Scientiae Mathematicae Japonicae Online, **Vol. 7**, (2002), 531–538

SEMILINEAR WAVE EQUATIONS WITH SOME KIND OF NONLINEAR DAMPING IN HIGHER SPACE DIMENSIONS

SOICHIRO KATAYAMA

Received September 6, 2002

ABSTRACT. We consider the initial – boundary value problem for semilinear wave equations with nonlinear damping:

$$u_{tt} - \Delta u + a|u|^{m-1}u_t = b|u|^{p-1}u \quad \text{in } (0,\infty)\times\Omega,$$

where Ω is a domain in $\mathbb{R}^n$ with a smooth boundary, $m > 1$ and $p > 1$, while a and b are positive constants. We impose the Dirichlet condition on the boundary. In this paper, we prove global existence and uniqueness of a certain weak solution to this problem under the condition $p \le m$.

1. INTRODUCTION

Let Ω be a domain in $\mathbb{R}^n$ with a smooth boundary. All the functions which will appear below are supposed to be real-valued. We consider the initial – boundary value problem in Ω for seminlinear wave equations with nonlinear damping:

$$\square u + Q(u, u_t) = F(u) \quad \text{in } (0,\infty)\times\Omega, \tag{1.1}$$

$$u(0,x) = u_0(x),\ u_t(0,x) = u_1(x) \text{ for } x \in \Omega, \tag{1.2}$$

$$u(t,x) = 0 \text{ for } (t,x) \in (0,\infty)\times\partial\Omega, \tag{1.3}$$

where $\square$ is the d'Alembertian ($\square = \partial_t^2 - \Delta_x$), and Q represents nonlinear damping, i.e., we assume

$$Q(u,v)v \ge 0 \text{ for any } u, v \in \mathbb{R}. \tag{1.4}$$

If $\Omega = \mathbb{R}^n$, we regard the problem as the initial value problem, and always neglect the boundary condition (1.3) in the following.

We want to look for solutions to (1.1) – (1.2), which are at least in $H_0^1(\Omega)$ for almost every $t > 0$.

For a while, let $Q(u,v) = a|v|^{m-1}v$ and $F(u) = b|u|^{p-1}u$, where m, p, a and b are constants satisfying $m > 1$, $p > 1$ and $a \ge 0$. The case where $a \ge 0$ and $b \le 0$, or the case where $a = 0$ and $b > 0$ are studied by many authors (for example, see Lions – Strauss [8], Haraux – Zuazua [3] and Glassey [2]). Roughly speaking, Q makes the solution exist globally when $a > 0$, while F may make the solution blow up in finite time when $b > 0$. Therefore it is interesting to see the relationship between Q and F for the case where both of a and b are positive. This problem under the assumption

$$\begin{cases} 1 < p \le \dfrac{n}{n-2}, & n \ge 3, \\ 1 < p < \infty, & n = 1, 2, \end{cases} \tag{1.5}$$

2000 *Mathematics Subject Classification.* 35L70.

Key words and phrases. wave equations, nonlinear damping, global existence.

was studied by Georgiev–Todorova [1] when Ω is bounded, and by Todorova [10] when $\Omega = \mathbb{R}^n$ (see also Levine – Park – Serrin [7], Ikehata [4] and Ono [9]). We note that the condition on p is imposed to ensure $|u|^{2p} \in L^2$. In [1] and [10], they proved that for any $u_0 \in H_0^1(\Omega)$ and $u_1 \in L^2(\Omega)$, there exists a global solution u to (1.1) – (1.2) which satisfies

$$u \in C\big([0,\infty); H_0^1(\Omega)\big) \cap C^1\big([0,\infty); L^2(\Omega)\big),$$
$$u_t \in L^{m+1}\big((0,T)\times\Omega\big) \text{ for any } T>0,$$

provided that $p \le m$. In contrast to this, when $p > m$, they also proved that the solution blows up in finite time for some data, and there exists *no* global solution of the above regularity for such data.

Now we want to consider another kind of nonlinear damping. More precisely, we treat the problem with $Q(u,v) = a|u|^{m-1}v$, where $a > 0$ and $m > 1$. Lions – Strauss ([8]) considered the equation

$$\Box u + a|u|^{m-1}u_t = f(t,x) \tag{1.6}$$

with initial data $u = u_0$ and $u_t = u_1$ at $t = 0$, and showed that there exists a unique global (weak) solution. More precisely, they proved that if $u_0 \in H_0^1(\Omega) \cap L^{2m}(\Omega)$, $u_1 \in L^2(\Omega)$ and $f \in W^{1,\infty}\big(0,T; L^2(\Omega)\big)$, then for any $T > 0$, (1.6) admits a unique weak solution $u \in L^\infty\big((0,T); H_0^1(\Omega) \cap L^{2m}(\Omega)\big)$ with $u_t \in L^\infty\big((0,T); L^2(\Omega)\big)$. Note that the initial data make sense in this framework, because from the equation we find $u \in C\big([0,T); L^2(\Omega)\big)$ and $u_t \in C\big([0,T); H^{-1}(\Omega) + L^q(\Omega)\big)$, where $q = 1$ when $m \ge 2$, and $q = 2/m$ when $1 < m < 2$. Here $v \in H^{-1}(\Omega) + L^q(\Omega)$ means that v can be written as $v = v_1 + v_2$ using some $v_1 \in H^{-1}(\Omega)$ and $v_2 \in L^q(\Omega)$. This space is a normed vector space endowed with the norm $\|v\|_{H^{-1}+L^q} = \inf_{(v_1,v_2)\in A(v)} \|v_1\|_{H^{-1}(\Omega)} + \|v_2\|_{L^q(\Omega)}$, where $A(v) = \{(v_1,v_2) \in H^{-1}(\Omega) \times L^q(\Omega); v_1 + v_2 = v\}$ for $v \in H^{-1}(\Omega) + L^q(\Omega)$.

Their idea of the proof is based on the fact that we have

$$\partial_t\big(|\widetilde{u}_t|^{m-1}\widetilde{u}_t\big) = m|\widetilde{u}_t|^{m-1}(\widetilde{u}_t)_t$$

for a sufficiently smooth function $\widetilde{u}$. Roughly speaking, by introducing a new unkown $\widetilde{u}$ satisfying $\widetilde{u}_t = u$, they reduced the problem to the known case where the nonlinear damping has the form $\frac{a}{m}|\widetilde{u}_t|^{m-1}\widetilde{u}_t$.

We want to investigate a similar problem to [1] and [10] for this $Q(u,u_t) = a|u|^{m-1}u_t$ with nonlinear force terms. More precisely, we consider the semilinear wave equations of the type

$$\begin{cases} \Box u + a|u|^{m-1}u_t = b|u|^{p-1}u & \text{in } (0,\infty)\times\Omega, \\ u(0,x) = u_0(x),\ u_t(0,x) = u_1(x) & \text{for } x \in \Omega, \\ u(t,x) = 0 & \text{for any } (t,x) \in (0,\infty)\times\partial\Omega, \end{cases} \tag{1.7}$$

where Ω is a domain in $\mathbb{R}^n$ with a smooth boundary (or $\Omega = \mathbb{R}^n$), $a > 0$, $b > 0$, $m > 1$ and p satisfies (1.5). In [5] the author et al. considered this problem in one space dimension ($n = 1$), and showed that the condition $m \ge p$ implies the existence and the uniqueness of a global solution in the class $C\big([0,\infty); H_0^1(\Omega)\big) \cap C^1\big([0,\infty); L^2(\Omega)\big)$, while $p > m$ implies blowing up of the solution for some data. In the proof, a similar idea to that in [8] was used. Unfortunately, our proof in [5] is only applicable to the case $n = 1$, because the proof relies on the embedding $H_0^1 \subset L^\infty$, which is only available for the case $n = 1$.

In this paper, we will prove existence and uniqueness of a global weak solution, which is similar to that in [8], for higher space dimensional cases under the condition $m \ge p$. Our main result is the folllowing:

Theorem 1.1. *Assume that* (1.5) *is fulfilled. Suppose that* $u_0 \in H_0^1(\Omega) \cap L^{2m}(\Omega)$ *and* $u_1 \in L^2(\Omega)$.

If $m \geq p$, *then there exists a unique solution* u *to the problem* (1.7), *satisfying*

$$u \in L^\infty\big(0,T; H_0^1(\Omega) \cap L^{2m}(\Omega)\big),\ u_t \in L^\infty\big(0,T; L^2(\Omega)\big) \text{ for any } T > 0,$$

and

$$u \in C\big([0,\infty); L^2(\Omega)\big),\ u_t \in C\big([0,\infty); H^{-1}(\Omega) + L^q(\Omega)\big),$$

where

$$q = \begin{cases} 1 & \text{when } m \geq 2, \\ 2/m & \text{when } 1 < m < 2. \end{cases}$$

Our strategy is as follows: First, we will introduce a reduced problem, following the method of Lions – Strauss [8]. The existence part of the theorem is proved by applying the compactness method of Lions – Strauss to the reduced problem, with the help of an *a priori* estimate which was essentially used in [5]. The proof will be given in Section 3.

The uniqueness part of our theorem is rather delicate, because our solution is not regular enough for us to apply the classical uniqueness result directly, and careful treatment of nonlinear terms is needed. The proof of uniqueness will be given in Setion 2.

2. Proof of the Uniqueness

In this section, we will give a proof for the uniqueness.

For $y \in \mathbb{R}$ and $m > 1$, we define $Q_m(y) = |y|^{m-1}y$. Note that Q_m belongs to C^1, and we have $Q_m'(y) = m|y|^{m-1}$. Hence Q_m is an increasing function of y, and we have

$$\{Q_m(y) - Q_m(z)\}(y - z) \geq 0 \tag{2.1}$$

for any y, $z \in \mathbb{R}$.

Let u be a solution to (1.7) with the regulaity mentioned in Theorem 1.1. Set

$$v(t,x) = \int_0^t u(\tau,x)d\tau. \tag{2.2}$$

Then we have

$$\begin{cases} v \in C\big([0,\infty); H_0^1(\Omega) \cap L^{2m}(\Omega)\big) \cap C^1\big([0,\infty); L^2(\Omega)\big), \\ v_t \in L^\infty\big(0,T; H_0^1(\Omega) \cap L^{2m}(\Omega)\big) \cap C\big([0,\infty); L^2(\Omega)\big), \\ v_{tt} \in L^\infty\big(0,T; L^2(\Omega)\big) \end{cases} \tag{2.3}$$

for any $T > 0$. By integrating (1.7), we obtain

$$\begin{aligned} \Box v(t,x) + \frac{a}{m}|v_t(t,x)|^{m-1}v_t(t,x) =& b\int_0^t |v_t(\tau,x)|^{p-1}v_t(\tau,x)d\tau \\ &+ \frac{a}{m}|u_0(x)|^{m-1}u_0(x) + u_1(x) \end{aligned} \tag{2.4}$$

with $v = 0$ and $v_t = u_0$ at $t = 0$.

Now let $\widetilde{u}$ be another solution to (1.7), and set $\widetilde{v} = \int_0^t \widetilde{u}(\tau,x)d\tau$ as above. Since we have the same equation as (2.4) for $\widetilde{v}$, we obtain

$$\begin{aligned} \Box\,(v - \widetilde{v})(t,x) &+ \frac{a}{m}\left\{Q_m\big(v_t(t,x)\big) - Q_m\big(\widetilde{v}_t(t,x)\big)\right\} \\ &= b\int_0^t \left\{Q_p\big(v_t(\tau,x)\big) - Q_p\big(\widetilde{v}_t(\tau,x)\big)\right\} d\tau. \end{aligned} \tag{2.5}$$

We write $v(t,x)-\widetilde{v}(t,x)$ as $w(t,x)$, and the right-hand side of (2.5) as $G(t,x)$, respectiveily. By integrating the equation (2.5) multiplied by w_t over Ω, we find

$$
\begin{aligned}
\frac{d}{dt}\int_\Omega \frac{w_t^2+|\nabla_x w|^2}{2}(t,x)dx + \frac{a}{m}\int_\Omega \left\{Q_m(v_t)-Q_m(\widetilde{v}_t)\right\} w_t(t,x)dx \\
= \int_\Omega G(t,x)w_t(t,x)dx.
\end{aligned} \tag{2.6}
$$

Thanks to (2.1), we see that the second term in the left-hand side of (2.6) is non-negative. Therefore, noting that $w=w_t=0$ at $t=0$, we obtain

$$
E(t)^2 \le \int_0^t \int_\Omega G(s,x)w_t(s,x)dxds, \tag{2.7}
$$

where

$$
E(t)^2 = \frac{1}{2}\int_\Omega \left\{|w_t(t,x)|^2 + |\nabla_x w(t,x)|^2\right\} dx.
$$

The above derivation of (2.7) is formal, but we can justify it because, as we will see in the below, (2.3) implies $w \in C\big([0,T); H_0^1(\Omega)\big) \cap C^1\big([0,T); L^2(\Omega)\big)$, $w_t \in L^\infty(0,T;H_0^1)$ and $G_t \in L^\infty\big(0,T; L^{\frac{2p}{2p-1}}\big)$.

Now we would like to estimate the right-hand side of (2.7). By integration by parts, we obtain

$$
\int_0^t\int_\Omega (Gw_t)(s,x)dxds = \int_\Omega G(t,x)w(t,x)dx - \int_0^t\int_\Omega G_t(s,x)w(s,x)dxds. \tag{2.8}
$$

By Hölder's inequality, we have

$$
\int_0^t\int_\Omega |G_t(s,x)w(s,x)|dxds \le \int_0^t \|G_t(s,\cdot)\|_{L^{\frac{2p}{2p-1}}} ds \sup_{0\le s<t} \|w(s,\cdot)\|_{L^{2p}}. \tag{2.9}
$$

Similarly we have

$$
\begin{aligned}
\int_\Omega |G(t,x)w(t,x)|dx \le & C\|G(t,\cdot)\|_{L^{\frac{2p}{2p-1}}} \|w(t,\cdot)\|_{L^{2p}} \\
\le & C\int_0^t \|G_t(s,\cdot)\|_{L^{\frac{2p}{2p-1}}} ds \|w(t,\cdot)\|_{L^{2p}}.
\end{aligned} \tag{2.10}
$$

From the definition of G, using the mean value theorem and Hölder's inequality, we find

$$
\begin{aligned}
\|G_t(t,\cdot)\|_{L^{\frac{2p}{2p-1}}} \le & C\left\|\big(|v_t(t,\cdot)|^{p-1} + |\widetilde{v}_t(t,\cdot)|^{p-1}\big)\right\|_{L^{\frac{2p}{p-1}}} \|(v_t-\widetilde{v}_t)(t,\cdot)\|_{L^2} \\
\le & C\big(\|v_t(t,\cdot)\|_{L^{2p}} + \|\widetilde{v}_t(t,\cdot)\|_{L^{2p}}\big)^{p-1} \|w_t(t,\cdot)\|_{L^2}.
\end{aligned} \tag{2.11}
$$

Because of (1.5), we have $2 < 2p \le n/(n-2)$ for $n \ge 3$, and $2 < 2p < \infty$ for $n=1,2$. Hence by Sobolev's embedding theorem, we have

$$
\|f\|_{L^{2p}(\Omega)} \le C\|f\|_{H_0^1(\Omega)} \text{ for any } f \in H_0^1(\Omega). \tag{2.12}
$$

From (2.8) – (2.11) with the help of (2.12), we obtain

$$
\left|\int_0^t\int_\Omega G(s,x)w(s,x)dxds\right| \le CM_T^{p-1}\int_0^t \|w_t(s,\cdot)\|_{L^2} ds \sup_{0\le s\le t} \|w(s,\cdot)\|_{H_0^1} \tag{2.13}
$$

for $0 \le t \le T$, where $M_T = \sup_{s\in[0,T]}\left\{\|v_t(s,\cdot)\|_{H_0^1(\Omega)} + \|\widetilde{v}_t(s,\cdot)\|_{H_0^1(\Omega)}\right\}$.

Set $E_*(t) = \sup_{0\le s\le t} E(s)$. We have $\|w_t(t,\cdot)\|_{L^2} \le CE_*(t)$ and $\|\nabla_x w(t,\cdot)\|_{L^2} \le CE_*(t)$. We also have

$$
\|w(t,\cdot)\|_{L^2} \le \int_0^t \|w_t(s,\cdot)\|_{L^2} ds \le TE_*(t) \text{ for } 0 \le t \le T,
$$

and hence we get $\sup_{0\le s\le t}\|w(s,\cdot)\|_{H_0^1}\le C(1+T)E_*(t)$. Now, using (2.7) and (2.13), we obtain

$$E_*(t)^2 \le CM_T^{p-1}(1+T)E_*(t)\int_0^t E_*(s)ds \text{ for } 0\le t\le T. \tag{2.14}$$

Gronwall's lemma applied to (2.14) implies $E_*(t)=0$ for any $t\in[0,T]$, which leads to $u(t,\cdot)=\widetilde{u}(t,\cdot)$ for $0\le t\le T$, since we have

$$\|(u-\widetilde{u})(t,\cdot)\|_{L^2(\Omega)}=\|(v_t-\widetilde{v}_t)(t,\cdot)\|_{L^2(\Omega)}\le E_*(t)$$

by the definition of v and $\widetilde{v}$. This completes the proof of uniqueness.

3. Proof of the Existence

In the following, we write v' for v_t, and v'' for v_{tt}. Fix arbitrary $T>0$. We will construct a global weak solution v of (2.4) satisfying

$$\begin{cases} v\in L^\infty\big(0,T;H_0^1(\Omega)\big),\\ v'\in L^\infty\big(0,T;H_0^1(\Omega)\big)\cap L^{m+1}\big((0,T)\times\Omega\big),\\ v''\in L^\infty\big(0,T;L^2(\Omega)\big).\end{cases} \tag{3.1}$$

For a while, let us assume that there exists a solution v of (2.4) satisfying (3.1). Define

$$f(t,x)=b\int_0^t |v'(\tau,x)|^{p-1}v'(\tau,x)d\tau+\frac{a}{m}|u_0(x)|^{m-1}u_0(x)+u_1(x). \tag{3.2}$$

Then we have

$$\|f(t,\cdot)\|_{L^2}\le CT\sup_{0\le s\le T}\|v'(\tau,\cdot)\|_{H_0^1}^p+C\|u_0\|_{L^{2m}}^m+\|u_1\|_{L^2}<\infty,$$

and we find that f belongs to $L^1\big(0,T;L^2(\Omega)\big)$. Similarly, we have $f'\in L^1\big(0,T;L^2(\Omega)\big)$. Therefore we can apply the regularity theorem of Lions –Strauss (see Theorem 1.2 in [8]) to conclude that the regularity of v in fact is (2.3). Set $u(t,x)=v'(t,x)$. It is not difficult to check that u is the desired solution to (1.7).

It remains to construct v satisfying (3.1). Since $H_0^1(\Omega)\cap L^{m+1}(\Omega)$ is separable, we can find a basis $W=\{w_i\}_{i=1}^\infty$ of $H_0^1\cap L^{m+1}$. In other words, we can find a subset W of $H_0^1\cap L^{m+1}$ such that elements in W are linearly independent, and that the set of functions which are finite combinations of elements in W is dense in $H_0^1\cap L^{m+1}$. We may assume $w_1=u_0$ unless $u_0=0$, because $u_0\in H_0^1\cap L^{2m}\subset H_0^1\cap L^{m+1}$.

We construct approximate solutions $\{v_N\}_{N=1}^\infty$ by solving the following:

$$v_N(t,x)=\sum_{i=1}^N a_{Ni}(t)w_i(x), \tag{3.3}$$

$$\langle v_N'',w_j\rangle+\sum_{i=1}^n\langle\partial_{x_i}v_N,\partial_{x_i}w_j\rangle+\frac{a}{m}\langle Q_m(v_N'),w_j\rangle=\langle f_N,w_j\rangle \quad \text{for } 1\le j\le N, \tag{3.4}$$

$$v_N(0,x)=0,\ v_N'(0,x)=u_0(x), \tag{3.5}$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product in $L^2(\Omega)$, and f_N is defined by

$$f_N(t,x)=b\int_0^t |v_N'(\tau,x)|^{p-1}v_N'(\tau,x)d\tau+\frac{a}{m}|u_0(x)|^{m-1}u_0(x)+u_1(x).$$

Writing $a_N(t) = (a_{N1}(t), \cdots, a_{NN}(t))^T$ and $B_N = (b_{ij})_{i,j=1,\cdots,N}$ with $b_{ij} = \langle w_j, w_i \rangle$, we can see (3.4) is a system of ordinary differential equations of the form

$$B_N a_N''(t) = C_N(a_N(t), a_N'(t)) + \int_0^t D_N(a_N'(\tau))d\tau, \tag{3.6}$$

where C_N and D_N are C^1-functions of their arguments. Observe that B_N is a regular matrix since elements in W are linearly independent. By the classical argument, we can easily show the existence of C^3-solution $a_N(t)$ to the systems of the form (3.6) in some time interval $[0, \delta_N]$, where δ_N depends only on $|a_N(0)|$ and $|a_N'(0)|$. Consequently we also get the solution v_N to (3.4) – (3.5) for $0 \le t \le \delta_N$.

Now we are going to get *a priori* estimates to show that $|a_N(t)|$ and $|a_N'(t)|$ stays bounded as far as a_N exists. Once we get such estimates, we can choose $\delta_N = T$. By (3.3) we have $a_N(t) = B_N^{-1} X_N$ with $X_N^T = (\langle v_N, w_j \rangle)_{j=1,\cdots,N}$. Therefore we get

$$|a_N(t)| \le c_N \|v_N(t,\cdot)\|_{L^2} \le c_N T \sup_{0\le s\le t} \|v_N'(t)\|_{L^2},\ |a_N'(t)| \le c_N \|v_N'(t,\cdot)\|_{L^2} \tag{3.7}$$

for $0 \le t < T$, where c_N is a constant depending only on N. It has turned out that our task is to get a bound for $\|v_N'(t,\cdot)\|_{L^2}$.

We differentiate (3.4) to get

$$\langle v_N''', w_j \rangle + \sum_{i=1}^n \langle \partial_{x_i} v_N', \partial_{x_i} w_j \rangle + a\langle |v_N'|^{m-1} v_N'', w_j \rangle = b\langle |v_N'|^{p-1} v_N', w_j \rangle \quad (j = 1, \cdots, N).$$

By multiplying each of the above equations by $a_j''(t)$ and then summing them over $1 \le j \le N$, we obtain

$$\langle v_N''', v_N'' \rangle + \sum_{i=1}^n \langle \partial_{x_i} v_N', \partial_{x_i} v_N'' \rangle + a\langle |v_N'|^{m-1} v_N'', v_N'' \rangle = b\langle |v_N'|^{p-1} v_N', v_N'' \rangle. \tag{3.8}$$

Now we define $H_N(t)$ by

$$H_N(t) = \frac{1}{2}\Big(\|v_N''(t)\|_{L^2}^2 + \sum_{i=1}^n \|\partial_{x_i} v_N'(t)\|_{L^2}^2 \Big) + \frac{b}{p+1} \|v_N'(t)\|_{L^{p+1}}^{p+1}. \tag{3.9}$$

Let P_N be the orthogornal projection in $L^2(\Omega)$ onto the subspace generated by $w_1, \cdots w_N$. Then we have $v_N''(0) = P_N u_1$ since we have $\langle v_N''(0), w_j \rangle = \langle u_1, w_j \rangle$ for $1 \le j \le N$ from (3.4) and (3.5). Therefore we conclude $\|v_N''(0)\|_{L^2} \le \|u_1\|_{L^2}$. We also have $\|u_0\|_{L^{p+1}} \le C\big(\|u_0\|_{L^{2p}} + \|u_0\|_{L^2}\big) \le C\|u_0\|_{H_0^1}$. Hence we get

$$H_N(0) \le C\left(\|u_0\|_{H_0^1}^2 + \|u_1\|_{L^2}^2 + \|u_0\|_{H_0^1}^{p+1} \right), \tag{3.10}$$

where C is a constant independent of N.

By straightforward calculation, we get

$$H_N'(t) = \langle v_N''', v_N'' \rangle + \sum_{i=1}^n \langle \partial_{x_i} v_N', \partial_{x_i} v_N'' \rangle + b\langle |v_N'|^{p-1} v_N', v_N'' \rangle. \tag{3.11}$$

From (3.8) and (3.11) we find

$$H_N'(t) = -a\langle |v_N'|^{m-1} v_N'', v_N'' \rangle + 2b\langle |v_N'|^{p-1} v_N', v_N'' \rangle. \tag{3.12}$$

Let $\varepsilon > 0$. Taking the condition $1 < p \le m$ into account, we get

$$\begin{aligned} \big||v_N'|^{p-1} v_N' v_N''\big| &\le \varepsilon |v_N'|^{p-1} |v_N''|^2 + (4\varepsilon)^{-1} |v_N'|^{p+1} \\ &\le \varepsilon\big(|v_N''|^2 + |v_N'|^{m-1}|v_N''|^2\big) + (4\varepsilon)^{-1}|v_N'|^{p+1}. \end{aligned} \tag{3.13}$$

Fix some ε sufficiently small to satisfy $2b\varepsilon \le a$. Then by (3.12) and (3.13) we obtain

$$H_N'(t) \le 2b\varepsilon \|v_N''\|_{L^2}^2 + b(2\varepsilon)^{-1}\|v_N'\|_{L^{p+1}}^{p+1} \le CH_N(t). \tag{3.14}$$

Now Gronwall's lemma implies $\|H_N(t)\|_{L^2} \le H_N(0)e^{CT}$ for $0 \le t < T$, which leads to

$$\sup_{0\le t\le T}\left(\|v_N'(t,\cdot)\|_{H_0^1} + \|v_N''(t,\cdot)\|_{L^2}\right) \le C_T \tag{3.15}$$

with some constant C_T which depends only on T, $\|u_0\|_{H_0^1}$ and $\|u_1\|_{L^2}$, since we have $\|v_N'(t)\|_{L^2} \le \|v_N'(0)\|_{L^2} + \int_0^t \|v_N''(s)\|_{L^2}ds$.

From this, as we have stated in the above, we see that the solution $v_N(t)$ to (3.4) – (3.5) exists in the time interval $0 \le t \le T$ for each N. We also have

$$\sup_{0\le t\le T}\|f_N(t,\cdot)\|_{L^2} \le C_T. \tag{3.16}$$

Next we want to take the limit of v_N to obtain the solution to (2.4). Returning to (3.4), and going in a similar way to the derivation of (3.8), we get

$$\begin{aligned}&\frac{1}{2}\Big(\|v_N'(t,\cdot)\|_{L^2}^2 + \sum_{i=1}^n \|\partial_{x_i}v_N(t,\cdot)\|_{L^2}^2\Big) + \frac{a}{m}\int_0^t\int_\Omega |v_N'(s,x)|^{m+1}dxds\\ &\qquad = \frac{1}{2}\|u_0\|_{L^2} + \int_0^t \langle f_N(s,\cdot), v_N'(s,\cdot)\rangle ds.\end{aligned} \tag{3.17}$$

From this, using (3.15) and (3.16), we find

$$\sup_{0\le t\le T}\|v_N(t,\cdot)\|_{H_0^1(\Omega)}^2 + \|v_N'\|_{L^{m+1}((0,T)\times\Omega)}^{m+1} \le C_T. \tag{3.18}$$

From the uniform bounds (3.15) and (3.18), we conclude that there exists some function v such that

$$\begin{cases} v_N \to v & \text{weakly star in } L^\infty\big(0,T;H_0^1(\Omega)\big),\\ v_N' \to v' & \text{weakly star in } L^\infty\big(0,T;H_0^1(\Omega)\big) \text{ and weakly in } L^{m+1}\big((0,T)\times\Omega\big),\\ v_N'' \to v'' & \text{weakly star in } L^\infty\big(0,T;L^2\big)\end{cases}$$

as $N \to \infty$, if we take an appropriate subsequence.

We also see that $|v_N'|^{m-1}v_N$ is bounded in $L^{\frac{m+1}{m}}\big((0,T)\times\Omega\big)$ from (3.18), and that f_N is bounded in $L^2\big((0,T)\times\Omega\big)$ from (3.16). Hence there exist two functions Φ and Ψ such that

$$\begin{cases} Q_m(v_N') = |v_N'|^{m-1}v_N' \to \Phi & \text{weakly in } L^{\frac{m+1}{m}}\big((0,T)\times\Omega\big),\\ f_N \to \Psi & \text{weakly in } L^2\big((0,T)\times\Omega\big),\end{cases} \tag{3.19}$$

as $N \to \infty$, if we take an appropriate subsequence. Let K be a compact subset of $(0,T)\times\Omega$. From (3.15) and (3.18), we also have $v_N' \to v'$ weakly in $H_0^1(K)$, and since $H_0^1(K)$ is compact in $L^2(K)$, taking a further subsequence if necessary, we get $v_N' \to v'$ strongly in $L^2(K)$. Hence, taking a further subsequence again if necessary, we find that $v_N' \to v'$ a.e. in $(0,T)\times\Omega$. This is sufficient to conclude that $\Phi = |v'|^{m-1}v'$ and $\Psi = f$ in the above, where f is given by (3.2). Now we get

$$\int_0^T \big\langle Q_m\big(v_N'(t,\cdot)\big), w_j\big\rangle\,\phi(t)dt \to \int_0^T \big\langle Q_m\big(v'(t,\cdot)\big), w_j\big\rangle\,\phi(t)dt, \tag{3.20}$$

$$\int_0^T \langle f_N(t,\cdot), w_j\rangle\phi(t)dt \to \int_0^T \langle f(t,\cdot), w_j\rangle\phi(t)dt \tag{3.21}$$

for any $\phi \in C_0^\infty\big((0,T)\big)$, since $w_j \in L^2(\Omega) \cap L^{m+1}(\Omega)$. Finally we can pass to the limit in (3.4) and we find that

$$\langle v''(t), w\rangle + \sum_{j=1}^{n} \langle \partial_{x_i} v(t), \partial_{x_i} w\rangle + \langle Q_m(v'(t)), w\rangle = \langle f(t), w\rangle \tag{3.22}$$

for any $w \in W$ in the sense of distribution over $(0,T)$. Since W is a basis of $H_0^1 \cap L^{m+1}$, (3.22) remains true for any $w \in H_0^1(\Omega) \cap L^{m+1}(\Omega)$, and we see that v satisfies the desired equation (2.4) in the sense of distribution over $(0,T) \times \Omega$. Obviously $v(0) = 0$, $v'(0) = u_0$, and v satisfies (3.1). This completes the proof of the existence part of Theorem 1.1.

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Department of Mathematics, Wakayama University,
930 Sakaedani,Wakayama 640-8510, JAPAN
E-mail address: soichi-k@math.edu.wakayama-u.ac.jp