PSEUDO-IDEALS OF PSEUDO-BCK ALGEBRAS

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Abstract. The notion of (positive implicative) pseudo-ideals in a pseudo-BCK algebra is introduced, and several properties are investigated. Characterizations of a pseudo-ideal are displayed. Conditions for a subset to be a pseudo-ideal are given. The concept of pseudo-homomorphism is discussed.

1. Introduction

G. Georgescu and A. Iorgulescu [1] introduced the notion of a pseudo-BCK algebra as an extended notion of BCK-algebras. In [4], Y. B. Jun, one of the present authors, gave a characterization of pseudo-BCK algebra, and provided conditions for a pseudo-BCK algebra to be $\wedge$-semi-lattice ordered (resp. $\cap$-semi-lattice ordered). In this paper, We introduce the notion of (positive implicative) pseudo-ideals in a pseudo-BCK algebra, and then we investigate some of their properties. We display characterizations of a pseudo-ideal, and provide conditions for a subset to be a pseudo-ideal. We also introduce the notion of pseudo-homomorphism between pseudo-BCK algebras. We prove that every pseudo-homomorphic image and preimage of a (positive implicative) pseudo-ideal is also a (positive implicative) pseudo-ideal.

2. Preliminaries

The notion of pseudo-BCK algebras is introduced by Georgescu and Iorgulescu [1] as follows:

Definition 2.1. A *pseudo-BCK algebra* is a structure $\mathfrak{X} = (X, \preceq, *, \diamond, 0)$, where "$\preceq$" is a binary relation on X, "$*$" and "$\diamond$" are binary operations on X and "0" is an element of X, verifying the axioms: for all $x, y, z \in X$,

(a1) $(x * y) \diamond (x * z) \preceq z * y$, $(x \diamond y) * (x \diamond z) \preceq z \diamond y$,
(a2) $x * (x \diamond y) \preceq y$, $x \diamond (x * y) \preceq y$,
(a3) $x \preceq x$,
(a4) $0 \preceq x$,
(a5) $x \preceq y, \quad y \preceq x \Longrightarrow x = y$,
(a6) $x \preceq y \Longleftrightarrow x * y = 0 \Longleftrightarrow x \diamond y = 0$.

If $\mathfrak{X}$ is a pseudo-BCK algebra satisfying $x * y = x \diamond y$ for all $x, y \in X$, then $\mathfrak{X}$ is a BCK-algebra (see [1, Remark 1.2]).

In a pseudo-BCK algebra we have (see [1])

(p1) $x \preceq y \Longrightarrow z * y \preceq z * x, \quad z \diamond y \preceq z \diamond x$.
(p2) $x \preceq y, \quad y \preceq z \Longrightarrow x \preceq z$.
(p3) $(x * y) \diamond z = (x \diamond z) * y$.

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(p4) $x * y \preceq z \Longleftrightarrow x \diamond z \preceq y$.
(p5) $x * y \preceq x, \quad x \diamond y \preceq x$.
(p6) $x * 0 = x = x \diamond 0$.
(p7) $x \preceq y \Longrightarrow x * z \preceq y * z, \quad x \diamond z \preceq y \diamond z$.
(p8) $x \wedge y$ (and $y \wedge x$) is a lower bound for $\{x, y\}$, where $x \wedge y := y \diamond (y * x)$ (and $y \wedge x := x \diamond (x * y)$).
(p9) $x \cap y$ (and $y \cap x$) is a lower bound for $\{x, y\}$ where $x \cap y := y * (y \diamond x)$ (and $y \cap x := x * (x \diamond y)$).

3. Pseudo-ideals

We first give condition(s) for a pseudo-BCK algebra to be a BCK-algebra.

Theorem 3.1. *Let $\mathfrak{X}$ be a pseudo-BCK algebra that satisfies the conditions*

$$(x * y) * z = (x * z) * y \text{ and } (x \diamond y) \diamond z = (x \diamond z) \diamond y$$

for all $x, y, z \in X$. Then $\mathfrak{X}$ is a BCK-algebra.

Proof. Let $x, y \in X$. Since $x \diamond (x * y) \preceq y$ by (a2), we have $0 = (x \diamond (x * y)) \diamond y = (x \diamond y) \diamond (x * y)$ and so $x \diamond y \preceq x * y$. From $x * (x \diamond y) \preceq y$, we get $0 = (x * (x \diamond y)) * y = (x * y) * (x \diamond y)$, i.e., $x * y \preceq x \diamond y$. It follows from (a5) that $x * y = x \diamond y$ so that $\mathfrak{X}$ is a BCK-algebra. □

Let $\mathfrak{X}$ be a pseudo-BCK -algebra. For any nonempty subset I of X and any element y of X, we denote

$$*(y, I) := \{x \in X \mid x * y \in I\} \text{ and } \diamond(y, I) := \{x \in X \mid x \diamond y \in I\}.$$

Note that $*(y, I) \cap \diamond(y, I) = \{x \in X \mid x * y \in I,\, x \diamond y \in I\}$.

Definition 3.2. A nonempty subset I of a pseudo-BCK algebra $\mathfrak{X}$ is called a *pseudo-ideal* of $\mathfrak{X}$ if it satisfies

(I1) $0 \in I$,
(I2) $\forall y \in I,\, *(y, I) \subseteq I$ and $\diamond(y, I) \subseteq I$.

Definition 3.3. A nonempty subset I of a pseudo-BCK algebra $\mathfrak{X}$ is called a *weak pseudo-ideal* of $\mathfrak{X}$ if it satisfies (I1) and

(I3) $\forall y \in I,\, *(y, I) \cap \diamond(y, I) \subseteq I$.

Obviously, every pseudo-ideal is a weak pseudo-ideal. Note that if $\mathfrak{X}$ is a pseudo-BCK algebra satisfying $x * y = x \diamond y$ for all $x, y \in X$, then the notion of a pseudo-ideal and an ideal coincide. Hence, in a positive implicative pseudo-BCK algebra, the notion of a pseudo-ideal and an ideal coincide (see [4]).

Proposition 3.4. *Let I be a pseudo-ideal of a pseudo-BCK algebra $\mathfrak{X}$. If $x \in I$ and $y \preceq x$, then $y \in I$.*

Proof. The proof is straightforward. □

Theorem 3.5. *For any element a of a pseudo-BCK algebra $\mathfrak{X}$, the initial section $\downarrow a := \{x \in X \mid x \preceq a\}$ is a pseudo-ideal of $\mathfrak{X}$ if and only if the following implications hold:*

(i) $\forall x, y, z \in X,\, x * y \preceq z,\, y \preceq z \Rightarrow x \preceq z$,
(ii) $\forall x, y, z \in X,\, x \diamond y \preceq z,\, y \preceq z \Rightarrow x \preceq z$.

Proof. Assume that for each $a \in X$, $\downarrow a$ is a pseudo-ideal of $\mathfrak{X}$. Let $x, y, z \in X$ be such that $x * y \preceq z$, $x \diamond y \preceq z$, and $y \preceq z$. Then $x * y \in \downarrow z$, $x \diamond y \in \downarrow z$, and $y \in \downarrow z$, that is, $y \in \downarrow z$, $x \in *(y, \downarrow z)$ and $x \in \diamond(y, \downarrow z)$. Since $\downarrow z$ is a pseudo-ideal of $\mathfrak{X}$, it follows from (I2) that $x \in \downarrow z$, i.e., $x \preceq z$. Conversely, consider $\downarrow z$ for any $z \in X$. Obviously $0 \in \downarrow z$. For every

$y \in \downarrow z$, let $a \in *(y, \downarrow z)$ and $b \in \diamond(y, \downarrow z)$. Then $a * y \in \downarrow z$ and $b \diamond y \in \downarrow z$, i.e., $a * y \preceq z$ and $b \diamond y \preceq z$. Since $y \in \downarrow z$, it follows from the hypothesis that $a \preceq z$ and $b \preceq z$, i.e., $a \in \downarrow z$ and $b \in \downarrow z$. This shows that $*(y, \downarrow z) \subseteq \downarrow z$ and $\diamond(y, \downarrow z) \subseteq \downarrow z$. Hence $\downarrow z$ is a pseudo-ideal of $\mathfrak{X}$ for every $z \in X$. This completes the proof. □

Theorem 3.6. *Let I be a nonempty subset of a pseudo-BCK algebra $\mathfrak{X}$. Then I is a pseudo-ideal of $\mathfrak{X}$ if and only if it satisfies:*

(i) $\forall x, y \in I,\ \forall z \in X,\ z * y \preceq x \Rightarrow z \in I,$
(ii) $\forall x, y \in I,\ \forall z \in X,\ z \diamond y \preceq x \Rightarrow z \in I.$

Proof. Suppose that I is a pseudo-ideal of $\mathfrak{X}$ and let $x, y \in I$ and $z \in X$ be such that $z * y \preceq x$ and $z \diamond y \preceq x$. Then $(z * y) \diamond x = 0 \in I$ and $(z \diamond y) * x = 0 \in I$, which imply that $z * y \in \diamond(x, I) \subseteq I$ and $z \diamond y \in *(x, I) \subseteq I$. It follows that $z \in *(y, I) \subseteq I$ and $z \in \diamond(y, I) \subseteq I$. Conversely, suppose (i) and (ii) are valid. Taking $x \in I$ because I is nonempty, we have $0 * x \preceq x$ and $0 \diamond x \preceq x$. Using (i) and (ii), we get $0 \in I$. For every $y \in I$, let $a \in *(y, I)$ and $b \in \diamond(y, I)$. Then $a * y \in I$ and $b \diamond y \in I$. Note from (a2) that $a \diamond (a * y) \preceq y$ and $b * (b \diamond y) \prec y$. Hence, by (i) and (ii), we have $a \in I$ and $b \in I$. Consequently, $*(y, I) \subseteq I$ and $\diamond(y, I) \subseteq I$. This completes the proof. □

Definition 3.7. A nonempty subset I of a pseudo-BCK algebra $\mathfrak{X}$ is called a *positive implicative pseudo-ideal* of $\mathfrak{X}$ if it satisfies (I1) and for all $x, y, z \in X$,

(I4) $(x * y) \diamond z \in I,\ y \diamond z \in I \Rightarrow x \diamond z \in I,$
(I5) $(x \diamond y) * z \in I,\ y * z \in I \Rightarrow x * z \in I.$

Theorem 3.8. *Any positive implicative pseudo-ideal is a pseudo-ideal.*

Proof. Let I be a positive implicative pseudo-ideal of a pseudo-BCK algebra $\mathfrak{X}$. For every $y \in I$, let $a \in *(y, I)$ and $b \in \diamond(y, I)$. Then $(a * y) \diamond 0 = a * y \in I$ and $(b \diamond y) * 0 = b \diamond y \in I$. Since $y * 0 = y \in I$ and $y \diamond 0 = y \in I$, it follows from (p6), (I4) and (I5) that $a = a \diamond 0 \in I$ and $b = b * 0 \in I$ so that $*(y, I) \subseteq I$ and $\diamond(y, I) \subseteq I$. Hence I is a pseudo-ideal of $\mathfrak{X}$. □

Theorem 3.9. *Let $\mathfrak{X}$ be a pseudo-BCK algebra. If I is a positive implicative pseudo-ideal of $\mathfrak{X}$, then for every $w \in X$, the set*

$$I_w := *(w, I) \cap \diamond(w, I)$$

is a weak pseudo-ideal of $\mathfrak{X}$.

Proof. Assume that I is a positive implicative pseudo-ideal of $\mathfrak{X}$. Obviously $0 \in I_w$. For every $y \in I_w$, let $x \in *(y, I_w) \cap \diamond(y, I_w)$. Then $x * y \in I_w$ and $x \diamond y \in I_w$, which imply that $(x * y) \diamond w \in I$ and $(x \diamond y) * w \in I$. Since $y * w \in I$ and $y \diamond w \in I$, it follows from (I4) and (I5) that $x \diamond w \in I$ and $x * w \in I$ so that $x \in *(w, I) \cap \diamond(w, I) = I_w$. This shows that $*(y, I_w) \cap \diamond(y, I_w) \subseteq I_w$. Hence I_w is a weak pseudo-ideal of $\mathfrak{X}$. □

Proposition 3.10. *Let I be a positive implicative pseudo-ideal of a pseudo-BCK algebra $\mathfrak{X}$. Then*

(I6) $\forall x, y \in X,\ (x * y) \diamond y \in I \Rightarrow x * y \in I,\ x \diamond y \in I.$

Proof. Let $x, y \in X$ be such that $(x * y) \diamond y \in I$. Then $(x \diamond y) * y \in I$ by (p3). Since $y \diamond y = 0 \in I$ and $y * y = 0 \in I$, it follows from (I4) and (I5) that $x \diamond y \in I$ and $x * y \in I$. This completes the proof. □

Proposition 3.11. *Let I be a pseudo-ideal of a pseudo-BCK algebra $\mathfrak{X}$ that satisfies the condition* (I6). *If $\mathfrak{X}$ satisfies the conditions*

$$(x * z) \diamond (y * z) \preceq x * y \text{ and } (x \diamond z) * (y \diamond z) \preceq x \diamond y$$

for all $x, y, z \in X$, then

(I7) $\forall x, y, z \in X$, $(x * y) \diamond z \in I \Rightarrow (x * z) \diamond (y * z) \in I$.
(I8) $\forall x, y, z \in X$, $(x \diamond y) * z \in I \Rightarrow (x \diamond z) * (y \diamond z) \in I$.

Proof. Let $x, y, z \in X$ be such that $(x * y) \diamond z \in I$. Since

$$((x \diamond (y * z)) * z) \diamond z = ((x * z) \diamond (y * z)) \diamond z \preceq (x * y) \diamond z,$$

it follows from Proposition 3.4 that $((x \diamond (y * z)) * z) \diamond z \in I$ so from (I6) and (p3) that $(x * z) \diamond (y * z) = (x \diamond (y * z)) * z \in I$. Assume that $(x \diamond y) * z \in I$ for all $x, y, z \in X$. Note that

$$((x * (y \diamond z)) \diamond z) * z = ((x \diamond z) * (y \diamond z)) * z \preceq (x \diamond y) * z.$$

Hence, by (p3) and Proposition 3.4, we have

$$((x * (y \diamond z)) * z) \diamond z = ((x * (y \diamond z)) \diamond z) * z \in I.$$

Using (p3) and (I6), we get $(x \diamond z) * (y \diamond z) = (x * (y \diamond z)) * z \in I$. This completes the proof. □

We give conditions for a subset of a pseudo-BCK algebra to be a pseudo-ideal.

Proposition 3.12. *Let I be a subset of a pseudo-BCK algebra $\mathfrak{X}$ satisfying* (I1) *and*

(I9) $\forall x, y, z \in X$, $((x * y) \diamond y) * z \in I$, $z \in I \Rightarrow x \diamond y \in I$,
(I10) $\forall x, y, z \in X$, $((x \diamond y) * y) \diamond z \in I$, $z \in I \Rightarrow x * y \in I$.

Then I is a pseudo-ideal of $\mathfrak{X}$.

Proof. For every $y \in I$, let $a \in *(y, I)$ and $b \in \diamond(y, I)$. Then $a * y \in I$ and $b \diamond y \in I$. Hence $((a * 0) \diamond 0) * y = a * y \in I$ and $((b \diamond 0) * 0) \diamond y = b \diamond y \in I$, which imply from (p6), (I9) and (I10) that $a = a \diamond 0 \in I$ and $b = b * 0 \in I$. Therefore $*(y, I) \subseteq I$ and $\diamond(y, I) \subseteq I$. Consequently, I is a pseudo-ideal of $\mathfrak{X}$. □

Definition 3.13. Let $\mathfrak{X}$ and $\mathfrak{Y}$ be pseudo-BCK algebras. A mapping $\mathfrak{f} : \mathfrak{X} \to \mathfrak{Y}$ is called a *pseudo-homomorphism* if $\mathfrak{f}(x * y) = \mathfrak{f}(x) * \mathfrak{f}(y)$ and $\mathfrak{f}(x \diamond y) = \mathfrak{f}(x) \diamond \mathfrak{f}(y)$ for all $x, y \in X$.

Note that if $\mathfrak{f} : \mathfrak{X} \to \mathfrak{Y}$ is a pseudo-homomorphism, then $\mathfrak{f}(0_{\mathfrak{X}}) = 0_{\mathfrak{Y}}$ where $0_{\mathfrak{X}}$ and $0_{\mathfrak{Y}}$ are zero elements of $\mathfrak{X}$ and $\mathfrak{Y}$, respectively.

Theorem 3.14. *Let $\mathfrak{f} : \mathfrak{X} \to \mathfrak{Y}$ be a pseudo-homomorphism of pseudo-BCK algebras $\mathfrak{X}$ and $\mathfrak{Y}$.* (i) *If J is a* (*positive implicative*) *pseudo-ideal of $\mathfrak{Y}$, then $\mathfrak{f}^{-1}(J)$ is a* (*positive implicative*) *pseudo-ideal of $\mathfrak{X}$.* (ii) *If $\mathfrak{f}$ is surjective and I is a pseudo-ideal of $\mathfrak{X}$, then $\mathfrak{f}(I)$ is a pseudo-ideal of $\mathfrak{Y}$.*

Proof. (i) Assume that J is a pseudo-ideal of $\mathfrak{Y}$. Obviously $0_{\mathfrak{X}} \in \mathfrak{f}^{-1}(J)$. For every $y \in \mathfrak{f}^{-1}(J)$, let $a \in *(y, \mathfrak{f}^{-1}(J))$ and $b \in \diamond(y, \mathfrak{f}^{-1}(J))$. Then $a * y \in \mathfrak{f}^{-1}(J)$ and $b \diamond y \in \mathfrak{f}^{-1}(J)$. It follows that $\mathfrak{f}(a) * \mathfrak{f}(y) = \mathfrak{f}(a * y) \in J$ and $\mathfrak{f}(b) \diamond \mathfrak{f}(y) = \mathfrak{f}(b \diamond y) \in J$ so that $\mathfrak{f}(a) \in *(\mathfrak{f}(y), J) \subseteq J$ and $\mathfrak{f}(b) \in \diamond(\mathfrak{f}(y), J) \subseteq J$ because J is a pseudo-ideal of $\mathfrak{X}$ and $\mathfrak{f}(y) \in J$. Hence $a \in \mathfrak{f}^{-1}(J)$ and $b \in \mathfrak{f}^{-1}(J)$, which shows that $*(y, \mathfrak{f}^{-1}(J)) \subseteq \mathfrak{f}^{-1}(J)$ and $\diamond(y, \mathfrak{f}^{-1}(J)) \subseteq \mathfrak{f}^{-1}(J)$. Hence $\mathfrak{f}^{-1}(J)$ is a pseudo-ideal of $\mathfrak{X}$. If J is positive implicative, let $x, y, z \in X$ be such that $(x * y) \diamond z \in \mathfrak{f}^{-1}(J)$ and $y \diamond z \in \mathfrak{f}^{-1}(J)$. Then

$$(\mathfrak{f}(x) * \mathfrak{f}(y)) \diamond \mathfrak{f}(z) = \mathfrak{f}((x * y) \diamond z) \in J$$

and $\mathfrak{f}(y) \diamond \mathfrak{f}(z) = \mathfrak{f}(y \diamond z) \in J$. It follows from (I4) that $\mathfrak{f}(x \diamond z) = \mathfrak{f}(x) \diamond \mathfrak{f}(z) \in J$ so that $x \diamond z \in \mathfrak{f}^{-1}(J)$. Suppose that $(x \diamond y) * z \in \mathfrak{f}^{-1}(J)$ and $y * z \in \mathfrak{f}^{-1}(J)$. Then

$$(\mathfrak{f}(x) \diamond \mathfrak{f}(y)) * \mathfrak{f}(z) = \mathfrak{f}((x \diamond y) * z) \in J$$

and $\mathfrak{f}(y) * \mathfrak{f}(z) = \mathfrak{f}(y * z) \in J$. Using (I5), we have $\mathfrak{f}(x * z) = \mathfrak{f}(x) * \mathfrak{f}(z) \in J$ and so $x * z \in \mathfrak{f}^{-1}(J)$. Therefore $\mathfrak{f}^{-1}(J)$ is a positive implicative pseudo-ideal of $\mathfrak{X}$.

(ii) Assume that $\mathfrak{f}$ is surjective and let I be a pseudo-ideal of $\mathfrak{X}$. Obviously, $0_{\mathfrak{Y}} \in \mathfrak{f}(I)$. For every $y \in \mathfrak{f}(I)$, let $a, b \in Y$ be such that $a \in *(y, \mathfrak{f}(I))$ and $b \in \diamond(y, \mathfrak{f}(I))$. Then $a * y \in \mathfrak{f}(I)$ and $b \diamond y \in \mathfrak{f}(I)$. It follows that there exist $x_*, x_\diamond \in I$ such that $\mathfrak{f}(x_*) = a * y$ and $\mathfrak{f}(x_\diamond) = b \diamond y$. Since $y \in \mathfrak{f}(I)$, there exists $x_y \in I$ such that $\mathfrak{f}(x_y) = y$. Also since $\mathfrak{f}$ is surjective, there exist $x_a, x_b \in X$ such that $\mathfrak{f}(x_a) = a$ and $\mathfrak{f}(x_b) = b$. Hence

$$\mathfrak{f}(x_a * x_y) = \mathfrak{f}(x_a) * \mathfrak{f}(x_y) = a * y \in \mathfrak{f}(I)$$

and

$$\mathfrak{f}(x_b \diamond x_y) = \mathfrak{f}(x_b) \diamond \mathfrak{f}(x_y) = b \diamond y \in \mathfrak{f}(I),$$

which imply that $x_a * x_y \in I$ and $x_b \diamond x_y \in I$. Since I is a pseudo-ideal of $\mathfrak{X}$, we get $x_a \in *(x_y, I) \subseteq I$ and $x_b \in \diamond(x_y, I) \subseteq I$, and thus $a = \mathfrak{f}(x_a) \in \mathfrak{f}(I)$ and $b = \mathfrak{f}(x_b) \in \mathfrak{f}(I)$. This shows that $*(y, \mathfrak{f}(I)) \subseteq \mathfrak{f}(I)$ and $\diamond(y, \mathfrak{f}(I)) \subseteq \mathfrak{f}(I)$. Therefore $\mathfrak{f}(I)$ is a pseudo-ideal of $\mathfrak{Y}$. □

Corollary 3.15. *Let* $\mathfrak{f} : \mathfrak{X} \to \mathfrak{Y}$ *be a pseudo-homomorphism of pseudo-BCK algebras* $\mathfrak{X}$ *and* $\mathfrak{Y}$. *Then the kernel* $\mathrm{Ker}(\mathfrak{f}) := \{x \in X \mid \mathfrak{f}(x) = 0_{\mathfrak{Y}}\}$ *of* $\mathfrak{f}$ *is a pseudo-ideal of* $\mathfrak{X}$

Proof. The proof is straightforward. □

Open Problem 3.16. (i) Under what condition(s), is the set I_w described in Theorem 3.9 a pseudo-ideal?

(ii) Is there a weak pseudo-ideal which is not a pseudo-ideal?

(iii) If I is a pseudo-ideal of a pseudo-BCK -algebra $\mathfrak{X}$ satisfying the condition (I6), then is I a positive implicative pseudo-ideal of $\mathfrak{X}$?

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