GENERALIZED K-ASSOCIATIVE BCI-ALGEBRAS

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Abstract. In this paper, we discuss the BCI-algebras satisfying $(x*y)*z^k \leq x*(y*z)$, where k is a fixed positive integer,and give some properties of such algebras.

1. Introduction and Preliminaries In [1], Q.P.Hu and K.Iseki discussed the BCI-algebras satisfying $(x*y)*z = x*(y*z)$, which is called an associative BCI-algebra. Moreover, it's proved that X is associative if and only if $0*x = x$ for all in X. In [2], W.Huang discussed the BCI-algebras satisfying $0*x^k = x$, which is called a K-associative BCI-algebra. Moreover, it's proved that X is k-associative if and only if $(x*y)*z^k = x*(y*z)$. In [3], C.C.Xi investigated the BCI-algebras satisfying $(x*y)*z \leq x*(y*z)$, which is called a quasi-associative BCI-algebra. In this paper, we discuss the BCI-algebras satisfying $(x*y)*z^k \leq x*(y*z)$, which is called a generalized K-associative BCI-algebra.

For any elements x, y in a BCI-algebra X, we use $x*y^n$ denotes the elment $(\cdots(x*y)*y\cdots)*y$, where y occurs n times.

A BCI-algebra is an algebra $(X;*,0)$ of type $(2,0)$ with the following conditions.

(I) $((x*y)*(x*z))*(z*y) = 0$

(II) $(x*(x*y))*y = 0$

(III) $x*x = 0$

(IV) $x*y = 0 = y*x$ implies $x = y$.

For a BCI-algebra $X, P(X) = \{x \in X \mid 0*x = 0\}$ is called p-radical of X. If $P(X) = 0$, then we call X is a p-semisimple BCI-algebra. For any positive integer k, put $N_k(X) = \{x \in X \mid 0*x^k = 0\}$.

Definition 1.1 A BCI-algebra X is called a generalized K-associative if $(x*y)*z^k \leq x*(y*z)$ for any $x, y, z \in X$.

It's clear that if X is a generalized K-associative BCI-algebra then $0*x^{k+1} = 0$. In fact, let $x = 0$ and $y = z = x$, we have $0*x^{k+1} \leq 0*(x*x) = 0$, that is, $0*x^{k+1} = 0$.

Lemma 1.2 ([2]) Let X be a BCI-algebra and $P(X)$ the p-radical of X. Then $X = N^{k+1}(X)$ for some positive integer k if and only if $X/P(X)$ is K-associative.

Lemma 1.3 ([2]) Let X be a BCI-algebra and k a positive integer, then the following conditions are equivalent:

(i) $0*x = 0*(0*x^k)$

(ii) $0*(0*x) = (0*x^k)$

(iii) $x \in N_{k+1}(X)$

Lemma 1.4 ([2]) Let X be a BCI-algebra and k a positive integer, then the following conditions are equivalent:

(i) X is K-associative

(ii) $(x*y)*z^k = x*(y*z)$ for all x, y and z in X.

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Example 1.5 (i) Any K-associative BCI-algebra is generalized K-associative.

(ii) Let $X = \{0, a, b\}$ and $*$ be given the table

$*$	0	a	b
0	0	0	b
a	0	0	b
b	b	b	0

Then X is generalized K-associative, but not K-associative.

2. Main Results

Theorem 2.1 Let X be a BCI-algebra and $P(X)$ the p-radical of X, then the following conditions are equivalent: (i) X is generalized K-associative.

(ii) $0 * x^k = 0 * (0 * x)$ for any $x \in X$.

(iii) $X/P(X)$ is k-associative.

Proof. (i) implies (ii) Assume X is generalized K-associative, we have $(0 * 0) * z^k \leq 0 * (z * z)$, that is, $0 * z^k \leq 0 * (0 * z)$. On the other hand, we have $(0 * (0 * z)) * (0 * z^k) = 0 * ((0 * z) * z^k) = 0 * (0 * z^{k+1}) = 0$

(ii) implies(i). Assume that $0 * x^k = 0 * (0 * x)$ holds for any $x \in X$, we have

$$\begin{aligned} ((x * y) * z^k) * (x * (y * z)) &= ((x * y) * (x * (y * z))) * z^k \\ &\leq ((y * z) * y) * z^k \\ &= ((y * y) * z) * z^k \\ &= 0 * z^{k+1} \\ &= 0 \end{aligned}$$

that is, $(x * y) * z^k \leq x * (y * z)$ for any $x, y, z \in X$. Therefore X is generalized K-associative.

(ii) implies (iii). Assume that $0 * x^k = 0 * (0 * x)$ holds for any $x \in X$, we have $x \in N^{k+1}(X)$ by Lemma 1.3. Hence $X/P(X)$ is K-associative by Lemma 1.2.

(iii) implies (ii). Assume that $X/P(X)$ is K-assoacitive, we have $X = N^{k+1}(X)$ by Lemma 1.2, and that $0 * (0 * x) = 0 * x^k$ for any $x \in X$ by Lemma 1.3.

Theorem 2.2 Every generalized k-associative BCI-algebra contains a K-associative BCI-algebra $A(X)$ such that $X/P(X) \cong A(X)$.

Proof. Put $A(X) = \{x \in X \mid 0 * x^k = x\}$, it's a k-associative subalgebra. Define a homomorphism by

$$\Phi : A(X) \to X/P(X)$$

Let $(X) = C_0$ for some x in $A(X)$, this means $C_x = C_0$ and $x \in P(X)$. Hence $x = 0 * x^k = 0$, and Φ is monic. From Theorem 2.1, we know that $X/P(X)$ is K-associative, therefore $C_x = C_0 * C_x^k = C_{0*x^k}$ holds for each x in $A(X)$. Which shows that is epimorphism, because $0 * x^k \in A(X)$. This completes the proof.

Definition 2.3 An ideal I is called generalized K-associative if for each x in I, we have $0 * x^k = 0 * (0 * x)$.

Theorem 2.4 Every BCI-algebras contains a maximal generalized K-associative ideal, which is also a subalgebra.

Proof. Put $Q(X) = \{x \in X \mid 0 * x^k = 0 * (0 * x)\}$, then it's a subalgebra. In fact, Assume $x, y \in Q(X)$, then $0 * x^k = 0 * (0 * x)$ and $0 * y^k = 0 * (0 * y)$. Hence $0 * (x * y)^k = (0 * x^k) * (0 * y^k) = (0 * (0 * x)) * (0 * (0 * y)) = 0 * ((0 * x) * (0 * y)) = 0 * (0 * (x * y))$ and that $x * y \in Q(X)$. Now we show that it's also an ideal of X. Assume $y, x * y \in Q(X)$, then $0 * y^{k+1} = 0$ and $0 * (x * y)^{k+1} = 0$. Hence $(0 * x)^k * (0 * y^k) = 0 * (x * y)^k \leq x * y$ and $(0 * x^k) * (x * y) \leq 0 * y^k$. That is, $(0 * (x * y)) * x^k \leq 0 * y^k$. Hence $0 * x^k = ((0 * (x * y)) * x^k) * (0 * (x * y)) \leq (0 * y^k) * (0 * (x * y)) \leq (x * y) * y^k$. Therefore $(0 * x^k) * x \leq ((x * y) * y^k) * x = ((x * x) * y) * y^k = 0 * y^{k+1} = 0$. This implies that $x \in Q(X)$. The proof is completed.

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