THE REPLENISHMENT POLICY FOR AN INVENTORY SYSTEM WITH POISSON ARRIVAL DEMANDS

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Abstract. The consumers arrive at a store according to a Poisson process and purchase a stochastic quantity of resources. A store with a bounded tank is forced to follow reasonable management in a balance between surplus orders and shortages. In this paper, we consider an inventory replenishment policy that the store establishes safety stock quantities to minimize the expected costs per unit time, and that it replenishes the stock at the time fallen to the safety stock level so as to keep the losses of costs for shortages and excess orders to a minimum. We also present a numerical example for the case that demand follows the exponential distribution.

1 Introduction. The consumers arrive at a store, which sells a liquescent resource expressed by continuous quantity, *e.g.* water, wine, gasoline and so on, according to a Poisson process and purchase a stochastic quantity of resources. This store possesses a tank with a certain maximum permissible quantity. If the stock-out occurs, *i.e.*, the stock level falls to zero, it is impossible for him to sell resources. It means that he loses confidence from consumers and sustains very great losses. A way to avoid these losses is to replenish a tank with resources before the stock level falls to zero. The ordering implies that he replenishes resources until his tank is filled. On the other hand, if he shortens intervals between orders, the number of ordering increases. Then he is charged ordering costs in surplus. Thus he must keep an efficient management in a balance between surplus orders and shortages.

This paper considers an inventory replenishment policy that the store establishes safety stock quantities to minimize the expected costs per unit time, and that it replenishes the stock at the time fallen to the safety stock level so as to keep the losses of costs for shortages and excess orders to a minimum. To avoid analytical complication in this model, we assume that ordered resources are instantaneously replenished and that the ordering costs and the loss of costs for being sold out are constant independently of the ordering quantities and the quantities of shortages, respectively. We formulate a special problem of the classical (S, s) policy (*e.g.* [1], [2],[10]) by using a renewal reward process [7]. It gives a very simple formulation.

The outline of this paper is as follows. In section 2, we give the details of the model formulation. Section 3 deals with the analysis of the optimal replenishment policy. In section 4, we also describe numerical examples when the quantity of resources purchased by each of consumers is exponentially distributed. At last, we give some conclusions in section 5.

2 Model. We consider a model such that a store with a tank of maximum permissible quantity U sells a liquescent resource expressed by continuous quantity. Each consumer arrives at the store according to a Poisson process with intensity function λ and purchases

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it. Let S_j $(j = 0, 1, 2, \cdots)$ be the random variable denoting the arrival time of j-th consumer with $S_0 = 0$. Then, the probability that consumers of j and over arrive during $(0, t]$, $F^{(j)}(t)$, is given by

$$F^{(j)}(t) \equiv Pr\{S_j \leq t\} = \sum_{i=j}^{\infty} \frac{(\lambda t)^i}{i\,!} e^{-\lambda t} \quad (j = 0, 1, 2, \cdots). \tag{1}$$

Let $N(t)$ be the total number of consumers arriving by time t. The probability that exactly j consumers arrive until time t, $H_j(t)$, is given by

$$H_j(t) \equiv Pr\{N(t) = j\} = F^{(j)}(t) - F^{(j+1)}(t) \quad (j = 0, 1, 2, \cdots). \tag{2}$$

The j-th consumer purchases a quantity Y_j $(j = 0, 1, 2, \cdots)$ of resources. Y_j is a sequence of identical and independent random variable having a distribution function $G(x)$, *i.e.*, $G(x) = Pr\{Y_j \leq x\}$. Let Z_j $(j = 0, 1, 2, \cdots)$ be the total sales quantity after the j-th purchase with $Z_0 = 0$. Then Z_j is a cumulative process with

$$Z_j \equiv \sum_{i=1}^{j} Y_i \quad (j = 1, 2, 3, \cdots) \tag{3}$$

and

$$Pr\{Z_j \leq x\} \equiv G^{(j)}(x) \qquad (j = 0, 1, 2, \cdots), \tag{4}$$

where $G^{(j)}(x)$ is the j-fold Stieltjes convolution of $G(x)$ with itself, and

$$G^{(0)}(x) \equiv \begin{cases} 1, & x \geq 0 \\ 0, & x < 0. \end{cases} \tag{5}$$

For a continuous distribution function $G(x)$, let $\bar{G}(x)$ and $g(x)$, respectively, denote the survivor and density functions expressed by $\bar{G}(x) = 1 - G(x) = Pr\{Y_j > x\}$ and $g(x) = \frac{d}{dx}G(x)$.

Since this store possesses a tank with maximum permissible quantity U, the resource runs out of stock if the total sales quantity is over U. Then the store must order it so as to fill the tank immediately. Because the stocking-out often makes a great loss, it is to be desired that the store orders resources in advance. If the order takes place before stocking-out, the losses for shortages do not occur, however it shortens the interval of ordering. Consequently, the frequency of ordering increases and the costs of ordering resources are charged in surplus. Therefore the manager must operate his store in a balance between surplus orders and shortages. The ordering before stocking-out optimally needs to take place so as to keep these losses as minimal as possible.

¿From the above arguments, we have the following policy:

> The ordering is carried out so as to fill the tank if the stock level falls to the safety stock level u $(0 \leq u \leq U)$,

where $u = 0$ means that the order takes place after stocking-out occurs, and $u = U$ means that the order always takes place after demand occurs. To simplify the analysis for this policy, we make the following assumptions:

1. The ordered resource arrives instantaneously.
2. The ordering cost is charged by C_r regardless of the order quantity, and the loss for stocking-out is charged by C_p regardless of the quantity of shortages.

The objective of this paper is to decide the optimal safety stock level u^* so as to minimize the expected cost, denoted by $C(u)$, per unit time for infinite periods.

3 Analysis. Since the stock level is renewed by replenishing, we define an interval of two adjacent renewal time as a cycle for the renewal process. From the renewal reward theorem, the expected cost per unit time for infinite periods, $C(u)$, can be expressed as the ratio of the expected cycle cost $R(u)$ and the expected cycle length $L(u)$, that is, $R(u)/L(u)$.

We begin with calculating the expected cycle cost $R(u)$ and the expected cycle length $L(u)$. The probability of occurring shortages, $\alpha(u)$, is given by

$$\begin{aligned}\alpha(u) &= \sum_{j=1}^{\infty} Pr\{Z_{j-1} \le U-u,\ Z_j > U\} \\ &= \sum_{j=0}^{\infty} \int_0^{U-u} \bar{G}(U-x)\ dG^{(j)}(x) \\ &= 1 + M(U-u) - \sum_{j=1}^{\infty} \int_0^{U-u} G(U-x)\ dG^{(j-1)}(x), \quad (6)\end{aligned}$$

where

$$M(x) \equiv \sum_{j=1}^{\infty} G^{(j)}(x). \quad (7)$$

The probability $\beta(u)$, that the order takes place on account of falling short of the safety stock level u when the stock level is positive, is given by

$$\begin{aligned}\beta(u) &= \sum_{j=1}^{\infty} Pr\{Z_{j-1} \le U-u < Z_j \le U\} \\ &= \sum_{j=0}^{\infty} \int_0^{U-u} [G(U-x) - G(U-u-x)]\ dG^{(j)}(x) \\ &= \sum_{j=1}^{\infty} \int_0^{U-u} G(U-x)\ dG^{(j)}(x) - M(U-u). \quad (8)\end{aligned}$$

It is easily seen that $\alpha(u) + \beta(u) = 1$!%

Using Eqs.(6) and (8), the expected cycle cost $R(u)$ can be written as

$$\begin{aligned}R(u) &= (C_r + C_p)\alpha(u) + C_r\beta(u) \\ &= C_r + C_p \left[1 + M(U-u) - \sum_{j=1}^{\infty} \int_0^{U-u} G(U-x)\ dG^{(j-1)}(x)\right]. \quad (9)\end{aligned}$$

Also, the expected cycle length $L(u)$ can be written as

$$\begin{aligned}L(u) &= \sum_{j=1}^{\infty} \int_0^{\infty} t\ dH_j(t) \int_0^{U-u} \bar{G}(U-x)\ dG^{(j-1)}(x) \\ &\quad + \sum_{j=1}^{\infty} \int_0^{\infty} t\ dH_j(t) \int_0^{U-u} [G(U-x) - G(U-u-x)]\ dG^{(j-1)}(x) \\ &= \frac{1 + M(U-u)}{\lambda}. \quad (10)\end{aligned}$$

Using Eqs.(9),(10) and the renewal reward theorem, $C(u)$, the expected cost per unit time for infinite periods, is established as

$$C(u) = \frac{R(u)}{L(u)}$$

$$(11) \quad = \quad \frac{\lambda}{1+M(U-u)}\left[C_r + C_p\left\{1+M(U-u)-\sum_{j=1}^{\infty}\int_0^{U-u} G(U-x)\ dG^{(j-1)}(x)\right\}\right].$$

We now investigate a safety stock level u minimizing the objective function $C(u)$. Differentiating $C(u)$ with respect to u and setting it to 0, we obtain

$$\sum_{j=1}^{\infty}\int_0^{U-u}[G(U-x)-G(u)]\ dG^{(j-1)}(x) = \frac{C_r}{C_p}. \tag{12}$$

Denoting the left side of Eq.(12) by $V(u)$, it follows

$$\frac{dV(u)}{du} = -g(u)[1+M(U-u)] < 0 \tag{13}$$

and

$$\lim_{u\to 0} V(u) = M(U),$$
$$\lim_{u\to U} V(u) = 0.$$

As a result, we know that $V(u)$ is a decreasing function having real values in $[0, M(U)]$. Therefore there exists a unique root u satisfying Eq.(12) when it holds $M(U) > C_r/C_p$.

These arguments give the following result.

Theorem. 1 The optimal replenishment policy is given as follows:

(i) If it holds that

$$M(U) > \frac{C_r}{C_p}, \tag{14}$$

let u^* denote the unique root satisfying Eq.(12). Then, the optimal replenishment policy is to replenish up to U as soon as the stock level falls to the safety level u^*. The optimal expected cost per unit time is given by

$$C(u^*) = \lambda C_p[1+G(u^*)]. \tag{15}$$

(ii) If there is no root u satisfying Eq.(12) in $[0, U]$, put $u^* = 0$. It means that the optimal replenishment policy is to order after stocking out. Then the optimal expected cost per unit time is given by

$$C(u^*) = \lambda\left[\frac{C_r + C_p}{1+M(U)}\right]. \tag{16}$$

4 Numerical Examples. In this section, we present two numerical examples to explain the results obtained in the previous section. Although we dealt with a general distribution $G(x)$ in analysis, we adopt an exponential distribution as an example.

Suppose that the distribution function $G(x)$ follows the exponential distribution with mean $1/\theta$, *i.e.*, $G(x) = 1 - e^{-\theta x}$. Then, Eq.(7) and Eq.(12), respectively, can be rewritten as

$$M(x) = \theta x, \tag{17}$$

$$\sum_{j=1}^{\infty}\left[e^{-\theta u} - \sum_{i=0}^{j-1}\frac{\{\theta(U-u)\}^i}{i\ !}e^{-\theta U}\right] = \frac{C_r}{C_p}. \tag{18}$$

Table 1: the optimal safety quantity u^* and the corresponding cost $C(u^*)$ ($\theta = 0.02, \lambda = 10, C_r = 1$)

	$U=500$		$U=5000$		$U=10000$		$U=15000$		$U=20000$	
C_p	u^*	$C(u^*)$	u^*	$C(u^*)$	u^*	$C(u^*)$	u^*	$C(u^*)$	u^*	$C(u^*)$
10	204.0	198.31	341.9	199.89	378.1	199.95	399.0	199.97	413.7	199.97
20	233.5	398.12	376.1	399.89	412.6	399.95	433.5	399.97	448.2	399.97
40	262.4	797.89	410.4	799.89	447.1	799.95	468.1	799.97	482.8	799.97
60	279.0	1197.74	430.5	1199.89	467.2	1199.95	488.3	1199.97	503.0	1199.97
80	290.7	1597.61	444.7	1599.89	481.6	1599.95	502.6	1599.97	517.4	1599.97
100	299.7	1997.50	455.7	1999.89	492.7	1999.95	513.7	1999.97	528.5	1999.97

Table 2: the optimal safety quantity u^* and the corresponding cost $C(u^*)$ ($\lambda = 1, C_p = 10, C_r = 1$)

	$U=500$		$U=5000$		$U=10000$		$U=15000$		$U=20000$	
θ	u^*	$C(u^*)$	u^*	$C(u^*)$	u^*	$C(u^*)$	u^*	$C(u^*)$	u^*	$C(u^*)$
.005	371.7	184.41	1056.8	199.49	1217.0	199.77	1305.8	199.85	1367.4	199.89
.01	299.7	195.01	608.5	199.77	683.7	199.89	726.4	199.93	756.2	199.95
.015	243.4	197.40	435.3	199.85	484.2	199.93	512.3	199.95	532.0	199.97
.02	204.0	198.31	341.9	199.89	378.1	199.95	399.0	199.97	413.7	199.97
.025	175.8	198.77	282.9	199.92	311.7	199.96	328.3	199.97	340.0	199.98
.03	154.7	199.04	242.1	199.93	266.0	199.97	279.8	199.98	289.5	199.98

Example 1. Suppose that the parameters of the arrival interval and the order quantity, respectively, are $\lambda = 10$ and $\theta = 0.02$. That is, the expected number of consumers arriving per unit time is 10, and the expected quantity purchased by each consumer is 50. In an experimental example, the permissible quantity of a tank U was chosen at five levels ($U = 500, 5000, 10000, 15000$ and 20000). Then, Table 1 represents the optimal safety stock level u^* and the corresponding expected cost per unit time, $C(u^*)$, obtained for $C_r = 1$ and C_p taking values $10, 20, 40, 60, 80, 100$.

¿From Table 1, followings are observed: When the value of C_r/C_p decreases for fixed U, that is, the value C_p increases for fixed U and C_r, the optimal value of u and the corresponding cost $C(u^*)$ increase. Then, for large U, the optimal total cost $C(u^*)$ linearly increases with respect to C_p. For the fixed ratio C_r/C_p, there is an increase in the optimal value of u with an increase in the value of U. Then the increment for u is less than that for U. Therefore, for large U, u^* is scarcely affected by U. For small U, however, U effectively plays a part of the upper limit. If the safety stock level is optimally chosen for each U, there is hardly change in the optimal total cost $C(u^*)$.

Example 2. Suppose that the model parameters are given by $\lambda = 10$!$C_p = 10$, $C_r = 1$, and U taking values $500, 5000, 10000, 15000, 20000$. Then, Table 2 gives the optimal safety stock level u^* and the corresponding expected cost per unit time, $C(u^*)$, obtained for θ taking values $0.005, 0.01, 0.015, 0.02, 0.025, 0.03$.

For Table 2, following are observed: An increase in the value of $C(u^*)$ does not change as much as that of θ in the same way of Example 1. However, for fixed U, θ gives a great influence to u^*. Hence, in deciding the optimal value of u, the value of θ has to be looked upon as important. If the product of θ and U is constant, the optimal safety stock level u^* is in inverse proportion to θ, but the corresponding cost $C(u^*)$ remains unchanged.

Eq.(18) also states that u^* is unchanged if both C_r and C_p increase k times without change of the value of ratio C_r/C_p. Then, $C(u^*)$ also increases k times because it is a linear function in C_p. Obviously, $C(u^*)$ is a linear function in λ and C_p.

By the above statements, we conclude that u^* is very sensitive to θ, and $C(u^*)$ is sensitive to λ and C_p.

5 Conclusion. This paper considered a model under the assumptions such that the consumers arrive at a store according to a Poisson process, that the ordered resource arrives instantaneously, and that the constant ordering cost and the constant loss for stocking-out are charged regardless of the order quantity and the quantity of shortages, respectively. As a result, we knew that the optimal safety stock level u^*, which is important on operating an efficient management, is deeply concerned with the ratio of the ordering cost and the loss for stocking-out, C_r/C_p, from Eqs. (12) and (14). Furthermore, we saw that it is deeply with θ. Hence the manager should establish u^* in enough consideration of these values. We also knew that the optimal expected cost $C(u^*)$ increases in proportion to λ and C_p.

On the other hand, more general models will have the following assumptions: The necessary cost on ordering is in proportion to the order quantity. It takes much time in the arrival of commodities. The holding cost is charged for on-hand inventory quantity. These are left as further research problems.

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