n fixed by σ_0. Let C be the component of H which contains vertex v_0. $H - C$ naturally becomes K_m-spanning subgraph of K_{n-m} fixed by σ_1 when we change the name of the vertices. Conversely, let H be a K_m-spanning subgraph of K_{n-m} fixed by σ_1. Since $n-m$ is even, the axis of the line symmetry is not passing any vertices. If we take one vertex of K_m in the position of v_0 of the graph which we will construct and divide the remaining vertices of K_m into halves and distribute them between the vertices of H permitting redundancy and symmetrically regarding the axis then the resulting graph becomes a K_m-spanning subgraph of K_n fixed by σ_0. The number of ways to distribute the vertices of K_m is $\binom{\frac{n-1}{2}}{\frac{m-1}{2}}$. Then we have the results. □

Lemma 7. If n is even and m is odd then

$$S^m_{n;0} = \binom{\frac{n-2}{2}}{\frac{m-1}{2}} \times S^m_{n-m;0}$$

Proof. Let H be a K_m-spanning subgraph of K_n fixed by σ_0. Since n is even, the axis of σ_0 passes v_0 and $v_{\frac{n}{2}}$. Let C be the component of H which contains vertex $v_{\frac{n}{2}}$. Since m is odd, C does not contain the vertex v_0. $H - C$ naturally becomes K_m-spanning subgraph of K_{n-m} fixed by σ_0 when we change the name of the vertices. Conversely, let H be a K_m-spanning subgraph of K_{n-m} fixed by σ_0. Since $n-m$ is odd, the axis of σ_0 passes the vertex v_0. If we take one vertex of K_m in the position of $v_{\frac{n}{2}}$ of the graph which we will construct and divide the remaining vertices of K_m into halves and distribute them between the vertices of H permitting redundancy and symmetrically regarding the axis then the resulting graph becomes a K_m-spanning subgraph of K_n fixed by σ_0. The number of ways to distribute the vertices of K_m is $\binom{\frac{n-2}{2}}{\frac{m-1}{2}}$. Then we have the results. □

Lemma 8. If n is even and m is odd then

$$S^m_{2m;1} = 2^{m-1} \qquad \text{and}$$
$$S^m_{n;1} = \left(\sum_{k=0}^{m-1} \frac{\left(\frac{n-2}{2}\right)!}{k!(m-k-1)!\left(\frac{n-2m}{2}\right)!}\right) \times S^m_{n-2m;1} \qquad \text{if } n \geq 4m$$

Proof. We assume that $n = 2m$. If we take one vertex of K_m in the position of $v_{\frac{n}{2}}$ and one vertex of another K_m in the position of $v_{\frac{n}{2}+1}$ of the graph which we will construct and distribute the remaining vertices of two K_m to both sides of the perpendicular bisector of $v_{\frac{n}{2}-1}$ and $v_{\frac{n}{2}-1}$ permitting redundancy and symmetrically regarding the line then the resulting graph becomes a K_m-spanning subgraph of K_{2m} fixed by σ_1. The number of ways to distribute the vertices of two K_m is $\displaystyle\sum_{k=0}^{m-1} \frac{(m-1)!}{k!(m-k-1)!} = 2^{m-1}$. We assume that $n \geq 4m$. Let H be a K_m-spanning subgraph of K_n fixed by σ_1. Since n is even, the axis of σ_1 does not pass any vertices. Since m is odd, there is no component which contains both $v_{\frac{n}{2}}$ and $v_{\frac{n}{2}+1}$. Let C_0 be a component which contains vertex $v_{\frac{n}{2}}$ and C_1 be a component which contains vertex $v_{\frac{n}{2}+1}$. $H - C_0 - C_1$ naturally becomes K_m-spanning subgraph of K_{n-2m} fixed by σ_1 when we change the name of the vertices. Conversely, let H be a K_m-spanning subgraph of K_{n-2m} fixed by σ_1. Since $n-2m$ is even, the axis of σ_1 does not pass any vertices. If we take one vertex of K_m in the position of $v_{\frac{n}{2}}$ and one vertex of another K_m in the position of $v_{\frac{n}{2}+1}$ of the graph which we will construct and distribute the remaining vertices of two K_m between the vertices of H permitting redundancy and symmetrically regarding the axis then the resulting graph becomes a K_m-spanning subgraph of K_n fixed by σ_1. The number of ways to distribute the vertices of two K_m is $\displaystyle\sum_{k=0}^{m-1} \frac{\left(\frac{n-2}{2}\right)!}{k!(m-k-1)!\left(\frac{n-2m}{2}\right)!}$. Then we have the results. □

Lemma 9. If n is even and m is even then

$$S^m_{m;0} = 1 \qquad \text{and}$$
$$S^m_{n;0} = \binom{\frac{n-2}{2}}{\frac{m-2}{2}} \times S^m_{n-m;1} \qquad \text{if } n \geq 2m$$

Proof. The K_m-spanning subgraph of K_m is K_m and K_m is fixed by σ_0. Then we have $S^m_{m;0} = 1$. We assume that $n \geq 2m$. Let H be a K_m-spanning subgraph of K_n fixed by σ_0. Since n is even, the axis of σ_0 passes v_0 and $v_{\frac{n}{2}}$. Let C be the component of H which contains vertex v_0 and $v_{\frac{n}{2}}$. $H - C$ naturally becomes K_m-spanning subgraph of K_{n-m} fixed by σ_1 when we change the name of the vertices. Conversely, let H be a K_m-spanning subgraph of K_{n-m} fixed by σ_1. Since $n-m$ is even, the axis of σ_1 does not pass any vertices. If we take two vertices of K_m in the positions of v_0 and $v_{\frac{n}{2}}$ of the graph which we will construct and divide the remaining vertices of K_m into halves and distribute them between the vertices of H permitting redundancy and symmetrically regarding the axis then the resulting graph becomes a K_m-spanning subgraph of K_n fixed by σ_0. The number of ways to distribute the vertices of K_m is $\binom{\frac{n-2}{2}}{\frac{m-2}{2}}$. Then we have the results. □

Lemma 10. If n is even and m is even then

$$S^m_{m;1} = 1 \qquad \text{and}$$

$$S^m_{2m;1} = 2^{m-1} + \binom{\frac{2m-2}{2}}{\frac{m-2}{2}} \qquad \text{and}$$

$$S^m_{n;1} = \binom{\frac{n-2}{2}}{\frac{m-2}{2}} S^m_{n-m;1} + \left(\sum_{k=0}^{m-1} \frac{(\frac{n-2}{2})!}{k!(m-k-1)!(\frac{n-2m}{2})!} \right) \times S^m_{n-2m;1} \qquad \text{if } n \geq 3m$$

Proof. The K_m-spanning subgraph of K_m is K_m and K_m is fixed by σ_1. Then we have $S^m_{m;1} = 1$. We assume that $n \geq 3m$. We study two kinds of constitutions that compose K_m-spanning subgraphs of K_n fixed by σ_1 inductively.

The first method is the following:
Let H be a K_m-spanning subgraph of K_{n-m} fixed by σ_1. Since $n-m$ is even, the axis of σ_1 does not pass any vertices. If we take two vertices of K_m in the positions of $v_{\frac{n}{2}}$ and $v_{\frac{n}{2}+1}$ of the graph which we will construct and divide the remaining vertices of K_m into halves and distribute them between the vertices of H permitting redundancy and symmetrically regarding the axis then the resulting graph becomes a K_m-spanning subgraph of K_n fixed by σ_1. The number of ways to distribute the vertices of K_m is $\binom{\frac{n-2}{2}}{\frac{m-2}{2}}$. Similarly, if we take two vertices of K_m in the positions of v_0 and v_1 of the graph which we will construct and divide the remaining vertices of K_m into halves and distribute them between the vertices of H permitting redundancy and symmetrically regarding the axis then the resulting graph becomes a K_m-spanning subgraph of K_n fixed by σ_1. The number of ways to distribute the vertices of K_m is $\binom{\frac{n-2}{2}}{\frac{m-2}{2}}$. Accordingly, it is possible $2 \times \binom{\frac{n-2}{2}}{\frac{m-2}{2}} \times S^m_{n-m;1}$ K_m-spanning subgraph of K_n fixed by σ_1 as a whole with these constitutions.

The second method is the following:
Let H be a K_m-spanning subgraph of K_{n-2m} fixed by σ_1. Since $n-m$ is even, the axis of σ_1 does not pass any vertices. If we take one vertex of K_m in the position of $v_{\frac{n}{2}}$ and one vertex of another K_m in the position of $v_{\frac{n}{2}+1}$ of the graph which we will construct and distribute the remaining vertices of two K_m between the vertices of H permitting redundancy and symmetrically regarding the axis then the resulting graph becomes a K_m-spanning subgraph of K_n fixed by σ_1. The number of ways to distribute the vertices of two K_m is $\sum_{k=0}^{m-1} \frac{(\frac{n-2}{2})!}{k!(m-k-1)!(\frac{n-2m}{2})!}$. Similarly, if we take one vertex of K_m in the position of v_0 and one vertex of another K_m in the position of v_1 of the graph which we will construct and distribute the remaining vertices of two K_m between the vertices of H permitting redundancy and symmetrically regarding the axis then the resulting graph becomes a K_m-spanning subgraph of K_n fixed by σ_1. The number of ways to distribute the vertices of two K_m is $\sum_{k=0}^{m-1} \frac{(\frac{n-2}{2})!}{k!(m-k-1)!(\frac{n-2m}{2})!}$. Therefore, by this construction, we can construct $2 \times \sum_{k=0}^{m-1} \frac{(\frac{n-2}{2})!}{k!(m-k-1)!(\frac{n-2m}{2})!}$. K_m-spanning subgraph of K_n fixed by σ_1. By these two constructions, we can construct

$$2 \times \binom{\frac{n-2}{2}}{\frac{m-2}{2}} S^m_{n-m;1} + 2 \times \left(\sum_{k=0}^{m-1} \frac{(\frac{n-2}{2})!}{k!(m-k-1)!(\frac{n-2m}{2})!} \right) \times S^m_{n-2m;1}$$

K_m-spanning subgraphs of K_n fixed by σ_1. Clearly there are doubling two pieces of each. Also, it is clear to be able to compose all the K_m-spanning subgraphs of K_n fixed by σ_1

by these methods. We assume that n is equal to $2m$. Then we can similarly construct all K_m-spanning subgraphs of K_{2m} ¯xed by ¾$_1$ by these two constructions if we set H be a empty graph in the case of the second constitution. We have the results. □

Then we completely proved Theorem 2.

Remark 3. We calculated the non-equivarent K_4-spanning subgraphs of K_n, n · 16 by computer. The numbers agreed with the numbers that are given by Theorem 2. The results is as follows:

n=4	1
n=8	7
n=12	297
n=16	83488

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