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DECOMPOSITION AND CONTROL THEOREMS IN EFFECT ALGEBRAS

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ABSTRACT. Let L be a lattice ordered effect algebra. We prove that the lattice uniformities on L which makes $\ominus$ and $\oplus$ uniformly continuous form a Boolean algebra isomorphic to the centre of a suitable complete effect algebra associated to L. As a consequence, we obtain decomposition theorems - such as Lebesgue and Hewitt-Yosida decompositions - and control theorems - such as Bartle-Dunford-Schwartz and Rybakov theorems - for modular measures on L.

Introduction. Effect algebras have been introduced by D.J. Foulis and M.K. Bennett in 1994 (see [B-F]) for modelling unsharp measurement in a quantum mechanical system. They are a generalization of many structures which arise in quantum physics (see [B-C]) and in Mathematical Economics (see [E-Z], [B-K]), in particular of orthomodular lattices in non-commutative measure theory and MV-algebras in fuzzy measure theory. After 1994, there have been a great number of papers concerning effect algebras (see [D-P] for a bibliography).

In this paper we study modular measures on lattice ordered effect algebras.

Starting point of our paper is observing that the lattice structure of lattice uniformities plays a key role in non-commutative measure theory and in fuzzy measure theory (see $[W_5]$, [B-W] and [G]). Since modular measures on effect algebras generate a D-uniformity, i.e. a lattice uniformity which makes $\ominus$ and $\oplus$ uniformly continuous, it seems reasonable to expect that D-uniformities play a similar role in the study of modular measures on effect algebras.

In this paper we prove that the exhaustive D-uniformities on a lattice ordered effect algebra L form a Boolean algebra isomorphic to the centre of a suitable complete effect algebra associated to L (see Theorem (2.9)). As a consequence, we can apply a result of $[W_3]$ (3.14) to obtain a decomposition theorem for modular measures on L, which contains as particular cases Lebesgue and Hewitt-Yosida type decompositions (see (3.5) and compare with [D-D-P]). Moreover, we can derive a technique which allows us to transfer control theorems known for measures on Boolean algebras - such as Bartle-Dunford-Schwartz and Rybakov theorems - to modular measures on L (see Section 4).

1. Preliminaries

An *effect algebra* $(L, \oplus, 0, 1)$ is a structure consisting of a set L, two special elements 0 and 1, and a partially defined binary operation $\oplus$ on $L \times L$ satisfying the following conditions for every $a, b, c \in L$:

(1) If $a \oplus b$ is defined, then $b \oplus a$ is defined and $a \oplus b = b \oplus a$.

(2) If $b \oplus c$ is defined and $a \oplus (b \oplus c)$ are defined, then $a \oplus b$ and $(a \oplus b) \oplus c$ are defined and $a \oplus (b \oplus c) = (a \oplus b) \oplus c$.

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(3) For every $a \in L$, there exists a unique $a^\perp \in L$ such that $a \oplus a^\perp$ is defined and $a \oplus a^\perp = 1$.

(4) If $a \oplus 1$ is defined, then $a = 0$.

In every effect algebra a dual operation $\ominus$ to $\oplus$ can be defined as follows: $a \ominus c$ exists and equals b if and only if $b \oplus c$ exists and equals a.

Moreover we can define a binary relation on L by $a \le b$ if and only if there exists $c \in L$ such that $c \oplus a = b$ and $\le$ is a partial ordering in L, with 0 as smallest element. We say that two elements $a, b \in L$ are *orthogonal,* and we write $a \perp b$, if $a \oplus b$ exists. Then $a \perp b$ if and only if $a \le b^\perp$. Moreover, for every $a, b \in L$, we have $a^\perp = 1 \ominus a$, $a \oplus b = (b^\perp \ominus a)^\perp$, and $a \le b$ if and only if $a^\perp \ge b^\perp$.

If $(L, \le)$ is a lattice, we say that the effect algebra is a *lattice ordered effect algebra* or a *D-lattice.*

Effect algebras are a common generalization of orthomodular posets and MV-algebras. For a study, we refer to [D-P].

If L is a D-lattice, we set $a \triangle b = (a \vee b) \ominus (a \wedge b)$.

It is helpful to recall from [D-P] and [A-V] (2.3) the following result.

Proposition (1.1) *Let a, b, c, d elements of an effect algebra L. Then:*

(1) *If $a \perp b$, then $a \le a \oplus b$ and $(a \oplus b) \ominus a = b$.*

(2) *If $a \perp b$ and $a \oplus b \le c$, then $c \ominus (a \oplus b) = (c \ominus a) \ominus b = (c \ominus b) \ominus a$.*

(3) *If $a \le b$ and $b \perp c$, then $a \oplus c \le b \oplus c$ and $(b \oplus c) \ominus (a \oplus c) = b \ominus a$.*

(4) *If $a \le b \le c$, then $a \oplus (c \ominus b) = c \ominus (b \ominus a)$ and $(c \ominus b) \oplus (b \ominus a) = c \ominus a$.*

(5) *If L is a D-lattice, $a \perp b$ and $a \wedge b = 0$, then $a \oplus b = a \vee b$.*

(6) *If $a \le b$, then $b \ominus a \le b$ and $b \ominus (b \ominus a) = a$.*

(7) *If $a \le b \le c$, then $c \ominus b \le c \ominus a$ and $(c \ominus a) \ominus (c \ominus b) = b \ominus a$.*

(8) *If L is a D-lattice, $c \le a \le d$ and $c \le b \le d$, then $(a \ominus c) \triangle (b \ominus c) = a \triangle b = (d \ominus a) \triangle (d \ominus b)$.*

(9) *If $\{b_\alpha\} \subseteq L$, $b = \sup_\alpha b_\alpha$ exists and $a \perp b$, then $\sup_\alpha (a \oplus b_\alpha)$ exists and $a \oplus b = \sup_\alpha (a \oplus b_\alpha)$.*

We write $a_n \uparrow a$ (respectively, $a_n \downarrow a$) whenever $\{a_n\}$ is an increasing sequence in L and $a = \sup_n a_n$ (respectively, $\{a_n\}$ is decreasing and $a = \inf_n a_n$).

L is said to be *complete* (*σ-complete*) if every (countable) set in L has a supremum and an infimum. We set $\Delta = \{(a, b) \in L \times L : a = b\}$. If $a, b \in L$ and $a \le b$, we set $[a, b] = \{c \in L : a \le c \le b\}$.

If $a_1, ..., a_n \in L$, we inductively define $a_1 \oplus ... \oplus a_n = (a_1 \oplus ... \oplus a_{n-1}) \oplus a_n$ provided that the right hand side exists. The definition is independent on permutations of the elements. We say that a finite subset $\{a_1, ..., a_n\}$ of L is *orthogonal* if $a_1 \oplus ... \oplus a_n$ exists.

For a sequence $\{a_n\}$, we say that it is *orthogonal* if, for every n, $\bigoplus_{i \le n} a_i$ exists. If, moreover, $\sup_n \bigoplus_{i \le n} a_i$ exists, we set $\bigoplus_{n \in \mathcal{N}} a_n = \sup_n \bigoplus_{i \le n} a_i$.

An element $a \in L$ is said to be *principal* if $b \perp c$, $b \le a$ and $c \le a$ imply $b \oplus c \le a$.

An element $a \in L$ is said to be *central* if, for every $b \in L$, $b = (b \wedge a) \vee (b \wedge a^\perp)$. The set $C(L)$ of all central element of L is said to be *the centre* of L. By $[R_1]$ (5.5), $a \in L$ is central if and only if a is principal and, for every $b \in L$, $b = (b \wedge a) \oplus (b \wedge a^\perp)$. By 1.9.14 of [D-P], $C(L)$ is a Boolean algebra.

If G is an Abelian group, a function $\mu : L \to G$ is said to be a *measure* if $a \perp b$ implies $\mu(a \oplus b) = \mu(a) + \mu(b)$. It is easy to see that μ is a measure if and only if $a \le b$ implies $\mu(b \ominus a) = \mu(b) - \mu(a)$.

If G is a topological group, μ is said to be *σ-additive* if, for every orthogonal sequence $\{a_n\}$ in L such that $\bigoplus_n a_n$ exists, $\mu(\bigoplus_n a_n) = \sum_{n=1}^{\infty} \mu(a_n)$, *exhaustive* if, for every monotone sequence $\{a_n\}$ in L, $\{\mu(a_n)\}$ is a Cauchy sequence in G, *σ-order continuous* (σ-o.c.) if $a_n \uparrow a$ or $a_n \downarrow a$ in L implies $\lim_n \mu(a_n) = \mu(a)$, and *order-continuous* (o.c.) if the same condition holds for nets. By 2.2 of [A-B], a measure μ is σ-additive if and only if μ is σ-o.c.

In the sequel, we denote by L a D-lattice.

A function $\mu : L \to G$ is said to be *modular* if, for every $a, b \in L$, $\mu(a \vee b) + \mu(a \wedge b) = \mu(a) + \mu(b)$. By [F-T], every modular function on any lattice generates a *lattice uniformity* $\mathcal{U}(\mu)$, i.e. a uniformity which makes uniformly continuous the lattice operations $\vee$ and $\wedge$, and $\mathcal{U}(\mu)$ is the weakest lattice uniformity which makes μ uniformly continuous (see 3.1 of [W_4]). Moreover, by 4.2 of [A-B], if μ is a modular measure on L, then $\mathcal{U}(\mu)$ is a *D-uniformity*, i.e. $\mathcal{U}(\mu)$ makes $\ominus$ (and therefore $\oplus$) uniformly continuous, too, and a base of $\mathcal{U}(\mu)$ is the family consisting of the sets $\{(a, b) \in L \times L : \mu(c) \in W$ for every $c \leq a\triangle b\}$, where W is a neighbourhood of 0 in G.

A D-uniformity is said to be *exhaustive* if every monotone sequence in L is a Cauchy sequence in $\mathcal{U}$, *σ-order-continuous* (σ-o.c.) if $a_n \uparrow a$ or $a_n \downarrow a$ in L implies that $\{a_n\}$ converges to a in $\mathcal{U}$, and *order-continuous* (o.c.) if the same condition holds for nets.

It is also helpful to recall the following result of [A-V] (2.3 and 2.4).

Theorem (1.2) *Let $\mathcal{U}$ be a D-uniformity on L and $\mathcal{F}$ the collection of neighbourhoods of 0 in $\mathcal{U}$. Then:*

(1) *A base of $\mathcal{U}$ is the collection consisting of the sets $F^{\triangle} = \{(a, b) \in L \times L : a\triangle b \in F\}$, where $F \in \mathcal{F}$.*

(2) *$\mathcal{F}$ has the following properties:*

  (a) *For every $F \in \mathcal{F}$, there exists $G \in \mathcal{F}$ such that $a\triangle b \in G$ implies $(a\vee c)\triangle(b\vee c) \in F$ for every $c \in L$.*
  (b) *For every $F \in \mathcal{F}$, there exists $G \in \mathcal{F}$ such that $a\triangle b \in G$ implies $(a\wedge c)\triangle(b\wedge c) \in F$ for every $c \in L$.*
  (c) *For every $F \in \mathcal{F}$, there exists $G \in \mathcal{F}$ such that $a\triangle b \in G$ and $b\triangle c \in G$ imply $a\triangle c \in F$.*
  (d) *For every $F \in \mathcal{F}$, there exists $G \in \mathcal{G}$ such that $a \in G$ implies $(a \vee c) \ominus c \in F$ for every $c \in L$.*
  (e) *For every $F \in \mathcal{F}$, there exists $G \in \mathcal{F}$ such that $a \in G$ and $b \leq a$ imply $b \in F$.*

2. D-uniformities on lattice ordered effect algebras

In this section we prove that the exhaustive D-uniformities on L form a Boolean algebra.

First we need some results.

Lemma (2.1) *If $a, b, c \in L$, $a \perp b$ and $c \leq a$, then $(a \oplus b) \ominus c = (a \ominus c) \oplus b$.*

Proof. By (1.1)-3, we have $(a \oplus b) \ominus (c \oplus b) = a \ominus c$. Since $a \ominus c \leq a$ and $a \perp b$, then $a \ominus c \perp b$. Therefore, by (1.1)-1, 4, we have $(a \ominus c) \oplus b = ((a \oplus b) \ominus (c \oplus b)) \oplus b = ((a \oplus b) \ominus (c \oplus b)) \oplus ((c \oplus b) \ominus c) = (a \oplus b) \ominus c$.

Lemma (2.2) *Let $a, b, c, d \in L$ be such that $a \perp b$, $c \leq a$ and $d \leq b$. Then $(a \oplus b) \ominus (c \oplus d) = (a \ominus c) \oplus (b \ominus d)$.*

Proof. By (1.1)-2 and (2.1), we have $(a \oplus b) \ominus (c \oplus d) = ((a \oplus b) \ominus d) \ominus c = (a \oplus (b \ominus d)) \ominus c = (a \ominus c) \oplus (b \ominus d)$.

Lemma (2.3) *Let p be a central element of L and $a, c \in L$ be with $c \leq a$. Then $(a \ominus c) \wedge p = (a \wedge p) \ominus (c \wedge p)$.*

Proof. The assertion can be obtained as a consequence of (2.1)-2 of [A-B-C]. Here we give a direct proof.

Since p is principal, we have $(c \wedge p) \oplus ((a \ominus c) \wedge p) \leq p$. Moreover, by (1.1)-1, we have $(c \wedge p) \oplus ((a \ominus c) \wedge p) \leq c \oplus (a \ominus c) = a$. Then we get $(c \wedge p) \oplus ((a \ominus c) \wedge p) \leq a \wedge p$, from which, by (1.1)-1, we obtain

$$* \qquad (a \wedge p) \ominus (c \wedge p) \geq (a \ominus c) \wedge p.$$

Since $p^\perp$ is principal, too, in similar way we obtain

$$** \qquad (a \wedge p^\perp) \ominus (c \wedge p^\perp) \geq (a \ominus c) \wedge p^\perp.$$

Moreover, since p is central, we have $a = (a \wedge p) \oplus (a \wedge p^\perp)$ and $c = (c \wedge p) \oplus (c \wedge p^\perp)$. Then, by (2.2), we have $a \ominus c = \big((a \wedge p) \oplus (a \wedge p^\perp)\big) \ominus \big((c \wedge p) \oplus (c \wedge p^\perp)\big) = \big((a \wedge p^\perp) \ominus (c \wedge p^\perp)\big) \oplus \big((a \wedge p) \ominus (c \wedge p)\big)$. By (1.1)-1, we obtain

$$*** \qquad (a \ominus c) \ominus \big((a \wedge p^\perp) \ominus (c \wedge p^\perp)\big) = (a \wedge p) \ominus (c \wedge p).$$

On the other hand, we have also $a \ominus c = \big((a \ominus c) \wedge p\big) \oplus \big((a \ominus c) \wedge p^\perp\big)$. Then, by (1.1)-1, by (**) and by (***) we get $(a \ominus c) \wedge p = (a \ominus c) \ominus \big((a \ominus c) \wedge p^\perp\big) \geq (a \ominus c) \ominus \big((a \wedge p^\perp) \ominus (c \wedge p^\perp)\big) = (a \wedge p) \ominus (c \wedge p)$. By (*) and the last inequality, we obtain the assertion.

A *D-congruence* on L is a lattice congruence with the following property: for every $a, b, c \in L$, $a \sim c, b \sim d, a \leq b$ and $c \leq d$ imply $b \ominus a \sim d \ominus c$.

A *D-ideal* is a lattice ideal I on L with the following properties:

(1)For every $a, b \in I$, with $a \perp b$, $a \oplus b \in I$.

(2)For every $a \in I$ and every $b \in L$, $(a \vee b) \ominus b \in I$.

It is easy to see that, if $\mathcal{U}$ is a D-uniformity, then $N(\mathcal{U}) = \bigcap\{U : U \in \mathcal{U}\}$ is a D-congruence and the closure of $\{0\}$ in $\mathcal{U}$ is a D-ideal.

Lemma (2.4) *If L is complete and $\mathcal{U}$ is a o.c. D-uniformity on L, then there exists a central element p in L such that the closure of $\{0\}$ in $\mathcal{U}$ coincides with $[0, p]$.*

Proof. Denote by I the closure of 0 in $\mathcal{U}$ and set $p = \sup I$. Since $\mathcal{U}$ is o.c., the increasing net $\{b : b \in I\}$ converges to p in $(L, \mathcal{U})$. Since I is closed, then $p \in I$. Since I is a lattice ideal, we get that $I = [0, p]$. We prove that p is central. It is clear that p is principal since I is a D-ideal. Then $p \wedge p^\perp = 0$. Hence, if $a \in L$, we have $a \geq (a \wedge p) \vee (a \wedge p^\perp) = (a \wedge p) \oplus (a \wedge p^\perp)$, from which $a \ominus (a \wedge p^\perp) \geq a \wedge p$. Moreover we have that, for every $a \in L$, $a \ominus (a \wedge p^\perp) \in I$ since by $(p, 0) \in N(\mathcal{U})$ we get $(a \wedge p^\perp, a) \in N(\mathcal{U})$, from which $(a \ominus (a \wedge p^\perp), 0) \in N(\mathcal{U})$. Therefore we have $a \ominus (a \wedge p^\perp) \leq a \wedge p$. Then we have $a \wedge p = a \ominus (a \wedge p^\perp)$ for every $a \in L$. By 2.5 of $[R_2]$, we get that p is central.

For the next results, we use the following:

Notation. If $\mathcal{W}$ is a D-uniformity, we denote by $(\hat{L}, \hat{\mathcal{W}})$ the quotient of $(L, \mathcal{W})$ with respect to the D-congruence $N(\mathcal{W}) = \bigcap\{W : W \in \mathcal{W}\}$, by $(\tilde{L}, \tilde{\mathcal{W}})$ the uniform completion of $(\hat{L}, \hat{\mathcal{W}})$, and by $\overline{\mathcal{W}}$ the restriction of $\tilde{\mathcal{W}}$ to the centre $C(\tilde{L})$ of $\tilde{L}$.

Moreover we denote by $\mathcal{LU}(L, \mathcal{W})$ the lattice of all lattice uniformities finer than $\mathcal{W}$ and by $DU(L, \mathcal{W})$ the set of all D-uniformities in $\mathcal{LU}(L, \mathcal{W})$.

It is easy to see that $DU(L,\mathcal{W})$ is a complete sublattice of $\mathcal{L}U(L,\mathcal{W})$.
We use similar notations for $\hat{L}$, $\tilde{L}$ and $C(\tilde{L})$.

It is also helpful to recall the following result.

Proposition (2.5) *Let $\mathcal{W}$ be a D-uniformity on L. Then:*

(1) *$\hat{L}$ and $\tilde{L}$ are D-lattices and $\hat{\mathcal{W}}$, $\tilde{\mathcal{W}}$, $\overline{\mathcal{W}}$ are Hausdorff D-uniformities.*

(2) *If $\mathcal{W}$ is exhaustive, then $\tilde{\mathcal{W}}$ and $\overline{\mathcal{W}}$ are exhaustive and o.c., and $(\tilde{L},\leq)$, $(C(\tilde{L}),\leq)$ are complete.*

Proof. (1) has been proved in 4.2 of [A-B].

(2) By 3.7 of $[W_4]$, $\tilde{\mathcal{W}}$ is exhaustive and o.c., and $(\tilde{L},\leq)$ is complete. Moreover, by 3.4 of $[W_6]$, $C(\tilde{L})$ is a complete Boolean algebra with the following property: if $a_\alpha \uparrow a$ (respectively, $a_\alpha \downarrow a$) in $C(\tilde{L})$, then $a_\alpha \uparrow a$ (respectively, $a_\alpha \downarrow a$) in $\tilde{L}$. Therefore it is clear that $\overline{\mathcal{W}}$ is exhaustive and o.c., too.

Recall also that, by 4.1 of [A-B], a lattice uniformity $\mathcal{W}$ is a D-uniformity if and only if, for every $U \in \mathcal{W}$, there exist $V \in \mathcal{W}$ such that $V \ominus V \subseteq U$, where

$$V \ominus V = \{(a \ominus c, b \ominus d) : (a,b) \in V, (c,d) \in V, c \leq a, d \leq b\}.$$

Moreover, by (1.2), a base of a D-uniformity $\mathcal{W}$ is the family consisting of the sets $\{(a,b) \in L \times L : a \triangle b \in U_0\}$, where U_0 is a neighbourhood of 0 in $\mathcal{W}$.

Proposition (2.6) *Let $\mathcal{W}$ be a D-uniformity on L. Then the lattices $DU(L,\mathcal{W})$ and $DU(\hat{L},\hat{\mathcal{W}})$ are isomorphic.*

Proof. For $a \in L$, denote by $\hat{a}$ the equivalence class of a in $\hat{L}$. For every $U \in \mathcal{L}U(L,\mathcal{W})$, set $\hat{U} = \{(\hat{a},\hat{b}) : (a,b) \in U\}$ for $U \in \mathcal{U}$ and $\hat{\mathcal{U}} = \{\hat{U} : U \in \mathcal{U}\}$. By $[W_1]$ (page 381), the map $\mathcal{U} \to \hat{\mathcal{U}}$ is a lattice isomorphism between $\mathcal{L}U(L,\mathcal{W})$ and $\mathcal{L}U(\hat{L},\hat{\mathcal{W}})$. Then we obtain the assertion observing that $\mathcal{U}$ is a D-uniformity if and only if $\hat{\mathcal{U}}$ is a D-uniformity.

Proposition (2.7) *Suppose that $\mathcal{W}$ is a Hausdorff exhaustive D-uniformity on L. Then the lattices $DU(L,\mathcal{W})$ and $DU(\tilde{L},\tilde{\mathcal{W}})$ are isomorphic.*

Proof. By 3.8 of $[W_4]$, the map $\tilde{\mathcal{U}} \in \mathcal{L}U(L,\mathcal{W}) \to \tilde{\mathcal{U}}_{|L} \in \mathcal{L}U(L,\mathcal{W})$ is a lattice isomorphism. Then we have only to prove that $\tilde{\mathcal{U}} \in DU(\tilde{L},\tilde{\mathcal{W}})$ if and only if $\tilde{\mathcal{U}}_{|L} \in DU(L,\mathcal{W})$.

Let $\tilde{\mathcal{U}}$ be a D-uniformity on $\tilde{L}$. If $\tilde{U},\tilde{V} \in \tilde{\mathcal{U}}$ and $\tilde{V} \ominus \tilde{V} \subseteq \tilde{U}$, then obviously we have $(\tilde{V} \cap (L\times L)) \ominus (\tilde{V} \cap (L \times L)) \subseteq \tilde{U} \cap (L \times L)$. Therefore, it is clear that $\tilde{\mathcal{U}}_{|L}$ is a D-uniformity, too.

Conversely, let $\mathcal{U}$ be a D-uniformity in $\mathcal{L}U(L,\mathcal{W})$. By 1.5 of $[W_1]$, $\mathcal{U}$ has an extension $\tilde{\mathcal{U}} \in \mathcal{L}U(\tilde{L},\tilde{\mathcal{W}})$ which has as base the family $\{\overline{U} : U \in \mathcal{U}\}$, where $\overline{U}$ denotes the closure of U in the product uniformity $\tilde{\mathcal{W}} \times \tilde{\mathcal{W}}$ on $L \times L$.

We have to prove that, for every $\tilde{U} \in \tilde{\mathcal{U}}$, there exists $\tilde{V} \in \tilde{\mathcal{U}}$ such that $\tilde{V} \ominus \tilde{V} \subseteq \tilde{U}$.

Let $U,V \in \mathcal{U}$ be such that $\overline{U} \subseteq \tilde{U}$ and $V \ominus V \subseteq U$. It is sufficient to prove that $\overline{V} \ominus \overline{V} \subseteq \overline{V \ominus V}$. Let $a,b,c,d \in \tilde{L}$ be such that $c \leq a$, $d \leq b$, $(a,b) \in \overline{V}$ and $(c,d) \in \overline{V}$. Recall that a base in $\tilde{\mathcal{W}} \times \tilde{\mathcal{W}}$ consists of all sets $\{((a,b),(c,d)) : (a,c) \in W, (b,d) \in W\}$, where $W \in \tilde{\mathcal{W}}$. Then, for each $W \in \tilde{\mathcal{W}}$, we may choose $a_W, b_W, c_W, d_W \in L$ such that $(a_W,a), (b_W,b), (c_W,c), (d_W,d) \in W$ and $(a_W,b_W), (c_W,d_W) \in V$. Since $\tilde{\mathcal{W}}$ is a lattice uniformity, we may assume $c_W \leq a_W$ and $d_W \leq b_W$. Clearly $\{a_W : W \in \tilde{\mathcal{W}}\}$, $\{b_W : W \in \tilde{\mathcal{W}}\}$, $\{c_W : W \in \tilde{\mathcal{W}}\}$ and $\{d_W : W \in \tilde{\mathcal{W}}\}$ are nets which converge, respectively, to a,b,c,d

in $\tilde{\mathcal{W}}$. Then we have that $\{a_W \ominus c_W\}$ converges to $a \ominus c$ in $\tilde{\mathcal{W}}$ and $\{b_W \ominus d_W\}$ converges to $b \ominus d$ in $\tilde{\mathcal{W}}$. Therefore $(a \ominus c, b \ominus d) \in \overline{V \ominus V}$.

Theorem (2.8) *Suppose that L is complete. Let $\mathcal{W}$ be a o.c. Hausdorff D-uniformity on L. Then $DU(L, \mathcal{W})$ is a Boolean algebra isomorphic to the centre $C(L)$ of L.*

Proof. For every $\mathcal{U} \in DU(L, \mathcal{W})$, denote by $I(\mathcal{U})$ the closure of $\{0\}$ in $\mathcal{U}$, and let

$$\phi : \mathcal{U} \in DU(L, \mathcal{W}) \to \sup I(\mathcal{U}) \in C(L).$$

By (2.4), ϕ is well defined. We prove that ϕ is a dual lattice isomorphism.

Let $p \in C(L)$. We first show that there exists $\mathcal{U} \in DU(L, \mathcal{W})$ such that $I(\mathcal{U}) = p$.

Set $\mathcal{B}_0 = \{U_W : W \in \mathcal{W}\}$, where, for each $W \in \mathcal{W}$,

$$U_W = \{(a, b) \in L \times L : (a \wedge p^{\perp}, b \wedge p^{\perp}) \in W\}.$$

We prove that $\mathcal{B}_0$ is a base of a D-uniformity.

It is clear that, for every $W_1, W_2 \in \mathcal{W}$, we have

$$U_{W_1} \cap U_{W_2} = U_{W_1 \cap W_2},\ U_{W_1^{-1}} = (U_{W_1})^{-1},\ U_{W_1} \circ U_{W_2} \subseteq U_{W_1 \circ W_2}.$$

Now let $W, W' \in \mathcal{W}$ be such that $W' \ominus W' \subseteq W$. We show that $U_{W'} \ominus U_{W'} \subseteq U_W$.

Let $(a, b), (c, d) \in U_{W'}$ be such that $c \leq a$ and $d \leq b$. Then we have

$$\Big((a \wedge p^{\perp}) \ominus (c \wedge p^{\perp}), (b \wedge p^{\perp}) \ominus (d \wedge p^{\perp})\Big) \in W' \ominus W' \subseteq W.$$

By (2.3), we obtain $\big((a \ominus c) \wedge p^{\perp}, (b \ominus d) \wedge p^{\perp}\big) \in W' \ominus W' \subseteq W$, i.e. $(a \ominus c, b \ominus d) \in U_W$.

Now let $W, W' \in \mathcal{W}$ be such that $W' \wedge W' \subseteq W$. We have $U_{W'} \wedge U_{W'} \subseteq U_W$ since, if $(a, b), (c, d) \in U_{W'}$, we have $(a \wedge c \wedge p^{\perp}, b \wedge d \wedge p^{\perp}) = (a \wedge p^{\perp}, b \wedge p^{\perp}) \wedge (c \wedge p^{\perp}, d \wedge p^{\perp}) \in W' \wedge W' \subseteq W$, from which we get $(a, b) \wedge (c, d) \in U_W$.
Since $a \vee b = 1 \ominus ((1 \ominus a) \wedge (1 \ominus b))$, we have that also $\vee$ is uniformly continuous.
Hence $\mathcal{B}_0$ is a base for a D-uniformity $\mathcal{U}_0$.

Moreover we see that $\mathcal{U}_0 \leq \mathcal{W}$. Indeed, for each $W \in \mathcal{W}$, we can find $V \in \mathcal{W}$ such that and $V \wedge \Delta \subseteq W$, whence $V \subseteq U_W$.

Finally we have $I(\mathcal{U}_0) = [0, p]$. Indeed, since p is central and therefore $a = (a \wedge p) \oplus (a \wedge p^{\perp})$ for every $a \in L$, we have that $c \in I(\mathcal{U}_0) \Leftrightarrow (c, 0) \in N(\mathcal{U}_0) \Leftrightarrow (c \wedge p^{\perp}, 0) \in N(\mathcal{W}) = \Delta \Leftrightarrow c \wedge p^{\perp} = 0 \Leftrightarrow c \leq p$.

Now let $\mathcal{U}, \mathcal{V} \in DU(L, \mathcal{W})$. It is clear that $\mathcal{U} \leq \mathcal{V}$ implies $\phi(\mathcal{U}) \geq \phi(\mathcal{V})$. Conversely, let $\phi(\mathcal{U}) \leq \phi(\mathcal{V})$. By (2.4) we get that $I(\mathcal{U}) \subseteq I(\mathcal{V})$. Since $(a, b) \in N(\mathcal{U})$ if and only if $a \triangle b \in I(\mathcal{U})$, we have $N(\mathcal{U}) \subseteq N(\mathcal{V})$. By 6.7 of $[W_2]$, we obtain that the topology generated by $\mathcal{U}$ is finer then the topology generated by $\mathcal{V}$. By (1.2)-1, we obtain that $\mathcal{U} \geq \mathcal{V}$.

Corollary (2.9) *Let $\mathcal{W}$ be an exhaustive D-uniformity on L. Then the lattices $DU(L, \mathcal{W})$, $DU(\hat{L}, \hat{\mathcal{W}})$, $DU(\tilde{L}, \tilde{\mathcal{W}})$ and $DU(C(\tilde{L}), \overline{\mathcal{W}})$ are Boolean algebras isomorphic to $C(\tilde{L})$. In particular, the lattice of all exhaustive D-uniformities on L is a Boolean algebra.*

Proof. By (2.6) and (2.7), $DU(L, \mathcal{W})$, $DU(\hat{L}, \hat{\mathcal{W}})$ and $DU(\tilde{L}, \tilde{\mathcal{W}})$ are isomorphic. Moreover, by (2.5), $\tilde{\mathcal{W}}$ and $\overline{\mathcal{W}}$ are o.c. and Hausdorff, and $(\tilde{L}, \leq)$, $(C(\tilde{L}), \leq)$ are complete. Then, by (2.8), $DU(\tilde{L}, \tilde{\mathcal{W}})$ and $DU(C(\tilde{L}), \overline{\mathcal{W}})$ are isomorphic to the Boolean algebra $C(\tilde{L})$.

For the second part of the statement, take as $\mathcal{W}$ the supremum of all exhaustive D-uniformities.

Remark (2.10). Let $\mathcal{W}$ be an exhaustive D-uniformity on L. For $\mathcal{U} \in DU(L, \mathcal{W})$, denote by $\overline{\mathcal{U}}$ the element of $DU(C(\tilde{L}), \overline{\mathcal{W}})$ which correspond to $\mathcal{U}$ in the isomorphism of (2.9) between $DU(L,\mathcal{W})$ and $DU(C(\tilde{L}),\overline{\mathcal{W}})$. By the proofs of (2.8) and (2.9), we obtain that the isomorphism between $DU(L,\mathcal{W})$ and $C(\tilde{L})$ in (2.9) is the map

$$\phi : \mathcal{U} \in DU(L,\mathcal{W}) \to \left(\sup \overline{\{0\}}^{\mathcal{U}}\right)^{\perp} \in C(\tilde{L}).$$

3. A decomposition theorem

In this section we derive from the results of Section 2 and a result of H. Weber a decomposition theorem for modular measures on L, which contains as particular cases Lebesgue and Hewitt-Yosida type decompositions.

We denote by G, G' complete Hausdorff topological Abelian groups.

If $\mu : L \to G$ is a modular function, we denote by $\mathcal{U}(\mu)$ the lattice uniformity generated by μ (see Section 1).

We say that two lattice uniformities $\mathcal{U}$ and $\mathcal{V}$ are *permutable* if, for every $U \in \mathcal{U}$ and $V \in \mathcal{V}$, there exist $U' \in \mathcal{U}$ and $V' \in \mathcal{V}$ such that $V' \circ U' \subseteq U \circ V$.

$\mathcal{U} \wedge \mathcal{V} = 0$ means that the infimum of $\mathcal{U}$ and $\mathcal{V}$ is the trivial uniformity.

By 3.14 of $[W_3]$, the following result holds.

Theorem (3.1) *Let L' be a lattice, $\mathcal{LU}(L')$ be the lattice of all lattice uniformities on L' and $\mathcal{B}$ be a Boolean sublattice of $\mathcal{LU}(L')$ which contains the trivial uniformity and a greatest element $\mathcal{W}$. Assume that every two elements of $\mathcal{B}$ are permutable. Let $\mathcal{U} \in \mathcal{B}$ and $\mu : L' \to G$ be a $\mathcal{W}$-uniformly continuous modular function. Then there exists unique modular functions $\lambda, \nu : L' \to G$ such that $\mu = \lambda + \nu$, $\mathcal{U}(\lambda) \leq \mathcal{U}$, $\mathcal{U}(\nu) \wedge \mathcal{U} = 0$, and $\mathcal{U}(\mu) = \mathcal{U}(\lambda) \vee \mathcal{U}(\nu)$. Moreover, if L' is a D-lattice and μ is a measure, then λ and μ are measures, too, and $\lambda(L'), \nu(L')$ are contained in the closure of $\mu(L')$.*

To derive by (3.1) a decomposition theorem in L, we need some definitions and results.

Proposition (3.2) *If $\mathcal{U}$ and $\mathcal{V}$ are D-uniformities on L, then $\mathcal{U}$ and $\mathcal{V}$ are permutable.*

Proof. Denote by $\mathcal{F}$ and $\mathcal{G}$ the systems of neighbourhoods of 0, respectively, in $\mathcal{U}$ and in $\mathcal{V}$. Let $U \in \mathcal{U}$ and $V \in \mathcal{V}$. By (1.2) we can choose $F \in \mathcal{F}$ and $G \in \mathcal{G}$ such that $F^{\Delta} \subseteq U$ and $G^{\Delta} \subseteq V$. Choose $F_1, F_2, F_3 \in \mathcal{F}$ such that F_1 satisfies (c) of (1.2), F_2 satisfies (b) of (1.2) with respect to F_1 and F_3 satisfies (a) of (1.2) with respect to F_2. In similar way we choose $G_1, G_2, G_3 \in \mathcal{G}$. Note that $F_2 \subseteq F_1$ and $G_2 \subseteq G_1$.

We prove that $G_3^{\Delta} \circ F_3^{\Delta} \subseteq F^{\Delta} \circ G^{\Delta} \subseteq U \circ V$.

Let $(a,c) \in G_3^{\Delta} \circ F_3^{\Delta}$ and choose $b \in L$ such that $(a,b) \in G_3^{\Delta}$ and $(b,c) \in F_3^{\Delta}$. Then we have $a \Delta (a \vee b) = (a \vee a) \Delta (b \vee a) \in G_2$ and $(a \vee b \vee c) \ominus (b \vee c) = (a \vee (b \vee c)) \Delta (b \vee (b \vee c)) \in G_2$. In similar way we obtain $(b \vee c) \Delta c \in F_2$ and $(a \vee b \vee c) \ominus (a \vee b) \in F_2$. Set

$$a_1 = (a \vee b \vee c) \ominus (b \vee c), \ c_1 = (a \vee b \vee c) \ominus (a \vee b).$$

By (1.1)-6, we have $(a \vee b \vee c) \ominus a_1 = b \vee c$ and $(a \vee b \vee c) \ominus c_1 = a \vee b$. Observe that $c_1 \perp a$ and $a_1 \perp c$, and set $e = a \oplus c_1$, $f = a_1 \oplus c$ and $d = e \wedge f$. By (1.1)-4, we have $e = a \oplus \big((a \vee b \vee c) \ominus (a \vee b)\big) = (a \vee b \vee c) \ominus \big((a \vee b) \ominus a\big) \leq a \vee b \vee c$ and, similarly, $f \leq a \vee b \vee c$. Moreover we have $a \Delta e = e \ominus a = c_1 \in F_2$ and, by (1.1)-8, $e \Delta (a \vee b \vee c) = (e \ominus c_1) \Delta \big((a \vee b \vee c) \ominus c_1\big) = a \Delta (a \vee b) \in G_2$. In the same way we obtain $f \Delta c = f \ominus c = a_1 \in G_2$ and $(a \vee b \vee c) \Delta f = (b \vee c) \Delta c \in F_2$. Therefore $e \Delta d = (e \wedge (a \vee b \vee c)) \Delta (e \wedge f) \in F_1$ and $a \Delta e \in F_2 \subseteq F_1$. Then we get $a \Delta d \in F$, i.e. $(a,d) \in F^{\Delta}$. Similarly we get $d \Delta f = (e \wedge f) \Delta \big((a \vee b \vee c) \wedge f\big) \in G_1$, i.e. $(d,c) \in G^{\Delta}$. We conclude that $(a,c) \in F^{\Delta} \circ G^{\Delta}$.

If $\mu : L \to G$ is a modular measure and $\mathcal{U}$ is a D-uniformity, we write $\mu << \mathcal{U}$ if μ is $\mathcal{U}$-continuous in 0.

If $\lambda : L \to G'$ is another modular measure, $\mu << \lambda$ means $\mu << \mathcal{U}(\lambda)$.

Recall that a base of neighbourhoods of 0 in $\mathcal{U}(\lambda)$ is the family consisting of the sets $\{a \in L : \lambda(b) \in W \ \forall\, b \leq a\}$, where W is a neighbourhood of 0 in G' (see Section 1). Then we have that $\mu << \lambda$ if and only if, for every neighbourhood W of 0 in G, there exists a neighbourhood W' of 0 in G' such that, for every $a \in L$, $\lambda(b) \in W'$ for every $b \leq a$ implies $\mu(a) \in W$.

Proposition (3.3) *Let $\mu : L \to G$ be a modular measure and $\mathcal{U}$ a D-uniformity. Then the following conditions are equivalent:*

(1) $\mu << \mathcal{U}$.

(2) μ *is $\mathcal{U}$-uniformly continuous.*

(3) $\mathcal{U}(\mu) \leq \mathcal{U}$.

Proof. It has been proved in 3.5 of [A] that μ is continuous in 0 if and only if μ is uniformly continuous. Since $\mathcal{U}(\mu)$ is the weakest D-uniformity which makes μ uniformly continuous, we have that $\mu << \mathcal{U}$ if and only if $\mathcal{U}(\mu) \leq \mathcal{U}$.

We say that two D-uniformities $\mathcal{U}$ and $\mathcal{V}$ are *singular* if, for all neighbourhoods F, G of 0, respectively, in $\mathcal{U}$ and in $\mathcal{V}$, there exists $a \in L$ such that $a \in F$ and $a^\perp \in G$. In particular, if $\mathcal{U}$ is generated by a modular measure $\mu : L \to G$, we write $\mu \perp \mathcal{V}$ if $\mathcal{U}(\mu) \perp \mathcal{V}$. If $\lambda : L \to G'$ is another modular measure, $\mu \perp \lambda$ means $\mathcal{U}(\mu) \perp \mathcal{U}(\lambda)$. In this case, we say that μ and λ are *singular.* By (1.2)-1, we have that $\mu \perp \lambda$ if and only if, for all neighbourhoods W and W' of 0, respectively, in G and in G', there exists $a \in L$ such that $\mu(b) \in W$ for every $b \leq a$ and $\lambda(c) \in W'$ for every $c \leq a^\perp$.

Proposition (3.4) *Let $\mathcal{U}, \mathcal{V}$ be D-uniformities on L. Then the following conditions are equivalent:*

(1) $\mathcal{U} \perp \mathcal{V}$.

(2) $\mathcal{U} \wedge \mathcal{V} = 0$.

Proof. Denote by $\mathcal{F}$ and $\mathcal{G}$ the systems of neighbourhoods of 0, respectively, in $\mathcal{U}$ and in $\mathcal{V}$. By 2.4-c and 2.6-a of [A-V], $\mathcal{U} \wedge \mathcal{V} = 0$ if and only if, for every $F \in \mathcal{F}$ and $G \in \mathcal{G}$, $F \oplus G = L$, where $F \oplus G = \{a \oplus b : a \in F, b \in G, a \perp b\}$. Hence (2) $\Rightarrow$ (1) immediately follows.

(1) $\Rightarrow$ (2) Let $F \in \mathcal{F}$, $G \in \mathcal{G}$ and $a \in L$. By (1.2) we can choose $F' \in \mathcal{F}$ such that $a \leq b \in F'$ implies $a \in F$ and $G' \in \mathcal{G}$ such that $a \in G'$ implies $(a \vee b) \ominus b \in G$ for every $a \in L$. By (1), we can find $c \in L$ such that $c \in F'$ and $c^\perp \in G'$. Set $b = a \wedge c$. Then we have $a = b \oplus (a \ominus b)$, where $b \in F$ since $b \leq c \in F'$ and $a \ominus b \in G$ since, by (1.1)-7, we have $a \ominus b = a \ominus (a \wedge c) = (c^\perp \vee a^\perp) \ominus a^\perp \in G$. Since a is arbitrary, we conclude that $F \oplus G = L$.

Now, as a consequence of the results of Section 2, by (3.1) we obtain the following decomposition theorem.

Theorem (3.5) *Let $\mu : L \to G$ be an exhaustive modular measure and $\mathcal{U}$ a D-uniformity on L. Then there exist unique modular measures $\lambda, \nu : L \to G$ such that $\mu = \lambda + \nu$, $\lambda << \mathcal{U}$ and $\nu \perp \mathcal{U}$. Moreover λ and ν are exhaustive and singular, $\lambda(L)$, $\nu(L)$ are contained in $\overline{\mu(L)}$, and $\mathcal{U}(\mu) = \mathcal{U}(\lambda) \vee \mathcal{U}(\nu)$.*

Proof. Denote by $\mathcal{B}$ the lattice of all exhaustive D-uniformities on L and let $\mathcal{W} = \sup \mathcal{B}$. By (2.9), $\mathcal{B}$ is a Boolean sublattice of the lattice of all lattice uniformities on L. Moreover, by

(3.2), any two elements of $\mathcal{B}$ are permutable and, since by 3.2 of $[W_4]$ $\mathcal{U}(\mu)$ is exhaustive, we have $\mathcal{U}(\mu) \leq \mathcal{W}$, and therefore by (3.3) μ is $\mathcal{W}$-uniformly continuous.

Set $\mathcal{V} = \mathcal{U}(\mu) \wedge \mathcal{U}$. Since $\mathcal{V} \in \mathcal{B}$, by (3.1) there exist unique modular measures $\lambda, \nu : L \to G$ such that $\mu = \lambda + \nu$, $\mathcal{U}(\lambda) \leq \mathcal{V}$ and $\mathcal{U}(\nu) \wedge \mathcal{V} = 0$. Moreover $\mathcal{U}(\mu) = \mathcal{U}(\lambda) \vee \mathcal{U}(\nu)$, $\lambda(L)$ and $\nu(L)$ are contained in $\mu(L)$ and λ, ν are exhaustive since $\mathcal{U}(\lambda) \leq \mathcal{U}(\mu)$ and $\mathcal{U}(\nu) \leq \mathcal{U}(\mu)$.

Since $\mathcal{V} \leq \mathcal{U}$, we have $\mathcal{U}(\lambda) \leq \mathcal{U}$. Moreover, since $\mathcal{U}(\nu) \leq \mathcal{U}(\mu)$, we have $\mathcal{U}(\nu) \wedge \mathcal{U} = \mathcal{U}(\nu) \wedge \mathcal{U}(\mu) \wedge \mathcal{U} = \mathcal{U}(\nu) \wedge \mathcal{V} = 0$. By (3.3) and (3.4) we have $\lambda << \mathcal{U}$ and $\nu \perp \mathcal{U}$. Moreover, since $\mathcal{U}(\lambda) \wedge \mathcal{U}(\nu) = \mathcal{U}(\lambda) \wedge \mathcal{U} \wedge \mathcal{U}(\nu) = 0$, we have $\lambda \perp \nu$.

Different choices of $\mathcal{U}$ in (3.5) give different decomposition theorems.

For example, if we take as $\mathcal{U}$ the uniformity generated by a modular measure $m : L \to G'$, we obtain a Lebesgue decomposition theorem.

Corollary (3.6) (Lebesgue decomposition theorem). *Let $\mu : L \to G$ be an exhaustive modular measure and $m : L \to G'$ a modular measure. Then there exist unique G-valued singular modular measures λ and ν on L such that $\mu = \lambda + \nu$, $\lambda << m$ and $\nu \perp m$. Moreover $\mathcal{U}(\mu) = \mathcal{U}(\lambda) \vee \mathcal{U}(\nu)$.*

Now we want derive by (3.5) a Hewitt-Yosida decomposition theorem. First we need some results.

Lemma (3.7) *Let $\mu : L \to G$ be an exhaustive modular measure. Then, for every D-uniformity $\mathcal{U} \leq \mathcal{U}(\mu)$, there exists a modular measure $\nu : L \to G$ such that $\nu << \mu$ and $\mathcal{U} = \mathcal{U}(\nu)$.*

Proof. By (3.5) we can find exhaustive modular measures λ and ν such that $\mu = \lambda + \nu$, $\mathcal{U}(\mu) = \mathcal{U}(\lambda) \vee \mathcal{U}(\nu)$, $\mathcal{U}(\lambda) \wedge \mathcal{U} = 0$ and $\mathcal{U}(\nu) \leq \mathcal{U}$. By (3.3) we have $\nu << \mu$. Moreover, since by (2.9) the lattice of all exhaustive D-uniformities is distributive, we get $\mathcal{U} = \mathcal{U} \wedge \mathcal{U}(\mu) = \mathcal{U} \wedge ((\mathcal{U}(\lambda) \vee \mathcal{U}(\nu)) = (\mathcal{U} \wedge \mathcal{U}(\lambda)) \vee (\mathcal{U} \wedge \mathcal{U}(\nu)) = \mathcal{U} \wedge \mathcal{U}(\nu) = \mathcal{U}(\nu)$.

A modular measure $\mu : L \to G$ is said to be *purely non σ-additive* if the zero-measure is the only σ-additive modular measure λ with $\lambda << \mu$.

Lemma (3.8) *Let $\mu : L \to G$ be a modular measure. Denote by $\mathcal{U}_\sigma$ the supremum of all σ-o.c. D-uniformities on L. Then the following conditions are equivalent:*

(1) *μ is purely non σ-additive.*

(2) $\mu \perp \mathcal{U}_\sigma$.

(3) *$\mu \perp \lambda$ for every σ-additive modular measure λ.*

Proof. (1) $\Rightarrow$ (2) By (3.7) we can find a modular measure ν such that $\nu << \mu$ and $\mathcal{U}(\mu) \wedge \mathcal{U}_\sigma = \mathcal{U}(\nu)$. Then $\mathcal{U}(\nu)$ is σ-o.c. and therefore, by (3.2) of $[W_4]$, ν is σ-o.c., too. Hence, by 2.2 of [A-B], ν is σ-additive. By (1), we get $\nu = 0$. Therefore $\mathcal{U}(\nu) = 0$, i.e. $\mu \perp \mathcal{U}_\sigma$ by (3.4).

(2) $\Rightarrow$ (3) If λ is a σ-additive modular measure, by 2.2 of [A-B] λ is σ-o.c. and therefore we have $\mathcal{U}(\mu) \wedge \mathcal{U}(\lambda) = \mathcal{U}(\mu) \wedge \mathcal{U}(\lambda) \wedge \mathcal{U}_\sigma = 0$. Hence, by (3.4), $\mu \perp \lambda$.

(3) $\Rightarrow$ (1) If λ is a σ-additive modular measure with $\lambda << \mu$, by (3.3) and (3.4) we have $\mathcal{U}(\lambda) = \mathcal{U}(\lambda) \wedge \mathcal{U}(\mu) = 0$, whence $\lambda = 0$.

Corollary (3.9) (Hewitt-Yosida decomposition theorem). *Let $\mu : L \to G$ be an exhaustive modular measure. Then there exist unique G-valued modular measures λ and ν on L such that $\mu = \lambda + \nu$, λ is σ-additive and ν is purely non σ-additive. Moreover $\mathcal{U}(\mu) = \mathcal{U}(\lambda) \vee \mathcal{U}(\nu)$.*

Proof. Take in (3.5) as $\mathcal{U}$ the supremum of all σ-o.c. D-uniformities on L, and apply (3.8).

4. Control theorems

In this section we derive from the results of Section 2 a technique which allows us to transfer control theorems known for measures on Boolean algebras - as Bartle-Dunford-Schwartz and Rybakov theorems - to control theorems for modular measures on L.

We denote by X, Y complete Hausdorff locally convex linear spaces.

If $\mu : L \to X$ and and $\nu : L \to Y$ are modular measures, we say that ν is a *control* for μ if $\mu << \nu$ and $\nu << \mu$ (see Section 3).

Recall that, by (3.3), ν is a control for μ if and only if $\mathcal{U}(\mu) = \mathcal{U}(\nu)$.

If M is a collection of X-valued modular measures on L, we say that a modular measure ν is a *control* for M if $\mathcal{U}(\nu) = \sup\{\mathcal{U}(\mu) : \mu \in M\}$.

If $\mathcal{W}$ is an exhaustive D-uniformity, we denote by

$$DU(\hat{L}, \hat{\mathcal{W}}),\ DU(\tilde{L}, \tilde{\mathcal{W}}),\ DU(C(\tilde{L}), \overline{\mathcal{W}})$$

the lattices introduced in Section 2. If $\mathcal{U} \in DU(L, \mathcal{W})$, we denote by $\hat{\mathcal{U}}$,$\tilde{\mathcal{U}}$ and $\overline{\mathcal{U}}$ the elements which correspond in the isomorphism between $DU(L, \mathcal{W})$ and the other lattices, respectively (see (2.9)). Moreover, as in Section 2, we denote by $\mathcal{U}(\mu)$ the D-uniformity generated by a X-valued modular measure μ on L.

If μ is a $\mathcal{W}$-continuous modular measure on L, we denote by $\hat{\mu}$ the modular measure defined by $\hat{\mu}(\hat{a}) = \mu(a)$ for $a \in \hat{a} \in \hat{L}$, by $\tilde{\mu}$ the uniformly continuous extension of $\hat{\mu}$ to $(\tilde{L}, \tilde{\mathcal{W}})$ and by $\overline{\mu}$ the restriction of $\tilde{\mu}$ to $C(\tilde{L})$. It is clear that $\tilde{\mu}$ is a o.c. modular measure (see (2.5)) and therefore $\overline{\mu}$ is a measure on a Boolean algebra.

An essential step to obtain control theorems is the following result.

Proposition (4.1) *Let $\mathcal{W}$ be an exhaustive D-uniformity on L. Then:*

(1) *The map $\phi : \mu \to \overline{\mu}$ is a monomorphism between the linear space of all $\mathcal{W}$-continuous X-valued modular measures on L and the linear space of all $\overline{\mathcal{W}}$- continuous X-valued measures on $C(\tilde{L})$.*

(2) *If $X = R$ and $\mathcal{W}$ is the supremum of the D-uniformities generated by all bounded real-valued modular measures on L, then the map ϕ in (1) is an isomorphism between the linear space of all bounded real-valued modular measures on L and the linear space of all completely additive measures on $C(\tilde{L})$.*

(3) *If $\mu : L \to X$ is a $\mathcal{W}$-continuous modular measure and $\mathcal{U} = \mathcal{U}(\mu)$, then $\tilde{\mathcal{U}} = \mathcal{U}(\tilde{\mu})$ and $\overline{\mathcal{U}} = \mathcal{U}(\overline{\mu})$.*

(4) *If $\mu : L \to X$ and $\nu : L \to Y$ are $\mathcal{W}$-continuous modular measures, then $\mu << \nu$ if and only if $\tilde{\mu} << \tilde{\nu}$ if and only if $\overline{\mu} << \overline{\nu}$.*

Proof. (1) It is clear that the maps $\mu \to \hat{\mu} \to \tilde{\mu}$ are isomorphisms. Moreover, if $X = R$, the map $\tilde{\mu} \to \overline{\mu}$ is injective since, by 2.7 of [A-B-V], $\tilde{\mu}$ attains its supremum on $C(\tilde{L})$. Then in the general case ϕ is injective since the dual space X' of X separates the points and, by 6.3 of [W_4], the topology generated by μ is the supremum of the topologies generated by the modular measures $x' \circ \mu$, with $x' \in X'$.

(2) has been proved in 4.3 of [A-B-V].

(3) The equality $\tilde{\mathcal{U}} = \mathcal{U}(\tilde{\mu})$ has been proved in 3.8 of [W_4]. By (2.9) applied with $C(\tilde{L})$ in place of L, we obtain that $\overline{\mathcal{U}} = \mathcal{U}(\tilde{\mu})_{|C(\tilde{L})}$. Then, to prove the other equality, we have to prove that $\mathcal{U}(\tilde{\mu})_{|C(\tilde{L})} = \mathcal{U}(\overline{\mu})$. By (2.10), it is sufficient to prove that the elements of $C(\tilde{L})$ which correspond to $\overline{\mathcal{U}}$ and to $\mathcal{U}(\overline{\mu})$ in the isomorphism of (2.10) are equal.

Recall that, for a X-valued modular measure λ, the closure of $\{0\}$ in $\mathcal{U}(\lambda)$ is the set $\{a \in L : \lambda(b) = 0 \ \forall \ b \in L, b \leq a\}$ (see Section 1).

Then, by (2.10), we have to prove that, if $a \in C(\tilde{L})$, from $\tilde{\mu}(b) = 0$ for every $b \in C(\tilde{L})$ with $b \leq a$ it follows $\tilde{\mu}(b) = 0$ for every $b \in \tilde{L}$ with $b \leq a$.

Let $a \in C(\tilde{L})$ and suppose $\tilde{\mu}(b) = 0$ for every $b \in C(\tilde{L})$ with $b \leq a$. Set $\tilde{\nu}(b) = \tilde{\mu}(b \wedge a)$ for $b \in \tilde{L}$. By 2.2 of [A-B-V], $\tilde{\nu}$ is a bounded modular measure on $\tilde{L}$. By assumption, we have that $\overline{\nu} = 0$. By (1) applied to $\tilde{L}$ in place of L, we get $\tilde{\nu} = 0$, i.e. $\tilde{\mu}(b) = 0$ for every $b \in \tilde{L}$ with $b \leq a$.

By (3.3), (2.9) and (3), we have that $\mu << \nu \Leftrightarrow \mathcal{U}(\mu) \leq \mathcal{U}(\nu) \Leftrightarrow \mathcal{U}(\overline{\mu}) \leq \mathcal{U}(\overline{\nu}) \Leftrightarrow \overline{\mu} << \overline{\nu}$.

By (4.1) an exhaustive modular measure ν on L is a control for an exhaustive modular measure μ if and only if $\overline{\nu}$ is a control for $\overline{\mu}$. This allows us to immediately prove theorems of existence of real-valued controls.

Theorem (4.2) (Bartle-Dunford-Schwartz theorem). *Let $\mu : L \to X$ be an exhaustive modular measure and suppose that X is metrizable. Then μ has a $[0,1]$-valued control.*

Proof. Let $\mathcal{W}$ be as in (4.1)-2. By Bartle-Dunford-Schwartz theorem for measures on Boolean algebras, we can find a [0,1]-valued measure $\overline{\nu}$ on $C(\tilde{L})$ such that $\mathcal{U}(\overline{\nu}) = \mathcal{U}(\overline{\mu})$. By (4.1)-2, we can find a real-valued modular measure λ on L such that $\overline{\lambda} = \overline{\nu}$ and $\mathcal{U}(\lambda) = \mathcal{U}(\mu)$.

In similar way we can prove the following result.

Theorem (4.3) (Rybakov theorem). *Let $\mu : L \to X$ be an exhaustive modular measure and suppose that X is a Banach space. Then there exists a continuous linear functional x' on X such that the modular measure $x' \circ \mu$ is a control for μ.*

Now we want extend to modular measures on L a control theorem proved by A. Basile in [B] (Theorem 2). In this case, the control takes values in X and then it is not possible to immediately transfer the result because we don't know if the map ϕ in (4.1)-(1) is surjective. Nevertheless it is possible to extend the result of [B] with the aid of the isomorphism of (2.9).

Recall (see [D]) that every X-valued measure on a Boolean algebra $\mathcal{A}$ generates a Freéchet-Nikodym topology (FN-topology) on $\mathcal{A}$, i.e. a group topology having as base of neighbourhoods of 0 a family consisting of sets U with the following property: if $a \leq b \in U$, then $a \in U$.

For the proof of our result we need the following lemma (see proof of Lemma 1 of [B]), which we will apply to $C(\tilde{L})$.

Lemma (4.4) *Let $\mathcal{A}$ be a complete Boolean algebra, τ_0 a o.c. Hausdorff FN-topology on $\mathcal{A}$ and M a collection of X-valued τ_0-continuous measures on $\mathcal{A}$. Denote by τ the supremum of all FN-topologies $\tau(\mu)$ generated by the elements μ of M. Set $a_M = (\sup \overline{\{0\}}^{\tau})^{\perp}$ and $a_\mu = (\sup \overline{\{0\}}^{\tau(\mu)})^{\perp}$ for $\mu \in M$. Suppose that, for each integer n, there exist $a_n \in \mathcal{A}$, $\mu_n \in M$ and a τ_0-continuous measure ν_n^* on $\mathcal{A}$ with the following properties:*

(1) *$\{a_n\}$ is disjoint.*

(2) *For each integer n, $a_n \leq a_{\mu_n}$.*

(3) *$a_M = \sup_n a_n$.*

$\tau(\nu_n^*) = \tau(\nu_n)$, where $\nu_n(a) = \mu_n(a \wedge a_n)$ for $a \in \mathcal{A}$.

(4) *The series $\sum_{n=1}^{\infty} \nu_n^*$ is uniformly convergent on $\mathcal{A}$.*

Then the measure $\gamma = \sum_{n=1}^{\infty} \nu_n^*$ *is a* X*-valued control for* M.

If M is a collection of X-valued modular measures on L, we say that M is *uniformly exhaustive* if, for every monotone sequence $\{a_n\}$ in L, $\{\mu(a_n)\}$ is a Cauchy sequence in X uniformly with respect to $\mu \in M$. Then, if we set $\lambda(a) = (\mu(a))_{\mu \in M}$ for $a \in L$, we have that M is uniformly exhaustive if and only if $\lambda : L \to (X^M, \tau_\infty)$ is exhaustive, where τ_∞ is the topology of the uniform convergence in X^M. Therefore, by 3.4 of [A], we obtain that M is uniformly exhaustive if and only if, for every orthogonal sequence $\{a_n\}$ in L, the sequence $\{\mu(a_n)\}$ converges to 0 in X uniformly with respect to $\mu \in M$.

Now we can extend to effect algebras Theorem 2 of [B].

Theorem (4.5) *Let* $\{\mu_n\}$ *be a uniformly exhaustive sequence of* X*-valued modular measures on* L*. Then* $\{\mu_n\}$ *has a control with values in* X.

Proof. Set $\mathcal{W} = \sup\{\mathcal{U}(\mu_n) : n \in N\}$. By (2.10), the map

$$\phi : \mathcal{U} \in DU(L, \mathcal{W}) \to \left(\sup \overline{\{0\}}^{\,\mathcal{U}}\right)^{\perp} \in C(\tilde{L})$$

is a lattice isomorphism. By (2.5), $C(\tilde{L})$ is a complete Boolean algebra.

(i) We prove that the assumptions of (4.4) are satisfied with respect to the family $\overline{M} = \{\overline{\mu}_n : n \in N\}$ of measures on $C(\tilde{L})$.

Set $\tau = \sup_n \tau_n$, where, for each $n \in \mathbb{N}$, τ_n is the topologies generated by $\mathcal{U}(\overline{\mu}_n)$ on $C(\tilde{L})$. By 6.10 of $[W_2]$, τ_n and the topology τ_0 generated by $\overline{\mathcal{W}}$ are FN-topologies. By (2.5), τ_0 is o.c. Note that, since by (2.9) $DU(L, \mathcal{W})$ and $DU(C(\tilde{L}), \overline{\mathcal{W}})$ are isomorphic, from the definition of $\mathcal{W}$ we get $\tau_0 = \tau$. Therefore, for each $n \in \mathbb{N}$, $\overline{\mu}_n$ is τ_0-continuous.

Now set

$$a_{\mu_n} = \phi(\mathcal{U}(\mu_n)) \ (n \in \mathbb{N}), \ a_1 = a_{\mu_1}, \ a_n = a_{\mu_n} \setminus \bigvee_{i \le n-1} a_{\mu_i} \ (n \ge 2).$$

Then, by (4.1), $\{a_n\}$ is a disjoint sequence in $C(\tilde{L})$, with $a_n \le a_{\mu_n} = (\sup \overline{\{0\}}^{\,\tau_n})^{\perp}$. Moreover, if we set $a_M = \left(\sup \overline{\{0\}}^{\,\tau}\right)^{\perp}$, since ϕ is a lattice isomorphism, we have $\sup_n a_n = \sup_n a_{\mu_n} = \sup\{\phi(\mathcal{U}(\mu_n)) : n \in N\} = \phi(\mathcal{W}) = a_M$.

Now, for $a \in \tilde{L}$ and $n \in N$, set $\tilde{\nu}_n(a) = \tilde{\mu}_n(a \wedge a_n)$ and $\tilde{\nu}_n^* = 2^{-n}\tilde{\nu}_n$.

Since $\{a_n\} \subseteq C(\tilde{L})$, by 2.1 of [A-B-V] $\tilde{\nu}_n$ is a $\tilde{\mathcal{W}}$-continuous modular measure. Since $\mathcal{U}(\tilde{\nu}_n^*) = \mathcal{U}(\tilde{\nu}_n)$, we have that $\tilde{\nu}^*_{n|C(\tilde{L})}$ is τ_0-continuous.

We prove that the series $\sum_{n=1}^{\infty} \tilde{\nu}_n^*$ is uniformly convergent on $\tilde{L}$.

It is clear that, since $\{\mu_n\}$ is uniformly exhaustive, $\{\tilde{\mu}_n : n \in N\}$ is uniformly exhaustive, too. Observe that, since $\{a_n\}$ is an orthogonal sequence in $C(\tilde{L})$, by (2.1)-1 of [A-B-V] we have $a \wedge \bigoplus_{i=1}^n a_i = \bigoplus_{i=1}^n (a \wedge a_i)$ for each $a \in \tilde{L}$ and $n \in \mathbb{N}$. Therefore, for each $a \in \tilde{L}$, $\{a \wedge a_n\}$ in an orthogonal sequence in $\tilde{L}$. Then we have that $\lim_n \tilde{\nu}_n(a) = 0$, and therefore $\{\tilde{\nu}_n(a)\}$ is bounded in X for each $a \in \tilde{L}$. Hence, if we set $\tilde{h}(a) = (\tilde{\nu}_n(a))_{n \in N}$ for $a \in \tilde{L}$ and $H = \{f \in X^N : f(N) \text{ is bounded}\}$, we have that $\tilde{h}$ is a H-valued modular measure and, since $\{\tilde{\mu}_n\}$ is uniformly exhaustive, $\tilde{h}$ it is exhaustive with respect to the topology τ_∞ of the uniform convergence in H. Then, by 2.3 of $[W_4]$, $\tilde{h}$ is bounded in (H, τ_∞), whence it follows that $\bigcup_{n \in N} \tilde{\nu}_n(\tilde{L})$ is bounded in X. Let W be an absolutely convex neighbourhood of 0 in X, and choose $\varepsilon > 0$ such that $\varepsilon(\bigcup_{n\in} \tilde{\nu}_n(\tilde{L})) \subseteq W$. Let $r \in N$ be such that $\sum_{i>r} 2^{-i} < \varepsilon$. Then, if $p > q \ge r$ and $a \in \tilde{L}$, we obtain

$$\sum_{i=q}^{p} \tilde{\nu}_i^*(a) = \sum_{i=q}^{p} \frac{1}{2^i \varepsilon}(\varepsilon \tilde{\nu}_i(a)) \in W.$$

Then all the assumptions of (4.4) are satisfied.

(ii) Now set, for $a \in \tilde{L}$, $\tilde{\gamma}(a) = \sum_{n=1}^{\infty} \tilde{\nu}_n^*(a)$. By (i) and (4.4), $\tilde{\gamma}_{|C(\tilde{L})}$ is a control for $\overline{M}$. Moreover, since $\tilde{\nu}_n^*$ is $\tilde{\mathcal{W}}$-continuous and the series $\sum_{n=1}^{\infty} \tilde{\nu}_n^*$ is uniformly convergent on $\tilde{L}$, we have that $\tilde{\gamma}$ is $\mathcal{W}$-continuous, too. Then, by (4.1), $\tilde{\gamma}$ is a control for $\tilde{M} = \{\tilde{\mu}_N : n \in N\}$. Hence $\{\mu_n : n \in N\}$ has a X-valued control.

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