PROPERTY(δ) AND NORMALITY OF PRODUCT SPACES

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Received August 16, 2002

ABSTRACT. In this paper we shall prove that under the assumption of some conditions, the property(δ) of $X \times Y$ implies the normality of $X \times Y$.

1. INTRODUCTION.

Throughout this paper we assume that each space is a Hausdorff space and each map is continuous and onto. $\mathbf{N}$ denotes the set of positive integers. We consider the following question.

Question. Let X be a space and $f : Y \to Z$ be a closed map. Suppose $X \times Y$ is normal. Is $X \times Z$ normal?

Concerning this, the followings are well known.
(1) In case X is compact, the above question solved affiermatively by M. E. Rudin [11].
(2) In case X is a metric space, the above question solved affiermatively by M. E. Rudin and M. Starbird [12].

In this paper we shall obtain some results in connection with the above question.

Refering to the equivalence of normality and countable paracompactness, many results are known. Let us remember the following.
(3)(Hoshina [4]). Suppose X is a paracompact σ-space and Y a normal P-space. Then $X \times Y$ is normal if and only if it is countably paracompact.
(4)(Bešlagić and Chiba [1]). Suppose X is a first countable paracompact P-space and Y the closed image of a normal M-space. Then $X \times Y$ is normal if and only if it is countably paracompact.

In the above theorem (3), Hoshina [4] essentially proved that $X \times Y$ has property (δ) if and only if it is normal if and only if it is countably paracompact.

In this note, we shall prove that in the above theorem (4), similar result holds.

2. PRELIMINARIES.

Definition 1. A space X has *property(δ)* [5, p. 145] if, for any open subset $U_n, n \in \mathbf{N}$, and any closed subset B such that $\cap_n \overline{U_n} \cap B = \emptyset, \cap_n U_n$ and B are separated by open subsets of X. A subset A of a space X is a regular G_δ*-set* if A is written as $A = \cap_n U_n = \cap_n \overline{U_n}$ with some open subsets U_n of X. According to Mack [7], X is δ*-normal* if, for every pair of disjoint closed subsets one of which is a regular G_δ-set, there are disjoint open subsets containing them.

Normality implies property(δ) and property(δ) implies δ-normality. By Mack [7], countable paracompactness implies δ-normality. Hoshina [5, 2.5.Lemma] improved this result to: countable paracompactness implies property (δ).

2000 *Mathematics Subject Classification.* 54B10,54D15.
Key words and phrases. Property(δ), product space, normal.

Definition 2. A space X is *collectionwise Hausdorff* if each discrete collection of points $\{x_\lambda|\lambda \in \Lambda\}$ is separated by open sets, i.e., there is a pairwise disjoint collection $\{U_\lambda|\lambda \in \Lambda\}$ of open sets in X such that $x_\lambda \in U_\lambda$ for each $\lambda \in \Lambda$.

Definition 3. A space X is *perfect* if each closed set is a G_δ-set. A space X is *perfectly normal* if X is normal and perfect.

Definition 4. ([8]). A space X is a P-space if for any set Ω and for any collection $\{G(\alpha_1, ..., \alpha_n)|\alpha_i \in \Omega, i = 1, 2, ..., n; n \in \mathbf{N}\}$ of open subsets of X such that $G(\alpha_1, ..., \alpha_n) \subset G(\alpha_1, ..., \alpha_n, \alpha_{n+1})$ for $\alpha \in \Omega, i = 1, 2, ..., n, n+1$, there exists a collection $\{F(\alpha_1, ..., \alpha_n)|\alpha_i \in \Omega, i = 1, 2, ..., n; n \in \mathbf{N}\}$ of closed subsets of X such that the conditions (1),(2) below are satisfied:
(1) $F(\alpha_1, ..., \alpha_n) \subset G(\alpha_1, ..., \alpha_n)$ for $\alpha_i \in \Omega, i = 1, 2, ..., n$;
(2) $X = \cup_{n \in \mathbf{N}} F(\alpha_1, ..., \alpha_n)$ for any sequence (α_n) such that $X = \cup_{n \in \mathbf{N}} G(\alpha_1, ..., \alpha_n)$.

Definition 5. ([8]). A space X is an M-space if there is a sequence $\{\mathcal{U}_m|m \in \mathbf{N}\}$ of locally finite open covers of X satisfying the following conditions: For every $x \in X$ and $\{K_m|m \in \mathbf{N}\}$ a decreasing sequence of nonempty closed subsets of X with $K_m \subset \cap\{U \in \mathcal{U}_m|x \in U\}$ for $m \in \mathbf{N}$, we have that $\cap_{m \in \mathbf{N}} K_m \neq \emptyset$.

Definition 6. ([9],[10]). A space X is a σ-space if it has a σ-locally finite net.

Fact 1. ([3]). Let X be a normal M-space and Y a first countable paracompact P-space. Then $X \times Y$ is collectionwise normal.

Fact 2. (1) ([2]). If X is a regular σ-space and $f : X \to Y$ is a closed map, then $Y = \cup_{n=0}^{\infty} Y_n, Y_n$ is closed discrete in Y for each $n \geq 1$, for each $y \in Y_0, f^{-1}(y)$ is compact and $Y_0 \cap (\cup_{n \geq 1} Y_n) = \emptyset$.
(2) ([6]). If X is a normal M-space and $f : X \to Y$ is a closed map, then $Y = \cup_{n=0}^{\infty} Y_n, Y_n$ is closed discrete in Y for each $n \geq 1$, for each $y \in Y_0, f^{-1}(y)$ is countably compact and $Y_0 \cap (\cup_{n \geq 1} Y_n) = \emptyset$.

Fact 3. Let A and B are disjoint closed subsets in X. If there are open sets U_n and $V_n, n = 0, 1, 2, ...$ such that $A \subset \cup_n U_n, B \subset \cup_n V_n$ and $\overline{U_n} \cap B = \emptyset, \overline{V_n} \cap A = \emptyset$ for each n, then A and B are separated by open sets in X. In fact, $U = \cup_{n=0}^{\infty}(U_n \setminus \cup_{i \leq n} \overline{V_i})$ and $V = \cup_{n=0}^{\infty}(V_n \setminus \cup_{i \leq n} \overline{U_i})$ are disjoint open sets in X such that $U \supset A$ and $V \supset B$.

3. Theorems

Theorem 1. *Let X be a first countable space and $f : Y \to Z$ be a closed map which satisfy the condition: $Z = \cup_{i=0}^{\infty} Z_i, Z_i$ is closed, discrete in Z for $i > 0$, for each $z \in Z_0, f^{-1}(z)$ is countably compact and $Z_0 \cap (\cup_{i=1}^{\infty} Z_i) = \emptyset$. Suppose $X \times Y$ is normal and Z is collectionwise Hausdorff and perfectly normal. If $X \times Z$ has property(δ), then $X \times Z$ is normal.*

Proof. (I). Let A and B are disjoint closed subsets in $X \times Z$ such that $A, B \subset X \times Z_0$. Then there are disjoint open sets U and V in $X \times Z$ such that $A \subset U$ and $B \subset V$.
(Proof). Put $g = 1_X \times f : X \times Y \to X \times Z$ and let $C = g^{-1}(A), D = g^{-1}(B)$. Then C and D are disjoint closed subsets in $X \times Y$. Since $X \times Y$ is normal, there are disjoint open sets G and H in $X \times Y$ such that $C \subset G$ and $D \subset H$.

For each $a = \langle x, z \rangle \in A, g^{-1}(a) = \{x\} \times f^{-1}(z) \subset G$. Since $f^{-1}(z)$ is countably compact and X is first countable, there are an open set V_a in X and an open set W_a in Y such that $\{x\} \times f^{-1}(z) \subset V_a \times W_a \subset G$. Put $W_a^{'} = Z \setminus f(Y \setminus W_a)$. Then $W_a^{'}$ is open in Z and $a \in V_a \times W_a^{'}$ and $f^{-1}(W_a^{'}) \subset W_a$. Put $U = \cup\{V_a \times W_a^{'}|a \in A\}$. Then U is open in $X \times Z$ such that $U \supset A$ and $g^{-1}(U) \subset G$.

Similarly we can define an open set V such that $V \supset B$ and $g^{-1}(V) \subset H$. Then it is obvious that $U \cap V = \emptyset$.

(II). Let A and B are disjoint closed subsets in $X \times Z$ such that $A \subset X \times Z_n$ for some $n > 0$. Then there is an open set U in $X \times Z$ such that $A \subset U, \overline{U} \cap B = \emptyset$.

(Proof). Put $B_n = B \cap (X \times Z_n)$. Since Z is collectionwise Hausdorff, there is a pairwise disjoint collection $\mathcal{U} = \{U(z) | z \in Z_n\}$ of open sets in Z such that $z \in U(z)$ for each $z \in Z_n$. For each $z \in Z_n$, put $A_z = \{x \in X | \langle x, z\rangle \in A\}$ and $B_z = \{x \in X | \langle x, z\rangle \in B\}$. Then A_z and B_z are disjoint closed subsets in X. Since X is normal, there are disjoint open sets G_z and H_z in X such that $A_z \subset G_z$ and $B_z \subset H_z$.

For each $z \in Z_n$, there are open sets K_z and L_z in $X \times Z$ such that
$A_z \times \{z\} \subset K_z \subset G_z \times U(z)$ and $K_z \cap B = \emptyset$,
$B_z \times \{z\} \subset L_z \subset H_z \times U(z)$ and $L_z \cap A = \emptyset$.

Put $K = \cup\{K_z | z \in Z_n\}$ and $L = \cup\{L_z | z \in Z_n\}$. Then K and L are disjoint open sets in $X \times Z$ such that $A \subset K$ and $B_n \subset L$.

Put $B' = B \setminus L$. Then B' and $X \times Z_n$ are disjoint closed subsets in $X \times Z$. Since Z is perfectly normal, Z_n is a regular G_δ-set. Therefore $X \times Z_n$ is a regular G_δ-set of $X \times Z$. Since $X \times Z$ has property (δ), there is an open set G in $X \times Z$ such that $X \times Z_n \subset G$ and $\overline{G} \cap B' = \emptyset$. Then $A \subset K \cap G$ and $\overline{K \cap G} \cap (L \cup B') = \emptyset$. Therefore $\overline{K \cap G} \cap B = \emptyset$. Let us put $U = K \cap G$. Then U is the desired one.

(III). Let A and B are disjoint closed subsets in $X \times Z$ such that $A \subset X \times Z_0$. Then there is an open set U in $X \times Z$ such that $A \subset U, \overline{U} \cap B = \emptyset$.

(Proof). For each $n > 0$, $X \times Z_n$ is a regular G_δ-set such that $A \cap (X \times Z_n) = \emptyset$. Since $X \times Z$ has property (δ), there are open sets M_n and M_n' such that $A \subset M_n, X \times Z_n \subset M_n'$ and $M_n \cap M_n' = \emptyset$. Then $A \subset \cap_{n=1}^{\infty} M_n, (\cap_{n=1}^{\infty} \overline{M_n}) \cap (\cup_{n=1}^{\infty} M_n') = \emptyset$.

Put $B_0 = B \setminus \cup_{n=1}^{\infty} M_n'$. Then B_0 is a closed set of $X \times Z$ such that $B_0 \subset X \times Z_0$. Since A and B_0 are disjoint closed sets in $X \times Z$ such that $A, B_0 \subset X \times Z_0$. By (I), there are disjoint open sets M_0 and M_0' in $X \times Z$ such that $A \subset M_0$ and $B_0 \subset M_0'$. Then $A \subset \cap_{n=0}^{\infty} M_n$ and $(\cap_{n=0}^{\infty} \overline{M_n}) \cap B = \emptyset$. The existence of U follows from the property (δ) of $X \times Z$.

(IV). Now we finish the proof. Let A and B are disjoint closed sets in $X \times Z$. Put $A_n = A \cap (X \times Z_n)$ and $B_n = B \cap (X \times Z_n)$ for each $n > 0$. By (II), there are open sets $U_n, n = 1, 2, ...$ such that $A_n \subset U_n, \overline{U_n} \cap B = \emptyset$. Put $A_0 = A \setminus \cup_{n=1}^{\infty} U_n$. Then, by (III), there is an open set U_0 in $X \times Z$ such that $A_0 \subset U_0, \overline{U_0} \cap B = \emptyset$. Then, U_n are open sets in $X \times Z$ such that $A \subset \cup_{n=0}^{\infty} U_n, \overline{U_n} \cap B = \emptyset$ for each $n \geq 0$. Similarly there are open sets $V_n, n = 0, 1, 2, ...$ such that $B \subset \cup_{n=0}^{\infty} V_n, \overline{V_n} \cap A = \emptyset$ for each n. Therefore, by Fact 3, A and B are separated by open sets.

By the similar proof of Theorem 1, we obtain

Theorem 2. *Let X be a space and $f : Y \to Z$ be a closed map which satisfy the condition: $Z = \cup_{i=0}^{\infty} Z_i, Z_i$ is closed, discrete in Z for $i > 0$, for each $z \in Z_0, f^{-1}(z)$ is compact and $Z_0 \cap (\cup_{i=1}^{\infty} Z_i) = \emptyset$. Suppose $X \times Y$ is normal, Z is collectionwise Hausdorff and perfectly normal. If $X \times Z$ has property(δ), then $X \times Z$ is normal.*

Corollary 1. Let X be a space and $f : Y \to Z$ be a closed map. Suppose $X \times Y$ is normal, Y is a σ-space and Z is a collectionwise Hausdorff space. If $X \times Z$ has property(δ), then $X \times Z$ is normal.

Proof. By Fact 2 (1), $Z = \cup_{i=0}^{\infty} Z_i, Z_i$ is closed, discrete in Z for $i > 0$, for each $z \in Z_0, f^{-1}(z)$ is compact and $Z_0 \cap (\cup_{i=1}^{\infty} Z_i) = \emptyset$. By [10, Th. 2.8], Z is perfectly normal. Therefore, Corollary 1 follows from Theorem 2.

Corollary 2. Let X be a space and $f : Y \to Z$ be a closed map. Suppose $X \times Y$ is normal and Y is paracompact σ-space. If $X \times Z$ has property(δ), then $X \times Z$ is normal.

Theorem 3. *Suppose X is a first countable paracompact P-space and Y the closed image of a normal M-space. Then the following are equivalent.*
(1) $X \times Y$ has property (δ).
(2) $X \times Y$ is normal.
(3) $X \times Y$ is countably paracompact.

It is sufficient to prove the following.

Proposition. Suppose X is a first countable paracompact P-space and Y the closed image of a normal M-space. If $X \times Y$ has property (δ), then $X \times Y$ is normal.

Proof. Let Z be a normal M-space and $f : Z \to Y$ a closed map. Then $X \times Z$ is normal by Fact 1. Moreover, by Fact 2 (2), $Y = \cup_{i=0}^{\infty} Y_i, Y_i$ is closed, discrete for each $i > 0$ and $f^{-1}(y)$ is countably compact for each $y \in Y_0$ and $Y_0 \cap (\cup_{i=1}^{\infty} Y_i) = \emptyset$.
(I). Let A and B are disjoint closed subsets in $X \times Y$ such that $A, B \subset X \times Y_0$. Then there are disjoint open sets U and V in $X \times Y$ such that $A \subset U$ and $B \subset V$.
Proof is quite similar to that of (I) in Theorem 1.
(II). Let A and B are disjoint closed subsets in $X \times Y$ such that $A \subset X \times Y_n$ for some $n > 0$. Then A and B are separated by open sets in $X \times Y$.
(Proof). Put $A_y = \{x \in X | \langle x, y \rangle \in A\}$ for each $y \in Y_n$. Since Y_n is discrete and Y is collectionwise normal, there is a discrete collection $\{U(y) | y \in Y_n\}$ of open sets in Y such that $y \in U(y)$ for each y. Then
(*) for each $y \in Y_n$, there is an open set G_y in $X \times Y$ such that $A_y \times \{y\} \subset G_y \subset X \times U(y)$ and $\overline{G_y} \cap B = \emptyset$.
(Proof of (*)). For each $x \in A_y$, there are open sets $V(x)$ in X and W_x in Y such that $x \in V(x), y \in W_x \subset U(y)$ and $\overline{V(x) \times W_x} \cap B = \emptyset$. Then $\{V(x) | x \in A_y\} \cup \{X \setminus A_y\}$ is an open cover of X. Since X is paracompact, there is a locally finite open cover $\mathcal{V} = \{V_x | x \in A_y\} \cup \{V'\}$ of X such that $V_x \subset V(x)$ and $V' \subset X \setminus A_y$. Put $G_y = \cup \{V_x \times W_x | x \in A_y\}$. Then G_y has the desired properties.

Let $G = \cup \{G_y | y \in Y_n\}$. Then G is open in $X \times Y$ such that $G \supset A$ and $\overline{G} \cap B = \emptyset$.
(III). Now we finish the proof. Let A and B are disjoint closed subsets in $X \times Y$. Put $A_n = A \cap (X \times Y_n)$ for each $n > 0$. Then, by (II), there are open sets G_n in $X \times Y$ such that $A_n \subset G_n$ and $\overline{G_n} \cap B = \emptyset$. Put $A_0 = A \setminus \cup_{n=1}^{\infty} G_n$. Then, for each $n > 0$, A_0 and $X \times Y_n$ are disjoint closed subsets in $X \times Y$. Therefore, by (II), there are open sets M_n, L_n in $X \times Y$ such that $A_0 \subset M_n, X \times Y_n \subset L_n$ and $M_n \cap L_n = \emptyset$. Hence $\overline{M_n} \cap L_n = \emptyset$. Put $B_0 = B \setminus \cup_{n=1}^{\infty} L_n$. Then A_0 and B_0 are disjoint closed subsets in $X \times Y$ such that $A_0, B_0 \subset X \times Y_0$. By (I), there are disjoint open sets M_0 and L_0 such that $A_0 \subset M_0$ and $B_0 \subset L_0$. Then $A_0 \subset \cap_{n=0}^{\infty} M_n$ and $(\cap_{n=0}^{\infty} \overline{M_n}) \cap (\cup_{n=0}^{\infty} L_n) = \emptyset$. Since $\cup_{n=0}^{\infty} L_n \supset B, (\cap_{n=0}^{\infty} \overline{M_n}) \cap B = \emptyset$. Since $X \times Y$ has property (δ), there is an open set G_0 in $X \times Y$ such that $\cap_{n=0}^{\infty} M_n \subset G_0$ and $\overline{G_0} \cap B = \emptyset$. Thus $A_0 \subset G_0$ and $\overline{G_0} \cap B = \emptyset$. Therefore $A \subset \cup_{n=0}^{\infty} G_n$ and $\overline{G_n} \cap B = \emptyset$ for each $n \geq 0$.

Similarly we can choose open sets H_n in $X \times Y$ such that $B \subset \cup_{n=0}^{\infty} H_n$ and $\overline{H_n} \cap B = \emptyset$ for each $n \geq 0$. Hence A and B are separated by open sets in $X \times Y$ by Fact 3.

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