ONE CLASS OF INVOLUTORY ANTIAUTOMORPHISM OF RATIONAL ROTATION ALGEBRAS

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Abstract. A rational rotation algebra A_θ is a universal C^*-algebra generated by two unitaries U, V with relation $VU = \rho UV$, where $\rho = e^{2\pi i\theta}, 0 \leq \theta \leq 1$ is rational. Any involutary antiautomorphism of a rational rotation algebra is corresponding to an involution of the torus T^2, the spectrum of rational rotation algebra. In this paper, we prove that there is no involutory antiautomorphism in A_θ associated with the involution $\tau_1 : (\lambda, \mu) \mapsto (-\lambda, \mu)$ of the torus.

Let A_θ be the universal C^*-algebra generated by the two unitaries U, V with $VU = \rho UV$, where $\rho = e^{2\pi i\theta}, 0 \leq \theta \leq 1$. When $\theta = p/q$ is rational, A_θ is called rational. A rational rotation algebra can be regarded as an algebra of continuous function from the square $\mathbf{I}^2$ to the matrix algebra $M_q(\mathbb{C})$. The spectrum of A_θ is the torus hence its centre is isomorphic to $C(\mathbf{T}^2)$. Given an antiautomorphism α, it gives rise to a homomorphism $\widetilde{\alpha}$ of $\mathbf{T}^2$ with $\alpha f(x) = f(\widetilde{\alpha}(x))$, for any $x \in \mathbf{T}^2, f \in A_\theta$. For any antiautomorphism α of $A_{\frac{p}{q}}$, let $\sigma(\alpha)$ be the associated homomorphism. Restricting the antiautomorphism α to the centre $C(\mathbf{T}^2)$ of $A_{\frac{p}{q}}$, then it establishes a bijection between the involutory antiautomorphism of $A_{\frac{p}{q}}$ and the involutions (including the identity homomorphism) of $\mathbf{T}^2$. Now any involution of $\mathbf{T}^2$ is conjugate to one of the following five ones

$\tau_1 : \tau_1(\lambda, \mu) = (-\lambda, \mu)$,
$\tau_2 : \tau_2(\lambda, \mu) = (\bar{\lambda}, \mu)$,
$\tau_3 : \tau_3(\lambda, \mu) = (-\lambda, \bar{\mu})$,
$\tau_4 : \tau_4(\lambda, \mu) = (\lambda, \bar{\mu})$,
$\tau_5 : \tau_5(\lambda, \mu) = (\mu, \lambda)$.

For convenience, we will denote the identity homomorphism of $\mathbf{T}^2$ by τ_0. In [3] we proved briefly there is no involutory antiautomorphism associated with τ_1. In this paper we will employ a more general approach, which applys to other cases, to show this theorem.

According to the analysis of the case $q = 2$ and $\sigma(\phi) = \tau_1$, in [3] and the relation between principal bundles and their associated fibre bundles, to investigate involutory antiautomorphism of $A_{\frac{1}{2}}$ associated with τ_1, we can start from studying the principal PU_2'-bundles over $\mathbf{T}^2$. As the first step we give the classification of principal PU_2'-bundles over $\mathbf{T}$ and the conjugacy homotopy classes of their automorphisms.

Lemma 1. Let k be the transformation of $\mathbb{C}^2$ with $k(x, y) = (\bar{x}, \bar{y})$. Then each principal PU_2'-bundle over $\mathbf{T}$ is either isomorphic to the trivial $F_1 = PU_2' \times \mathbf{T}$ or isomorphic to F_2 which is obtained from $PU_2' \times I$ by pasting $([u], 0)$ to $([ku], 1)$.

Proof. There are two connected components, one containing I_2 and one containing k. By Lemma 3.1 of [3], we obtain the principal PU_2'-bundles F_1 over $\mathbf{T}$.

As was shown in Proposition 2.1 and Lemma 3.2 of [2] the conjugacy homotopy classes of the automorphisms of a principal PU_2'-bundle over $\mathbf{T}$ are related to the fundamental group of PU_2'. The following Lemma gives $\pi_1(PU_q')$.

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Lemma 2. $\pi_1(PU_q') \cong \mathbb{Z}_q$.

Proof. From [4,8.12.3] we have $\pi_1(U_q) = \mathbb{Z}$. From the fibration $\mathbf{T} \to U_q \to PU_q$ we have $\pi_1(\mathbf{T}) \to \pi_1(U_q) \to \pi_1(PU_q) \to \pi_0(\mathbf{T})$. The map $\pi_1(\mathbf{T}) \cong \mathbb{Z} \to \mathbb{Z} \cong \pi_1(U_q)$ maps $n = deg(l)$ to $nq = deg(l^q)$ for each loop l. So $\pi_1(PU_q) \cong \mathbb{Z}/q\mathbb{Z} \cong \mathbb{Z}_p$. Hence $\pi_1(PU_q') \cong \mathbb{Z}_q$.

Lemma 3. Define automorphisms of F_1 by

(1) $\alpha_1([u],\lambda) = ([u],\lambda)$;

(2) $\alpha_2([u],\lambda) = ([\begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} u],\lambda)$;

(3) $\alpha_3([u],\lambda) = ([ku],\lambda)$;

(4) $\alpha_4([u],\lambda) = ([\begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} ku],\lambda)$. Then each automorphism α of F_1 is homotopic to α_i for some $i \in \{1,2,3,4\}$.

Similarly, if we define automorphisms of F_2 by

(5) $\beta_1[([u],s)] = [([u],s)]$;

(6) $\beta_2[([u],s)] = [([\begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} u],s)]$, where $\lambda = e^{2\pi is}$;

(7) $\beta_3[([u],s)] = [([ku],s)]$;

(8) $\beta_4[([u],s)] = [([\begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} ku],s)]$, where $\lambda = e^{2\pi is}$.

Then each automorphism β of F_2 is homotopic to β_i for some $i \in \{1,2,3,4\}$.

Proof. Let E be a principal PU_2'-bundle over $\mathbf{T}$. Then by lemma 3 .1 of [2] any automorphism α of E corresponds to $\widetilde{\alpha} \in Map(\mathbf{T}, PU_2')$ with $\widetilde{\alpha}(1) = e$ or k and with $\alpha[([u],s)] = [([\widetilde{\alpha}_\lambda u],s)]$, where $\lambda = e^{2\pi is}$. Furthermore for two automorphisms α,β of E, if $\widetilde{\alpha}$ is homotopic to $\widetilde{\beta}$ then α is homotopic to β.

Now $\pi_1(PU_2') \cong \mathbb{Z}_2$ and $l_1 : \lambda \mapsto [I_2], l_2 : \lambda_1 \mapsto [\begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}]$ are non-homotopic loops based on $[I_2]$. Also $l_3 : \lambda \mapsto [k], l_4 : \lambda \mapsto [\begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} k]$ are non-homotopic loops based on $[k]$. The corresponding antomorphisms of F_1 are $\alpha_1,\alpha_2,\alpha_3,\alpha_4$ respectively, and the corresponding automorphisms of F_2 are $\beta_1,\beta_2,\beta_3,\beta_4$ respectively. Given any automorphism α of F_1 and β of F_2, $\widetilde{\alpha}$ or $\widetilde{\beta}$ is homotopic to l_i for some $i \in \{1,2,3,4\}$. So α is homotopic to α_i for some $i \in \{1,2,3,4\}$ or β is homotopic to β_i for some $i \in \{1,2,3,4\}$.

Proposition 4. All principal PU_2'-bundles over $\mathbf{T}^2$ are isomorphic to one of the following

(1) $F_{1\alpha_1} = F_1 \times_{\alpha_1} \mathbf{T} = PU_2' \times \mathbf{T}^2$;

(2) $F_{1\alpha_2} = F_1 \times_{\alpha_2} \mathbf{T}$ which is obtained from $PU_2' \times \mathbf{T} \times I$ by pasting $([u],\lambda,0)$ to $([\begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} u],\lambda,1)$;

(3) $F_{1\alpha_3} = F_1 \times_{\alpha_3} \mathbf{T}$ which is obtained from $PU_2' \times \mathbf{T} \times I$ by pasting $([u],\lambda,0)$ to $([ku],\lambda,1)$;

(4) $F_{1\alpha_4} = F_1 \times_{\alpha_4} \mathbf{T}$ which is obtained from $PU_2' \times \mathbf{T} \times I$ by pasting $([u],\lambda,0)$ to $([\begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} ku],\lambda,1)$;

(5) $F_{2\beta_1} = F_2 \times_{\beta_1} \mathbf{T}$ which is obtained from $PU_2' \times I \times I$ by pasting $([u],0,t)$ to $([ku],1,t)$ and pasting $([u],s,0)$ to $([u],s,1)$;

(6) $F_{2\beta_2} = F_2 \times_{\beta_2} \mathbf{T}$ which is obtained from $PU_2' \times I \times I$ by pasting $([u],0,t)$ to $([ku],1,t)$ and pasting $([u],s,0)$ to $([\begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} u],s,1)$; where $\lambda = e^{2\pi is}$;

(7) $F_{2\beta_3} = F_2 \times_{\beta_3} \mathbf{T}$ which is obtained from $PU_2' \times I \times I$ by pasting $([u], 0, t)$ to $([ku], 1, t)$ and pasting $([u], s, 0)$ to $([ku], s, 1)$;

(8) $F_{2\beta_4} = F_2 \times_{\beta_4} \mathbf{T}$ which is obtained from $PU_2' \times I \times I$ by pasting $([u], 0, t)$ to $([ku], 1, t)$ and pasting $([u], s, 0)$ to $([\begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} ku], s, 1)$, $\lambda = e^{2\pi i s}$.

Proof. This is a consequence of Lemma 2.1 of [1] and Lemma 3.2 of [2].

Proposition 5. Let $F_{1\alpha_1}, F_{1\alpha_2}, F_{1\alpha_3}, F_{1\alpha_4}, F_{2\beta_1}, F_{2\beta_2}, F_{2\beta_3}, F_{2\beta_4}$ be the principal PU_2'-bundles over $\mathbf{T}^2$ defined in Proposition 4. Then

(1) $\Gamma(F_{1\alpha_1}(M_2(\mathbb{C}))) \cong C(\mathbf{T}^2, M_2(\mathbb{C}))$ with complexification isomorphic to $C(\mathbf{T}^2, M_2(\mathbb{C})) \oplus C(\mathbb{T}^2, M_2(\mathbb{C}))$;

(2) $\Gamma(F_{1\alpha_2}(M_2(\mathbb{C}))) \cong \{f \in C(\mathbf{T} \times I, M_2(\mathbb{C})) \mid f(\lambda, 0) = \begin{pmatrix} \bar\lambda & 0 \\ 0 & 1 \end{pmatrix} f(\lambda, 1) \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}$, $\lambda \in \mathbf{T}\}$ with complexification isomorphic to $A_{1/2} \oplus A_{1/2}$;

(3) $\Gamma(F_{1\alpha_3}(M_2(\mathbb{C}))) \cong \{f \in C(\mathbf{T} \times I, M_2(\mathbb{C})) \mid f(\lambda, 0) = \overline{f(\lambda, 1)}, \lambda \in \mathbf{T}\}$ with complexification isomorphic to $C(\mathbf{T}^2, M_2(\mathbb{C}))$;

(4) $\Gamma = (F_{1\alpha_4}(M_2(\mathbb{C}))) \cong \{f \in C(\mathbf{T}^2, M_2(\mathbb{C})) \mid f(\lambda, \mu) = \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} \overline{f(\lambda, -\mu)} \begin{pmatrix} \bar\lambda & 0 \\ 0 & 1 \end{pmatrix}$, $\lambda, \mu \in \mathbf{T}\}$ with complexification isomorphic to $C(\mathbf{T}^2, M_2(\mathbb{C}))$;

(5) $\Gamma(F_{2\beta_1}(M_2(\mathbb{C}))) \cong \{f \in C(\mathbf{T} \times I, M_2(\mathbb{C})) \mid f(\lambda, 0) = \overline{f(\lambda, 1)} \lambda \in \mathbf{T}\}$ with complexification isomorphic to $C(\mathbf{T}^2, M_2(\mathbb{C}))$;

(6) $\Gamma(F_{\alpha\beta_2}(M_2(\mathbb{C}))) \cong \{f \in C(\mathbf{T}^2, M_2(\mathbb{C})) \mid f(\lambda, \mu) = \begin{pmatrix} \bar\mu & 0 \\ 0 & 1 \end{pmatrix} \overline{f(-\lambda, \mu)} \begin{pmatrix} \bar\mu & 0 \\ 0 & 1 \end{pmatrix}$, $\lambda, \mu \in \mathbf{T}\}$ with complexification isomorphic to $C(\mathbf{T}^2, M_2(\mathbb{C}))$;

(7) $\Gamma(F_{2\beta_3}(M_2(\mathbb{C}))) \cong \{f \in C(I^2, M_2(\mathbb{C})) \mid f(s, 0) = \overline{f(s, 1)}, f(0, t) = \overline{f(1, t)}, s, t \in I\}$ with complexification isomorphic to $C(\mathbf{T}^2, M_2(\mathbb{C}))$;

(8) $\Gamma(F_{2\beta_4}(M_2(\mathbb{C}))) \cong \{f \in C(\mathbf{T}^2, M_2(\mathbb{C})) \mid f(0, t) = \overline{f(1, t)}, f(s, 0) = \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} \overline{f(s, 1)} \begin{pmatrix} \bar\lambda & 0 \\ 0 & 1 \end{pmatrix}$, $s, t \in I, \lambda = e^{2\pi i s}\}$ with complexification isomorphic to $C(\mathbf{T}^2, M_2(\mathbb{C}))$;

Proof. (1) The fibre bundle induced from $F_{1\alpha_1}$ with fibres isomorphic to $M_2(\mathbb{C})$ is the trivial $M_2(\mathbb{C}) \times \mathbf{T}^2$ which has cross-section algebra $C(\mathbf{T}^2, M_2(\mathbb{C}))$ with complexification $C(\mathbf{T}^2, M_2(\mathbb{C})) \oplus C(\mathbf{T}^2, M_2(\mathbb{C}))$.

(2) Since $F_{1\alpha_2}$ can be regarded as a principal PU_2'-bundle over $\mathbf{T}^2$ obtained from $PU_2' \times I \times I$ by pasting $([u], 0, t)$ to $([u], 1, t)$ and pasting $([u], s, 0)$ to $([\begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} u], s, 1)$, by Lemma 2.2 of [2] we have

$$\Gamma(F_{1\alpha_2}(M_2(\mathbb{C}))) \cong \left\{ f \in C(I \times I, M_2(\mathbb{C})) \mid f(0, t) = f(1, t), f(s, 0) = \begin{pmatrix} \bar\lambda & 0 \\ 0 & 1 \end{pmatrix} f(s, 1) \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}, \lambda = e^{2\pi i s} \right\}$$

$$\cong \left\{ f \in C(\mathbf{T} \times I, M_2(\mathbb{C})) \mid f(\lambda, 0) = \begin{pmatrix} \bar\lambda & 0 \\ 0 & 1 \end{pmatrix} f(\lambda, 1) \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}, \lambda \in \mathbf{T} \right\}$$

Define $U : \mathbf{T} \to \mathcal{U}_2$ by $U(\lambda) = \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}$. Then

$$A(U) = \left\{ f \in C(\mathbf{T} \times I, M_2(\mathbb{C})) \mid f(\lambda, 0) = \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} f(\lambda, 1) \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}, \lambda \in \mathbf{T} \right\}$$

and $deg(detU_\lambda)/2 = 1/2$ and $(deg(detU_\lambda, 2)) = 1$. Thus by Proposition 2.8 of [1] $\Gamma(F(_{1\alpha_2}(M_2(\mathbb{C})))) \cong A_{1/2}$ which has complexification $A_{1/2} \oplus A_{1/2}$.

(3) Similarly, $F_{1\alpha_3}$ can be regarded as a principal PU_2'-bundle over $\mathbf{T}^2$ obtained from $PU_2' \times I \times I$ by pasting $([u], 0, t)$ to $([u], 1, t)$ and pasting $([u], s, 0)$ to $([ku], s, 1)$. Making the homeomoephic transformation $(x, y) \mapsto (y, x)$ on $\mathbf{T}^2$ we get a weakly isomorphic principal PU_2'-bundle over $\mathbf{T}^2$ obtained from $PU_2' \times I \times I$ by pasting $([u], s, 0)$ to $([u], s, 1)$ and pasting $([u], 0, t)$ to $([ku], 1, t)$. By Lemma 2.2 of [2]we have

$$\begin{aligned} &\Gamma(F_{1\alpha_3}(M_2(\mathbb{C}))) \cong \{f \in C(I \times I, M_2(\mathbb{C})) \mid f(s,0) = f(s,1), f(0,t) = k^{-1} f(1,t) k\} \\ \cong\ & \{f \in C(I \times \mathbf{T}, M_2(\mathbb{C})) \mid f(0, \lambda) = k^{-1} f(1, \lambda) k\} \\ \cong\ & \{f \in C(\mathbf{T} \times I, M_2(\mathbb{C})) \mid f(\lambda, 0) = \overline{f(\lambda, 1)} \\ \cong\ & \{f \in C(I) \mid f(0) = \overline{f(1)}\} \otimes_{\mathbb{R}} C(\mathbf{T}, \mathbb{R}) \otimes_{\mathbb{R}} M_2(\mathbb{R}) \\ \cong\ & \{f \in C(\mathbf{T}) \mid f(-\lambda) = \overline{f(\lambda)}\} \otimes_{\mathbb{R}} C(\mathbf{T}, \mathbb{R}) \otimes_{\mathbb{R}} M_2(\mathbb{R}) \end{aligned}$$

Let $R = \{f \in C(\mathbf{T}) \mid f(-\lambda) = \overline{f(\lambda)}\}$. Thus it is to show that the complexification of R is isomorphic to $C(\mathbf{T})$. Thus $\Gamma(F_{1\alpha_3}(M_2(\mathbb{C})))$ has complexification isomorphic to $C(\mathbf{T}) \otimes C(\mathbf{T}) \otimes M_2(\mathbb{C}) \cong C(\mathbf{T}^2, M_2(\mathbb{C}))$.

(4) The same argument as in (3) shows that

$$\begin{aligned} &\Gamma(F_{1\alpha_4}(M_2(\mathbb{C}))) \\ \cong\ & \left\{ f \in C(I \times I, M_2(\mathbb{C})) \mid f(s,0) = f(s,1), f(0,t) = k^{-1} \begin{pmatrix} \bar\lambda & 0 \\ 0 & 1 \end{pmatrix} f(1,t) \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} k, \right. \\ & \left. \lambda = e^{2\pi i t} \in \mathbf{T} \right\} \\ \cong\ & \left\{ f \in C(\mathbf{T} \times I, M_2(\mathbb{C})) \mid f(\lambda, 0) = \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} \overline{f(\lambda,1)} \begin{pmatrix} \bar\lambda & 0 \\ 0 & 1 \end{pmatrix}, \lambda \in \mathbf{T} \right\} \end{aligned}$$

Let $R = \left\{ f \in C(\mathbf{T}^2, M_2(\mathbb{C})) \mid f(\lambda, \mu) = \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} \overline{f(\lambda, -\mu)} \begin{pmatrix} \bar\lambda & 0 \\ 0 & 1 \end{pmatrix} \right\}$.

Define $\Phi : R \to C(\mathbf{T} \times I, M_2(\mathbb{C}))$ by $\Phi f(\lambda, t) = f(\lambda, e^{\pi t})$. Then

$$\Phi f(\lambda, 0) = f(\lambda, 1) = \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} \overline{f(\lambda, -1)} \begin{pmatrix} \bar\lambda & 0 \\ 0 & 1 \end{pmatrix}$$

and $\Phi f(\lambda, 1) = f(\lambda, -1)$. So $\Phi f(\lambda, 0) = \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} \overline{\Phi f(\lambda, 1)} \begin{pmatrix} \bar\lambda & 0 \\ 0 & 1 \end{pmatrix}$.

Hence $\Phi f \in \Gamma(F_{1\alpha_4}(M_2(\mathbb{C})))$. Obviously Φ is injective. To show that Φ is onto $\Gamma(F_{1\alpha_4}(M_2(\mathbb{C})))$, let $g \in C(\mathbf{T}^2, M_2(\mathbb{C}))$ be defined by

$$g(\lambda, \mu) = \begin{cases} f(\lambda, t) & \text{if} \quad \mu = e^{2\pi t} \quad t \in I \\ \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} \overline{f(\lambda, t)} \begin{pmatrix} \bar\lambda & 0 \\ 0 & 1 \end{pmatrix} & \text{if} \quad \mu = -e^{2\pi t} \quad t \in I. \end{cases}$$

for any element $f \in \Gamma(F_{1\alpha_4}(M_2(\mathbb{C})))$. Then

$$g(\lambda, e^{\pi i0}) = f(\lambda, 0) = \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} \overline{f(\lambda, 1)} \begin{pmatrix} \bar{\lambda} & 0 \\ 0 & 1 \end{pmatrix} = g(\lambda, -e^{\pi i1})$$

and

$$g(\lambda, e^{\pi i1}) = f(\lambda, 1) = \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} \overline{f(\lambda, 0)} \begin{pmatrix} \bar{\lambda} & 0 \\ 0 & 1 \end{pmatrix} = g(\lambda, -e^{\pi i0})$$

So g is well-defined. As a function of μ, g is continuous at ± 1 and also we have $\Phi g(\lambda, t) = g(\lambda, e^{\pi it}) = f(\lambda, t)$. So we are left to show $g \in R$. However, when $\mu = e^{\pi it}$, we have

$$g(\lambda, \mu) = f(\lambda, t) = \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} \overline{g(\lambda, -\mu)} \begin{pmatrix} \bar{\lambda} & 0 \\ 0 & 1 \end{pmatrix}$$

and when $\mu = -e^{\pi it}$, we have

$$g(\lambda, \mu) = \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} \overline{f(\lambda, t)} \begin{pmatrix} \bar{\lambda} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} \overline{g(\lambda, -\mu)} \begin{pmatrix} \bar{\lambda} & 0 \\ 0 & 1 \end{pmatrix}.$$

Therefore $\Gamma(F_{1\alpha_4}(M_2(\mathbb{C})))$ is isomorphic to

$$\left\{ f \in C(\mathbf{T}^2, M_2(\mathbb{C})) \mid f(\lambda, \mu) = \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} \overline{f(\lambda, -\mu)} \begin{pmatrix} \bar{\lambda} & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

Define an autiantomorphism φ of $C(\mathbf{T}^2, M_2(\mathbb{C}))$ by

$$\varphi f(\lambda, \mu) = \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} f(\lambda, -\mu)^{tr} \begin{pmatrix} \bar{\lambda} & 0 \\ 0 & 1 \end{pmatrix}$$

Then

$$\begin{aligned} \varphi^2 f(\lambda, \mu) &= \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} \varphi f(\lambda, -\mu)^{tr} \begin{pmatrix} \bar{\lambda} & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} \left(\begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} f(\lambda, \mu)^{tr} \begin{pmatrix} \bar{\lambda} & 0 \\ 0 & 1 \end{pmatrix} \right)^{tr} \begin{pmatrix} \bar{\lambda} & 0 \\ 0 & 1 \end{pmatrix} \\ &= f(\lambda, \mu) \end{aligned}$$

So φ is involutory and $\varphi f(\lambda, \mu) = f^*(\lambda, \mu)$ if and only if

$$f(\lambda, \mu) = \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} \overline{f(\lambda, -\mu)} \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}$$

Hence the complexification of $\Gamma(F_{1\alpha_4}(M_2(\mathbb{C})))$ is isomorphic to $C(\mathbf{T}^2, M_2(\mathbb{C}))$.
(5) By Lemma 2.2 of [2] we have

$$\begin{aligned} \Gamma(F_{2\beta_1}(M_2(\mathbb{C}))) &\cong \{f \in C(I \times I, M_2(\mathbb{C})) \mid f(s, 0) = f(s, 1), f(0, t) = k^{-1} f(1, t) k\} \\ &\cong \Gamma(F_{1\alpha_3}(M_2(\mathbb{C}))) \end{aligned}$$

which has complexification isomorphic to $C(\mathbf{T}^2, M_2(\mathbb{C}))$.

(6) By Lemma 2.2 of [2] we have

$$\Gamma(F_{2\beta_2}(M_2(\mathbb{C}))) \cong \left\{ f \in C(I^2, M_2(\mathbb{C})) \mid f(s,0) = \begin{pmatrix} \bar{\lambda} & 0 \\ 0 & 1 \end{pmatrix} f(s,1) \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} f(0,t) = k^{-1} f(1,t) k, \lambda = e^{2\pi i s} \right\}$$

Let $u(s,t) = \begin{pmatrix} e^{-2\pi i s t} & 0 \\ 0 & 1 \end{pmatrix}$ and let $g(s,t) = u(s,t) f(s,t) u(s,t)^*$ for any $f \in \Gamma(F_{2\beta_2}(M_2(\mathbb{C})))$. Then

$$\begin{aligned} g(0,t) &= u(0,t) f(0,t) u(0,t)^* \\ &= f(0,t) = \overline{f(1,t)} \\ &= \overline{u(1,t)^* g(1,t) u(1,t)} \\ &= \begin{pmatrix} e^{-2\pi i t} & 0 \\ 0 & 1 \end{pmatrix} \overline{g(1,t)} \begin{pmatrix} e^{2\pi i t} & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} g(s,0) &= u(s,0) f(s,0) u(s,0)^* \\ &= f(s,0) = \begin{pmatrix} e^{-2\pi i s} & 0 \\ 0 & 1 \end{pmatrix} f(s,1) \begin{pmatrix} e^{2\pi i s} & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} e^{-2\pi i s} & 0 \\ 0 & 1 \end{pmatrix} u(s,1)^* g(s,1) u(s,1) \begin{pmatrix} e^{2\pi i s} & 0 \\ 0 & 1 \end{pmatrix} \\ &= g(s,1) \end{aligned}$$

Conversely, if $g \in C(I \times I, M_2(\mathbb{C}))$ with $g(s,0) = g(s,1)$ and $g(0,t) = \begin{pmatrix} e^{-2\pi i t} & 0 \\ 0 & 1 \end{pmatrix} \overline{g(1,t)} \begin{pmatrix} e^{2\pi i t} & 0 \\ 0 & 1 \end{pmatrix}$, let

$$f(s,t) = u(s,t)^* g(s,t) u(s,t),$$

then

$$\begin{aligned} f(0,t) &= u(0,t)^* g(0,t) u(0,t) \\ &= g(0,t) = u(1,t) \overline{g(1,t)} u(1,t)^* \\ &= u(1,t) \overline{u(1,t) f(1,t) u(1,t)^*} u(1,t)^* \\ &= \overline{f(1,t)} \end{aligned}$$

$$\begin{aligned} f(s,0) &= u(s,0)^* g(s,0) u(s,0) = g(s,0) = g(s,1) \\ &= u(s,1) f(s,1) u(s,1)^* \\ &= \begin{pmatrix} e^{-2\pi i s} & 0 \\ 0 & 1 \end{pmatrix} f(s,1) \begin{pmatrix} e^{2\pi i s} & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

Therefore

$$\begin{aligned}
&\Gamma(F_{2\beta_2}(M_2(\mathbb{C})))\\
\cong\ & \left\{ f \in C(I \times I, M_2(\mathbb{C})) \mid f(s,0) = f(s,1), f(0,t) = \begin{pmatrix} e^{-2\pi i t} & 0 \\ 0 & 1 \end{pmatrix} \overline{f(1,t)} \begin{pmatrix} e^{2\pi i t} & 0 \\ 0 & 1 \end{pmatrix} \right\}\\
\cong\ & \left\{ f \in C(I \times \mathbf{T}, M_2(\mathbb{C})) \mid f(0,\lambda) = \begin{pmatrix} \bar{\lambda} & 0 \\ 0 & 1 \end{pmatrix} \overline{f(1,\lambda)} \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} \right\}\\
\cong\ & \Gamma(F_{2\alpha_4}(M_2(\mathbb{C}))
\end{aligned}$$

which has complexification isomorphic to $C(\mathbf{T}^2, M_2(\mathbb{C})))$.

(7) By Lemma 2.2 of [2] we have

$$\begin{aligned}
\Gamma(F_{2\beta_3}(M_2(\mathbb{C}))) &\cong \{f \in C(I^2, M_2(\mathbb{C})) \mid f(s,0) = k^{-1} f(s,1) k, f(0,t) = k^{-1} f(1,t) k\}\\
&\cong \{f \in C(I^2, M_2(\mathbb{C})) \mid f(s,0) = \overline{f(s,1)}, f(0,t) = \overline{f(1,t)}\}
\end{aligned}$$

Let $C = \{f \in C(I^2, M_2(\mathbb{C})) \mid f(s,0) = f(s,1), f(0,t) = f(1,t)\}$. For any $f \in C$, define a function g by

$$g(s,t) = \begin{cases} f(s+\frac{1}{2},t)^{tr} & \text{if } \quad s \le \frac{1}{2} \\ f(s-\frac{1}{2},t)^{tr} & \text{if } \quad s \ge \frac{1}{2} \end{cases}$$

Then $f(s,0) = f(s,1), f(0,t) = f(1,t)$ shows that g is continuous and $g(s,0) = f(s \pm 1/2, 0)^{tr} = f(s \pm 1/2, 1) = g(s,1), g(0,t) = f(1/2,t)^{tr} = g(1,t)$. So $g \in C$.

Let $\Phi f = g$. Then

$$\begin{aligned}
\Phi^2 f &= \Phi g = \begin{cases} g(s+\frac{1}{2},t)^{tr} & \text{if } \quad 0 \le s \le \frac{1}{2} \\ g(s-\frac{1}{2},t)^{tr} & \text{if } \quad \frac{1}{2} \le s \le 1 \end{cases}\\
&= f(s,t)
\end{aligned}$$

So $\Phi f = g$ defines an involutory anti-homomorphism, hence surjective, from C onto itself. Clearly Φ is injective. Thus Φ is an involutory antiautomorphism of C. The associated real algebra is

$$R(\Phi) = \left\{ f \in C \mid f(s,t) = \begin{cases} \overline{f(s+\frac{1}{2},t)} & \text{if } \quad s \le \frac{1}{2} \\ \overline{f(s-\frac{1}{2},t)} & \text{if } \quad s \ge \frac{1}{2} \end{cases} \right\}$$

Let $\Delta = \{(s,t) \in I^2 \mid 1 \le 2s+t \le 2, 0 \le t \le 1\}$, and note that the map $(s,t) \mapsto (2s+t-1,t)$ is a homomorphism from Δ onto I^2. Then, noting that restriction to Δ is an isomorphism on $R(\Phi)$.

$$\begin{aligned}
R(\Phi) &\cong \left\{ f \in C(\Delta, M_2(\mathbb{C})) \mid f(s,0) = \overline{f(s-\frac{1}{2},1)}, f(\frac{1-t}{2},t) = \overline{f(1-\frac{1}{2},t)} \right\}\\
&\cong \{f \in C(I^2, M_2(\mathbb{C})) \mid f(s,0) = \overline{f(s,1)}, f(0,t) = \overline{f(1,t)}\}\\
&\cong \Gamma(F_{2\beta_3}(M_2(\mathbb{C})))
\end{aligned}$$

Thus the complexifiction of $\Gamma(F_{2\beta_3}(M_2(\mathbb{C})))$ is isomorphic to C which is isomorphic to $C(\mathbf{T}^2, M_2(\mathbb{C}))$.

(8) By Lemma 2.2 of [2] we have

$$\begin{aligned}
&\Gamma(F_{2\beta_4}(M_2(\mathbb{C}))) \\
\cong\ &\Big\{f \in C(I^2, M_2(\mathbb{C})) \mid f(0,t) \\
&= k^{-1}f(1,t)k, f(s,0) = k^{-1}\begin{pmatrix} \bar{\lambda} & 0 \\ 0 & 1 \end{pmatrix} f(s,1)\begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} k, \lambda = e^{2\pi i s}\Big\} \\
\cong\ &\Big\{f \in C(I^2, M_2(\mathbb{C})) \mid f(s,0) = \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} \overline{f(s,1)} \begin{pmatrix} \bar{\lambda} & 0 \\ 0 & 1 \end{pmatrix}, f(0,t) = \overline{f(1,t)}\Big\}.
\end{aligned}$$

Let C as in (7), let $\Delta = \{(s,t) \in I^2 \mid 1 \le s + 2t \le 2, 0 \le s \le 1\}$ and note that the map $(s,t) \mapsto (s, s+2t-1)$ is a homomorphism from Δ onto I^2. For any $f \in C$, define a function g by

$$g(s,t) = \begin{cases} \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} f(t, t+\frac{1}{2})^{tr} \begin{pmatrix} \bar{\lambda} & 0 \\ 0 & 1 \end{pmatrix} & \text{if} \quad t \le \frac{1}{2} \\ \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} f(s, t-\frac{1}{2})^{tr} \begin{pmatrix} \bar{\lambda} & 0 \\ 0 & 1 \end{pmatrix} & \text{if} \quad t \ge \frac{1}{2} \end{cases}$$

Where $\lambda = e^{2\pi i s}$. Then $f(s,0) = f(s,1), f(0,t) = f(1,t)$ show that g is continuous and

$$g(s,0) = \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} f(s, \frac{1}{2})^{tr} \begin{pmatrix} \bar{\lambda} & 0 \\ 0 & 1 \end{pmatrix} = g(s,1), g(0,t) = f(0, t \pm \frac{1}{2})^{tr} = g(1,t).$$

So $g \in C$. Let $\Phi f = g$, then

$$\begin{aligned}
\Phi^2 f = \Phi g \quad &= \begin{cases} \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} g(s, t+\frac{1}{2})^{tr} \begin{pmatrix} \bar{\lambda} & 0 \\ 0 & 1 \end{pmatrix} & \text{if} \quad 0 \le t \le \frac{1}{2} \\ \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} g(s, t-\frac{1}{2})^{tr} \begin{pmatrix} \bar{\lambda} & 0 \\ 0 & 1 \end{pmatrix} & \text{if} \quad \frac{1}{2} \le t \le 1 \end{cases} \\
&= \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} \left(\begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} f(s,t)^{tr} \begin{pmatrix} \bar{\lambda} & 0 \\ 0 & 1 \end{pmatrix} \right)^{tr} \begin{pmatrix} \bar{\lambda} & 0 \\ 0 & 1 \end{pmatrix} \\
&= f(s,t)
\end{aligned}$$

So, $\Phi f = g$ define as involutory anti-homomorphism hence surjective, from C onto itself. Clearly Φ is injective. Thus Φ is an involulory antiantomorphism of C. The associated real algebra is

$$\begin{aligned}
R(\Phi) \quad &= \quad \{f \in C \mid f(s,t) = \left\{ \begin{array}{ll} \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} \overline{f(s, t+\frac{1}{2})} \begin{pmatrix} \bar{\lambda} & 0 \\ 0 & 1 \end{pmatrix} & \text{if} \quad t \le \frac{1}{2} \\ \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} \overline{f(s, t-\frac{1}{2})} \begin{pmatrix} \bar{\lambda} & 0 \\ 0 & 1 \end{pmatrix} & \text{if} \quad t \ge \frac{1}{2} \end{array} \right\} \\
&\cong \Big\{f \in C(\Delta, M_2(\mathbb{C})) \mid f(s, \frac{1-s}{2}) = \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} \overline{f(s, 1-\frac{s}{2})} \begin{pmatrix} \bar{\lambda} & 0 \\ 0 & 1 \end{pmatrix}, \\
&\qquad f(0,t) = \overline{f(1, t-\frac{1}{2})}\Big\} \\
&\cong \Big\{f \in C(I^2, M_2(\mathbb{C})) \mid f(s,0) = \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} \overline{f(s,1)} \begin{pmatrix} \bar{\lambda} & 0 \\ 0 & 1 \end{pmatrix}, f(0,t) = \overline{f(1,t)}\Big\} \\
&= \Gamma(F_{2\beta_4}(M_2(\mathbb{C})))
\end{aligned}$$

Thus the complexification of $\Gamma(F_{2\beta_4}(M_2(\mathbb{C})))$ is isomorphic to C which is isomorphic to $C(\mathbf{T}^2, M_2(\mathbb{C}))$.

So all the cross-section algebras of fibre bundles over $\mathbf{T}^2$ with fibres ismorphic to $M_2(\mathbb{C})$ and with group PU_2' have complexification not isomorphic to $A_{1/2}$. Hence we have the following corollary.

Corollary 6. There is no involutory antiautomorphism in $A_{1/2}$ associated with τ_1 : $(\lambda, \mu) \mapsto (-\lambda, \mu)$.

Proof. Let Φ be an involutory antiautomorphism in $A_{1/2}$ associated with τ_1. Since τ_1 has no fixed point, by Proposition 2.7 of [3], $R(\Phi)$ is a complex type algebra with spectrum $\mathbf{T}^2/\tau_1$ which is homomorphic to $\mathbf{T}^2$. So, by Proposition 2.5 of [3], $R(\Phi) \cong \Gamma(R)$ for some fibre bundle over $\mathbf{T}^2$ with fibres isomorphic to $M_2(\mathbb{C})$ and with group PU_2' and the complexification of $R(\Phi)$ is isomorphic to $A_{1/2}$. This contradicts Proposition 5.

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