CHARACTERIZATIONS OF $K-$FOLD POSITIVE IMPLICATIVE BCK-ALGEBRAS

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Abstract. In this paper, we introduce the concept of $K-$maps of BCK-algebras and discuss the characterizations of $K-$fold positive implicative BCK-algebras.

1. Introduction and Preliminaries By a BCK-algebra we mean an algebra $(X;*,0)$ of type (2,0) satisfying the following conditions:

(I) $(x*y)*(x*z) \leqslant z*y$

(II) $x*(x*y) \leqslant y$

(III) $x*x=0$

(IV) $0*x=0$

(V) $x*y=0$ and $y*x=0$ imply $x=y$

where $x \leqslant y$ is defined by $x*y=0$

For any elements x and y of a BCK-algebra $X, x*y^k$ denotes

$$(\cdots((x*y)*y)*\cdots)*y$$

in which y occurs K times.

A BCK-algebra X is called $K-$fold positive implicative if for any x, y and z in $X, (x*y)*z^k = (x*z^k)*(y*z^k)$. It has been proved that a BCK-algebra X is $K-$fold positive implicative if and only if $x*y^{k+1} = x*y^k$ for any x, y in X.

For any BCK-algebra X and element a in X, denote by ρ_a^k the $K-$map of X defined by $\rho_a^k(x) = x*a^k$ for all $x \in X$ [5]. Let $M^k(X)$ be the set of finite products $\rho_{a_1}^k \rho_{a_2}^k \cdots \rho_{a_n}^k$ of $K-$maps of X, where $a_1, a_2 \cdots a_n \in X$. It's clear that $M^k(X)$ is a commutative monoid under the multiplication of $K-$maps. For any $\sigma_k = \rho_{a_1}^k \rho_{a_2}^k \cdots \rho_{a_n}^k \in M^k(X)$, the subset $Im\sigma_k = \{\sigma_k(x) | x \in X\}, ker\sigma_k = \{x \in X \mid \sigma_k(x) = 0\}$ and $S(\sigma_k) = \{x \in X \mid \sigma_k(x) = x\}$ of X are called respectively the $K-$Image, the $K-$kernel and the $K-$stabilizer of σ_k.

2. Main Results

Lemma 2.1 If X is a $K-$fold positive implicative BCK-algebra, then $(x*y^k)*y^k = x*y^k$ for any x, y in X.

It's obvious.

Theorem 2.2 Let X be a BCK-algebra. Then the following conditions are equivalent:

(i) X is $K-$fold positive implicative

(ii) $\sigma_k^2 = \sigma_k$ for any $\sigma_k \in M^k(X)$

(iii) $ker\rho_a^k \cap Im\rho_a^k = \{0\}$ for any $a \in X$

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Proof. (i) implies (ii). Assume X is $K-$fold positive implicative, then by Lemma 2.1, we have $(x*y^k)*y^k = x*y^k$ for any x, y in X. For any $\sigma_k = \rho^k_{a_1}\rho^k_{a_2}\cdots\rho^k_{a_n} \in M^k(X)$, we have

$$\begin{aligned}
\sigma_k^2(x) &= \rho^k_{a_1}\cdots\rho^k_{a_n}(\rho^k_{a_1}\cdots\rho^k_{a_n})(x) \\
&= (\rho^k_{a_1})^2\cdots(\rho^k_{a_n})^2(x) \\
&= ((\cdots((x*a_n^k)*a_n^k)*\cdots)*a_1^k)*a_1^k \\
&= (\cdots(x*a_n^k)*\cdots)*a_1^k \\
&= \rho^k_{a_1}\rho^k_{a_2}\cdots\rho^k_{a}\rho^k_{a_n}(x) \\
&= \sigma_k(x)
\end{aligned}$$

that is $\sigma_k^2 = \sigma_k$

(ii)implies (iii) Assume $\sigma_k^2 = \sigma_k$ for any $\sigma \in M^k(X)$. Then $(\rho^k_a)^2 = \rho^k_a$ for any $a \in X$. If $x \in ker\rho^k_a \cap Im\rho^k_a$. Then $x*a^k = 0$ and there exists $y \in x$ such that $y*a^k = x$. Hence $x = y*a^k = (y*a^k)*a^k = x*a^k = 0$ That is $ker\rho^k_a \cap Im\rho^k_a = \{0\}$

(iii) implies (i). Assume (iii) holds. For any $x, y \in X$, we have $ker\rho^k_a \cap Im\rho^k_a = \{0\}$. Since $((x*y^k)*y^k)*(x*y^{k+1}) = ((x*y^{k+1})*y^{k-1})*(x*y^{k+1}) = 0, (x*y^k)*(x*y^{k+1}) \in ker\rho^k_y$. Moreover, $(x*y^k)*(x*y^{k+1}) = (x*(x*y^{k+1})*y^k \in Im\rho^k_y$. Hence $(x*y^k)*(x*y^{k+1}) \in ker\rho^k_y \cap Im\rho^k_y = \{0\}$ that is $(x*y^{k+1})*(x*y^k)*(x*y^{k+1}) = 0$. On the other hard, $(x*y^{k+1})*(x*y^k) = 0$ is obvious. Therefore $x*y^{k+1} = x*y^k$ and consequently X is $K-$fold positive implicative. The proof is complete.

Lemma 2.3 Let X be a BCK-algebra, then the following conclusion hold for any $\sigma_k \in M^k(X)$.

(i) $S(\sigma_k) \subseteq Im\sigma_k$

(ii) $S(\sigma_k) \cap ker\sigma_k = \{0\}$

Proof (i) If $x \in S(\sigma_k)$. then $\sigma_k(x) = x$. It's clear that $x \in Im\sigma_k$, that is $S(\sigma_k) \subseteq Im\sigma_k$

(ii) Suppose $x \in S(\sigma_k) \cap ker\sigma_k$, then $\sigma_k(x) = x$ and $\sigma_k(x) = 0$, that is $x = 0$. Hence $S(\sigma_k) \cap ker(\sigma_k) = \{0\}$.

Theorem 2.4 Let X be a BCK-algebra, then the following conditions are equivalent:

(i) X is $K-$fold positive implicative

(ii) $Im\sigma_k = S(\sigma_k)$ for any $\sigma_k \in M^k(X)$

(iii) $Im\rho^k_a = S(\rho^k_a)$ for any $a \in X$

Proof (i) implies (ii). Assume X is $K-$fold positive implicative. By Theorem 2.3 we have $\sigma_k^2 = \sigma_k$ for any $\sigma_k \in M^k(X)$. If $x \in Im\sigma_k$, then there exists some $y \in X$ such that $\sigma_k(y) = x$, hence $\sigma_k(x) = \sigma_k(\sigma_k(y)) = \sigma_k^2(y) = \sigma_k(y) = x$, that is $x \in S(\sigma_k)$, and so $Im\sigma_k \subseteq S(\sigma_k)$. Therefore, $Im\sigma_k = S(\sigma_k)$ by Lemma 2.3(i).

(ii) implies (iii) It's trivial.

(iii) implies (i) If $Im\rho^k_a \cap S(\rho^k_a) = \{0\}$ by Lemma 2.3 (ii), and consequently X is $K-$fold positive implicative by Theorem 2.2. The proof is complete.

Definition 2.5 ([4]) A nonempty subset I of a BCK-algebra X is called a $K-$ideal of X if (i) $0 \in I$ (ii) $x*y^k \in I$ and $y \in I$ imply $x \in I$.

Lemma 2.6 If X is a $K-$fold positive implicative BCK-algebra and $\sigma_k \in M^k(X)$, then $\sigma_k(x)*\sigma_k(y) = \sigma_k(x*y)$ for any x, y in X.

Proof Let $\sigma_k = \rho^k_{a_1}\rho^k_{a_2}\cdots\rho^k_{a_n}$, then

$$\begin{aligned}\sigma_k(x)*\sigma_k(y) &= \rho^k_{a_1}\rho^k_{a_2}\cdots\rho^k_{a_n}(x)*\rho^k_{a_1}\rho^k_{a_2}\cdots\rho^k_{a_n}(y)\\ &= ((\cdots((x*a^k_n)*a^k_{n-1}*\cdots)*a^k_1)*((\cdots((y*a^k_n)*a^k_{n-1})*\cdots)*a^k_1)\\ &= (\cdots(x*y)*a^k_n)*\cdots)*a^k_1\\ &= \rho^k_{a_1}\rho^k_{a_2}\cdots\rho^k_{a_n}(x*y)\\ &= \sigma_k(x*y)\end{aligned}$$

Theorem 2.7 Let X be a BCK-algebra, then the following conditions are equivalent:

(i) X is $K-$fold positive implicative,

(ii) $ker\sigma_k$ is a $K-$ideal of X, for any $\sigma_k \in M^k(X)$

(iii) $ker\rho^k_a$ is a $K-$ideal of X, for any $a \in X$.

Proof (i) implies (ii) Assume X is $K-$fold positive implicative and $\sigma_k \in M^k(X)$, then we have $0 \in ker\sigma_k$. If $x*y^k, y \in ker\sigma_k$, then $\sigma_k(x*y^k) = \sigma_k(y) = 0$.

Hence $\sigma_k(x) = \sigma_k(x)*0^k = \sigma_k(x)*\sigma_k(y)^k = \sigma_k(x*y^k) = 0$ That is $x \in ker\sigma_k$. Hence $ker\sigma_k$ is a $K-$ideal of X.

(ii) implies (iii), It's trivial

(iii) implies (i), Assume $ker\rho^k_a$ is a $K-$ideal of X, for any a in X then $ker\rho^k_a \cap Im\rho^k_a = \{0\}$. In fact, if $x \in ker\rho^k_a \cap Im\rho^k_a$, then we have $x*a^k = 0$ and there exists some $y \in X$, such that $y*a^k = x$, Hence $x*a^k = (y*a^k)*a^k = 0 \in ker\rho^k_a$. Since ρ^k_a is a $K-$ideal of X and $a \in ker\rho^k_a$, then we have $y \in ker\rho^k_a$ and therefore $x = y*a^k = 0$. By Theorem 2.2, X is $K-$fold positive implicative. The proof is complete.

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