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METRIZABILITY AND A QUESTION OF J.NAGATA

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Abstract. In this note, we give an answer to a problem about metrizability which was posed by J.Nagata in [2].

The purpose of this note is to consider a question raised in [3]. All spaces in this note are regular and T_1, and $\mathbf{N}$ denotes the set of all nature numbers. A function g from $X \times \mathbf{N}$ to τ, where τ denotes the topology of X, is called a g-function if $x \in g(n,x)$ for each $x \in X$ and $n \in \mathbf{N}$. A g-function is said to be decreasing if $g(n+1,x) \subseteq g(n,x)$ for each $n \in \mathbf{N}$ and $x \in X$.

The following Theorem was proved by Nagata:

Theorem 1 ([3] Theorem 7) *A topological space X is metrizable if and only if there is a decreasing g-function on X satisfying the following conditions* (1) *and* (2):
(1) *For any $x \in X$ and any neighborhood U of x, there is an $n \in \mathbf{N}$ such that*

$$x \notin [\cup\{g(n,y) : y \in X \setminus U\}]^-$$

(2) *For any $Y \subseteq X$ and each $n \in \mathbf{N}$, $\overline{Y} \subseteq \cup\{g^2(n,y) : y \in Y\}$,*
where $g^2(n,y) = \bigcup\{g(n,z) : z \in g(n,y)\}$.

After proving the above theorem, Nagata raised the question if the condition (2) in Theorem 1 can be weaken. In [2], Z. Gao proved the following result and gave an answer to Nagata's question:

Theorem 2 ([2] Theorem 3) *A topological space X is metrizable if and only if there is a decreasing g-function on X satisfying* (1) *and the following weaker condition* (3):
(3) *For any $Y \subseteq X$ and each $n \in \mathbf{N}$, $\overline{Y} \subseteq \cup\{\overline{g^2(n,y)} : y \in Y\}$.*

In this note, we show the condition (3) in Theorem 2 also can be weaken, and hence answer Nagata's question further.

Theorem 3 *A topological space X is metrizable if and only if there is a decreasing g-function on X satisfying* (1) *and the following weaker condition* (4):
(4) *There is a $k \in \mathbf{N}$, $k \geq 2$, such that for any $Y \subseteq X$ and each $n \in \mathbf{N}$,*
$\overline{Y} \subseteq \bigcup\{\overline{g^k(n,y)} : y \in Y\}$, *where $g^k(n,y) = \bigcup\{g^{k-1}(n,z) : z \in g(n,y)\}$ when $k > 2$.*

Proof: By Theorem 2, we know that the condition is necessary. So we only need prove the *sufficiency.*

It is clear that condition (1) implies the following condition (5):
(5) *If $x_n \to x$ when $n \to \infty$ and $x_n \in g(n,y_n)$ for each $n \in \mathbf{N}$, then $y_n \to x$ when $n \to \infty$.*

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Let $h(n,x)=\{y\in X: x\in \overline{g^k(n,y)}\}$, $o(n,x)=g(n,x)\cap h(n,x)$ and $O_n=\{o(n,x): x\in X\}$.

By condition (4), $h(n,x)$ is a neighborhood (not necessarily open) of x and so is $o(n,x)$. Therefore, in virtue of the Moore metrization theorem ([1] page 409 theorem 5.4.2), We only need prove that $\{st^2(x,O_n): n\in N\}$ is a neighborhood base for each $x\in X$.

In fact, if $\{st^2(x,O_n): n\in N\}$ is not a local neighborhood base for some $x\in X$, then there exists a neighborhood U of x such that $st^2(x,O_n)\setminus U\neq\emptyset$ for $n\in N$. Take $y_n\in st^2(x,O_n)\setminus U$, $n\in N$. That means we can find z_n, $w_n\in X$ such that $y_n\in o(n,z_n)$, $o(n,z_n)\cap o(n,w_n)\neq\emptyset$ and $x\in o(n,w_n)$. Take $v_n\in o(n,z_n)\cap o(n,w_n)$. By $x\in o(n,w_n)\subseteq g(n,w_n)$ and condition (5), we obtain that $\{w_n\}\to x$, and by $v_n\in o(n,w_n)\subseteq h(n,w_n)$, we get $w_n\in\overline{g^k(n,v_n)}$. We prove that $\{v_n\}\to x$.

Otherwise, there exists some open neighborhood V of x such that for each $n\in N$, there exists $i_n\in N$, $i_n\geq n$ and $v_{i_n}\notin V$. By (1), there exists $n_x\in N$ such that $x\notin[\cup g(n_x,y): y\in X-V]^-$. Since $\{v_{i_n}: n\in N\}\cap V=\emptyset$, $x\notin\overline{\bigcup_{n\in N} g(n_x,v_{i_n})}$.

Let $V_1=[\overline{\bigcup_{n\in N} g(n_x,v_{i_n})}]^c$. V_1 is a neighborhood of x. By(1), there exists $m_1\in N$ such that $x\notin[\cup\{g(m_1,y): y\in\overline{\bigcup_{n\in N} g(n_x,v_{i_n})}\}]^-$. Take $n_2\in N$ such that $n_2>$max(m_1,n_x). Since $g(n,x)$ is a decreasing g-function, $x\notin\overline{\bigcup_{n\in N} g^2(n_2,v_{i_n})}$.

Let $V_2=[\overline{\bigcup_{n\in N} g^2(n_2,v_{i_n})}]^c$. V_2 is a neighborhood of x. By(1), there exists $m_2\in N$ such that $x\notin[\cup\{g(m_2,y): y\in\overline{\bigcup_{n\in N} g^2(n_2,v_{i_n})}\}]^-$. Take $n_3\in N$ such that $n_3>$max(m_2,n_2). Since $g(n,x)$ is a decreasing g-function, $x\notin\overline{\bigcup_{n\in N} g^3(n_3,v_{i_n})}$.

Continue in this way, finally we can take $n_{k-1}\in N$ such that $x\notin\overline{\bigcup_{n\in N} g^{k-1}(n_{k-1},v_{i_n})}$.

Let $V_{k-1}=[\overline{\bigcup_{n\in N} g^{k-1}(n_{k-1},v_{i_n})}]^c$. By(1), there exists $m_{k-1}\in N$ such that $x\notin[\cup\{g(m_{k-1},y): y\in\overline{\bigcup_{n\in N} g^{k-1}(n_{k-1},v_{i_n})}\}]^-$. Since $w_n\to x$, there exists $m_0\in N$ such that $\{x,w_n: n>m_0\}\cap[\cup\{g(m_{k-1},y), y\in\overline{\bigcup_{n\in N} g^{k-1}(n_{k-1},v_{i_n})}\}]^-=\emptyset$. Take $n_k\in N$ such that $n_k>$max(m_0,m_{k-1},n_{k-1}). Since $g(n,x)$ is a decreasing g-function, $w_{n_k}\notin\overline{g^k(n_k,v_{n_k})}$. But $w_n\in\overline{g^k(n,v_n)}$ for each $n\in N$, which is contradiction. So $\{v_n\}\to x$.

By $v_n\in o(n,z_n)\subseteq g(n,z_n)$ and condition (5), we obtain that $\{z_n\}\to x$. Similarly, from $y_n\in o(n,z_n)\subseteq h(n,z_n)$ we have $\{y_n\}\to x$. But $y_n\notin U$ for $n\in N$, which is contradiction. This contradiction implies that $\{st^2(x,O_n): n\in N\}$ is a neighborhood base for each $x\in X$. So we complete the proof.

The following example shows that conditions of Theorem 3 are essentially weaker than those of Theorem 2:

Example: *A g-function in a metric space which satisfies the conditions of Theorem 3 but does not satisfy the conditions of Theorem 2.*

Let $X=[0,+\infty)$ with the usually topology.

If $x=0$ or $x\geq 1$, let $g(n,x)=B_{\frac{1}{3^{n+1}}}(x)=\{y\in X: |x-y|<\frac{1}{3^{n+1}}\}$.

If $x\in[\frac{1}{3^{k+1}},\frac{1}{3^k})$, $k=0,1,2...$, let

$$g(n,x)=\begin{cases} B_{\frac{1}{3^{k+1}}}(x), & n\leq k\\ B_{\frac{1}{3^{n+1}}}(x), & n>k\end{cases}.$$

It is easy to see that $g(n,x)$ is a decreasing g-function and satisfies (1) and (4) when k=3. But $g(n,x)$ doesn't satisfies (3) (We can see that by taking $Y=(\frac{2}{3^{k+1}},\frac{1}{3^k})$).

Remark:

We can let $X=[0,+\infty)$ and give usually topology on X. Let $k_0\geq 2$

If $x=0$ or $x\geq 1$, let $g_{k_0}(n,x)=B_{\frac{1}{{k_0}^{n+1}}}(x)$

If $x \in [\frac{1}{{k_0}^{m+1}}, \frac{1}{{k_0}^m})$, $m = 0, 1, 2, \cdots$, then

$$g_{k_0}(n,x) = \begin{cases} B_{\frac{1}{{k_0}^{m+1}}}(x), & n \leq m \\ B_{\frac{1}{{k_0}^{n+1}}}(x), & n > m \end{cases} .$$

It is easy to see that $g_{k_0}(n,x)$ is a decreasing g-function and satisfies (1) and (4) when k=k_0. But $g_{k_0}(n,x)$ doesn't satisfy (4) when k=$k_0 - 1$ if we take $Y_{k_0} = (\frac{k_0-1}{{k_0}^{m+1}}, \frac{1}{{k_0}^m})$.

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