SOME RESULTS ON IDEALS OF BCK-ALGEBRAS

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Abstract. In this paper, we introduce a special set in a BCK-algebra, and we give an example in which special set is not an ideal. We obtain a condition for this special set to be an ideal. Using this special set, we establish an equivalent condition of an ideal.

1. Introduction and Preliminaries The notion of BCK-algebras was proposed by Imai and Ieski in 1966. For the general development of BCK-algebras, the ideal theory plays an important role. In this paper, we introduce a special set in a BCK-algebra, and give an example in which special set is not an ideal. We obtain a condition for this special set to be an ideal. Using this special set, we establish an equivalent condition of an ideal.

By a BCK-algebra we mean algebra $(X, *, 0)$ of type $(2, 0)$ satisfying the following conditions:

(I) $((x * y) * (x * z)) * (z * y) = 0$
(II) $(x * (x * y)) * y = 0$
(III) $x * x = 0$
(IV) $0 * x = 0$
(V) $x * y = 0$ and $y * x = 0$ imply $x = y$
for all $x, y, z \in X$.

We can define a partial ordering "$\leq$" on X by $x \leq y$ if and only if $x * y = 0$. Let $\mathbb{N}$ denote the set of all positive integers. For any x and y of a BCK-algebra X, let $x*y^k$ denote $(\cdots\cdots((x * y) * y)\cdots\cdots) * y$ in which y occurs k times, where $k \in \mathbb{N}$. A BCK-algebra X is said to be K-fold positive implicative if $(x * z^k) * (y * z^k) = (x * y) * z^k$ for all $x, y, z \in X$ and $k \in \mathbb{N}$. A nonempty subset S of a BCK-algebra X is called a subalgebra of X if $x * y \in S$ whenever $x, y \in S$

A nonempty subset I of a BCK-algebra X is called an ideal of X if
(I1) $0 \in I$
(I2) $x * y \in I$ and $y \in I$ imply $x \in I$.

2. Main Results

Definition 2.1 For any $a, b \in X$ and $k \in \mathbb{N}$, we define $(a^k; b^k) = \{x \in X \mid (x * a^k) * b^k = 0\}$

Obviously, $0 \in (a^k; b^k)$ for all $a, b \in X$ and $k \in \mathbb{N}$.

Proposition 2.2 Let $a, b \in X$ and $k \in \mathbb{N}$. If $x \in (a^k; b^k)$, then $x * y \in (a^k; b^k)$ for all $y \in X$, and so $(a^k; b^k)$ is a subalgebra of X.

Proof. Assume that $x \in (a^k; b^k)$. Then $((x * y) * a^k) * b^k = ((x * a^k) * y) * b^k = 0 * y = 0$ for all $y \in X$. Hence $x * y \in (a^k; b^k)$ for all $y \in X$.

The following example shows that there exist $a, b \in X$ and $k \in \mathbb{N}$ such that $(a^k; b^k)$ is not an ideal of X.

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Example 2.3 Consider a BCK-algebra $X = \{0, a, b, c, d\}$ with the following Cayley table

0	0	0	0	0	0
a	a	0	a	0	0
b	b	b	0	b	b
c	c	a	c	0	a
d	d	d	d	d	0

Then $(d;b) = \{0, a, b, d\}$ is not an ideal of X because $c * d = a \in (d;b)$ and $d \in (d;b)$, but $c \notin (d;b)$.

We state a condition for a set $(a^k;b^k)$ to be an ideal.

Theorem 2.4 If X is K-fold positive implicative BCK-algebra, then $(a^k;b^k)$ is an ideal of X for all $a, b \in X$ and $k \in \mathbb{N}$.

Proof. Let $x, y \in X$ be such that $x*y \in (a^k;b^k)$ and $y \in (a^k;b^k)$. Then $0 = ((x*y)*a^k)*b^k = ((x*a^k)*(y*a^k))*b^k = ((x*a^k)*b^k)*((y*a^k)*b^k) = ((x*a^k)*b^k)*0 = (x*a^k)*b^k$ and so $x \in (a^k;b^k)$. Therefore $(a^k;b^k)$ is an ideal of X.

Using the set $(a^k;b^k)$, we establish a condition for a subset I of X to be an ideal of X.

Theorem 2.5 Let I be a nonempty subset of X. Then I is an ideal of X if and only if $(a^k;b^k) \subseteq I$ for every $a, b \in I$ and $k \in \mathbb{N}$.

Proof. Assume that I is an ideal of X and let $a, b \in I$ and $k \in \mathbb{N}$. If $x \in (a^k;b^k)$, then $(x * a^k) * b^k = 0 \in I$. Since $a, b \in I$, by using (I2) repeatedly we get $x \in I$. Hence $(a^k;b^k) \subseteq I$.

Conversely, suppose that $(a^k;b^k) \subseteq I$ for all $a, b \in I$ and $k \in \mathbb{N}$. Note that $0 \in (a^k;b^k) \subseteq I$. Let $x, y \in X$ be such that $x * y \in I$ and $y \in I$. Then

$$\begin{aligned} (x*(x*y)^k)*y^k &= ((x*(x*y)^k)*y)*y^{k-1} \\ &= ((x*y)*(x*y)^k)*y^{k-1} \\ &= (((x*y)*(x*y))*(x*y)^{k-1})*y^{k-1} \\ &= (0*(x*y)^{k-1})*y^{k-1} = 0 \end{aligned}$$

and thus $x \in ((x*y)^k;y^k) \subseteq I$. Hence I is an ideal of X. This completes the proof.

We use the notation $x \wedge y$ instead of $y * (y * x)$ for all $x, y \in X$.

Definition 2.6([5]) A nonempty subset I of X is called a quasi-ideal of X if

(i) $0 \in I$

(ii) $x \in I$ and $y \in X$ imply $y \wedge x \in I$ and $x \wedge y \in I$

Theorem 2.7 For every $a, b \in X$ and $k \in \mathbb{N}$, the set $(a^k;b^k)$ is a quasi-ideal of X.

Proof. Note that $0 \in (a^k;b^k)$. Let $x \in (a^k;b^k)$ and $y \in X$. Then $((y \wedge x) * a^k) * b^k = ((x*(x*y))*a^k)*b^k = ((x*a^k)*b^k)*(x*y) = 0*(x*y) = 0$, and so $y \wedge x \in (a^k;b^k)$. Now we get

$$\begin{aligned} ((x \wedge y)*a^k)*b^k &= ((y*(y*x))*a^k)*b^k \\ &= ((y*a^k)*(y*x))*b^k \\ &= (((y*a)*(y*x))*a^{k-1})*b^k \\ &\leq ((x*a)*a^{k-1})*b^k \\ &= (x*a^k)*b^k = 0 \end{aligned}$$

and hence $((x \wedge y)*a^k)*b^k = 0$ which shows that $x \wedge y \in (a^k;b^k)$. Therefore $(a^k;b^k)$ is a quasi-ideal of X. This completes the proof.

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